



SIGUENOS EN:



LIBROS UNIVERISTARIOS Y SOLUCIONARIOS DE
MUCHOS DE ESTOS LIBROS GRATIS EN
DESCARGA DIRECTA

VISITANOS PARA DESCARGARLOS GRATIS.

CHAPTER 1

Limits and Their Properties

Section 1.1	A Preview of Calculus	305
Section 1.2	Finding Limits Graphically and Numerically	305
Section 1.3	Evaluating Limits Analytically	309
Section 1.4	Continuity and One-Sided Limits	315
Section 1.5	Infinite Limits	320
Review Exercises		324
Problem Solving		327

CHAPTER 1

Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Even-Numbered Exercises

2. Calculus: velocity is not constant

$$\text{Distance} \approx (20 \text{ ft/sec})(15 \text{ seconds}) = 300 \text{ feet}$$

4. Precalculus: rate of change = slope = 0.08

6. Precalculus: Area = $\pi(\sqrt{2})^2$
= 2π

8. Precalculus: Volume = $\pi(3)^2 6 = 54\pi$

10. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

$$\text{Area} \approx \frac{1}{2} \left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5} \right) \approx 9.145$$

(b) You could improve the approximation by using more rectangles.

Section 1.2 Finding Limits Graphically and Numerically

2.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} \approx 0.25 \quad (\text{Actual limit is } \frac{1}{4}.)$$

4.

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.2485	-0.2498	-0.2500	-0.2500	-0.2502	-0.2516

$$\lim_{x \rightarrow -3} \frac{\sqrt{1-x}-2}{x+3} \approx -0.25 \quad (\text{Actual limit is } -\frac{1}{4}.)$$

6.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$$\lim_{x \rightarrow 4} \frac{[x/(x+1)] - (4/5)}{x-4} \approx 0.04 \quad (\text{Actual limit is } \frac{1}{25}.)$$

8.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \approx 0.0000 \quad (\text{Actual limit is } 0.) \quad (\text{Make sure you use radian mode.})$$

10. $\lim_{x \rightarrow 1} (x^2 + 2) = 3$

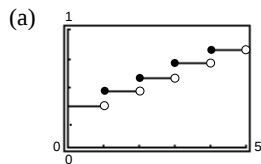
12. $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x^2 + 2) = 3$

14. $\lim_{x \rightarrow 3} \frac{1}{x - 3}$ does not exist since the function increases and decreases without bound as x approaches 3.

16. $\lim_{x \rightarrow 0} \sec x = 1$

18. $\lim_{x \rightarrow 1} \sin(\pi x) = 0$

20. $C(t) = 0.35 - 0.12\lfloor -(t - 1) \rfloor$



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
$C(t)$	0.59	0.71	0.71	0.71	0.71	0.71	0.71

$$\lim_{t \rightarrow 3.5} C(t) = 0.71$$

(c)

t	3	2.5	2.9	3	3.1	3.5	4
$C(t)$	0.47	0.59	0.59	0.59	0.71	0.71	0.71

$$\lim_{t \rightarrow 3.5} C(t) \text{ does not exist. The values of } C \text{ jump from } 0.59 \text{ to } 0.71 \text{ at } t = 3.$$

22. You need to find δ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 &< x^2 - 4 < 0.2 \\ 4 - 0.2 &< x^2 < 4 + 0.2 \\ 3.8 &< x^2 < 4.2 \\ \sqrt{3.8} &< x < \sqrt{4.2} \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$\begin{aligned} -(\sqrt{4.2} - 2) &< x - 2 < \sqrt{4.2} - 2 \\ \sqrt{3.8} - 2 &< x - 2 < \sqrt{4.2} - 2. \end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < \epsilon = 0.2.$$

24. $\lim_{x \rightarrow 4} \left(4 - \frac{x}{2}\right) = 2$

$$\left| \left(4 - \frac{x}{2}\right) - 2 \right| < 0.01$$

$$\left| 2 - \frac{x}{2} \right| < 0.01$$

$$\left| -\frac{1}{2}(x - 4) \right| < 0.01$$

$$0 < |x - 4| < 0.02 = \delta$$

Hence, if $0 < |x - 4| < \delta = 0.02$, you have

$$\left| -\frac{1}{2}(x - 4) \right| < 0.01$$

$$\left| 2 - \frac{x}{2} \right| < 0.01$$

$$\left| \left(4 - \frac{x}{2}\right) - 2 \right| < 0.01$$

$$|f(x) - L| < 0.01$$

$$26. \lim_{x \rightarrow 5} (x^2 + 4) = 29$$

$$|(x^2 + 4) - 29| < 0.01$$

$$|x^2 - 25| < 0.01$$

$$|(x + 5)(x - 5)| < 0.01$$

$$|x - 5| < \frac{0.01}{|x + 5|}$$

If we assume $4 < x < 6$, then $\delta = 0.01/11 \approx 0.0009$.

Hence, if $0 < |x - 5| < \delta = \frac{0.01}{11}$, you have

$$|x - 5| < \frac{0.01}{11} < \frac{1}{|x + 5|}(0.01)$$

$$|x - 5||x + 5| < 0.01$$

$$|x^2 - 25| < 0.01$$

$$|(x^2 + 4) - 29| < 0.01$$

$$|f(x) - L| < 0.01$$

$$30. \lim_{x \rightarrow 1} \left(\frac{2}{3}x + 9\right) = \frac{2}{3}(1) + 9 = \frac{29}{3}$$

Given $\epsilon > 0$:

$$\left|\left(\frac{2}{3}x + 9\right) - \frac{29}{3}\right| < \epsilon$$

$$\left|\frac{2}{3}x - \frac{2}{3}\right| < \epsilon$$

$$\frac{2}{3}|x - 1| < \epsilon$$

$$|x - 1| < \frac{3}{2}\epsilon$$

Hence, let $\delta = (3/2)\epsilon$.

Hence, if $0 < |x - 1| < \delta = \frac{3}{2}\epsilon$, you have

$$|x - 1| < \frac{3}{2}\epsilon$$

$$\left|\frac{2}{3}x - \frac{2}{3}\right| < \epsilon$$

$$\left|\left(\frac{2}{3}x + 9\right) - \frac{29}{3}\right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$34. \lim_{x \rightarrow 4} \sqrt{x} = \sqrt{4} = 2$$

Given $\epsilon > 0$:

$$|\sqrt{x} - 2| < \epsilon$$

$$|\sqrt{x} - 2| |\sqrt{x} + 2| < \epsilon |\sqrt{x} + 2|$$

$$|x - 4| < \epsilon |\sqrt{x} + 2|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\epsilon$. Then,

$$0 < |x - 4| < \delta = 3\epsilon \Rightarrow |x - 4| < \epsilon |\sqrt{x} + 2| \\ \Rightarrow |\sqrt{x} - 2| < \epsilon.$$

$$28. \lim_{x \rightarrow -3} (2x + 5) = -1$$

Given $\epsilon > 0$:

$$|(2x + 5) - (-1)| < \epsilon$$

$$|2x + 6| < \epsilon$$

$$2|x + 3| < \epsilon$$

$$|x + 3| < \frac{\epsilon}{2} = \delta$$

Hence, let $\delta = \epsilon/2$.

Hence, if $0 < |x + 3| < \delta = \frac{\epsilon}{2}$, you have

$$|x + 3| < \frac{\epsilon}{2}$$

$$|2x + 6| < \epsilon$$

$$|(2x + 5) - (-1)| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$32. \lim_{x \rightarrow 2} (-1) = -1$$

Given $\epsilon > 0$: $|-1 - (-1)| < \epsilon$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|(-1) - (-1)| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$36. \lim_{x \rightarrow 3} |x - 3| = 0$$

Given $\epsilon > 0$:

$$|(x - 3) - 0| < \epsilon$$

$$|x - 3| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence for $0 < |x - 3| < \delta = \epsilon$, you have

$$|x - 3| < \epsilon$$

$$||x - 3| - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

38. $\lim_{x \rightarrow -3} (x^2 + 3x) = 0$

Given $\epsilon > 0$:

$$|(x^2 + 3x) - 0| < \epsilon$$

$$|x(x + 3)| < \epsilon$$

$$|x + 3| < \frac{\epsilon}{|x|}$$

If we assume $-4 < x < -2$, then $\delta = \epsilon/4$.

Hence for $0 < |x - (-3)| < \delta = \frac{\epsilon}{4}$, you have

$$|x + 3| < \frac{1}{4}\epsilon < \frac{1}{|x|}\epsilon$$

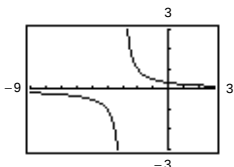
$$|x(x + 3)| < \epsilon$$

$$|x^2 + 3x - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

42. $f(x) = \frac{x - 3}{x^2 - 9}$

$$\lim_{x \rightarrow 3} f(x) = \frac{1}{6}$$



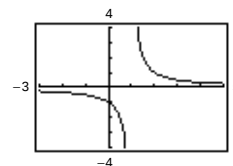
The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $(3, \frac{1}{6})$.

46. Let $p(x)$ be the atmospheric pressure in a plane at altitude x (in feet).

$$\lim_{x \rightarrow 0^+} p(x) = 14.7 \text{ lb/in}^2$$

40. $f(x) = \frac{x - 3}{x^2 - 4x + 3}$

$$\lim_{x \rightarrow 3} f(x) = \frac{1}{2}$$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $(3, \frac{1}{2})$.

44. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x \rightarrow 2} f(x) = 4$ has no bearing on the value of f at 2.

50. True

52. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

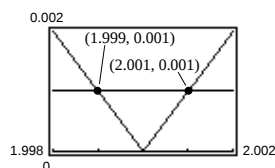
$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 - 4x) = 0 \text{ and } f(4) = 10 \neq 0$$

54. $\lim_{x \rightarrow 4} \frac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

48.



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x - 2| < 0.001, \left| \frac{x^2 - 4}{x - 2} - 4 \right| < 0.001.$$

56. $f(x) = mx + b$, $m \neq 0$. Let $\epsilon > 0$ be given. Take $\delta = \frac{\epsilon}{|m|}$.

If $0 < |x - c| < \delta = \frac{\epsilon}{|m|}$, then

$$|m||x - c| < \epsilon$$

$$|mx - mc| < \epsilon$$

$$|(mx + b) - (mc + b)| < \epsilon$$

which shows that $\lim_{x \rightarrow c} (mx + b) = mc + b$.

58. $\lim_{x \rightarrow c} g(x) = L$, $L > 0$. Let $\epsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|g(x) - L| < \epsilon = \frac{1}{2}L$.

That is,

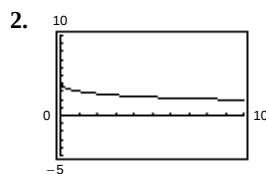
$$-\frac{1}{2}L < g(x) - L < \frac{1}{2}L$$

$$\frac{1}{2}L < g(x) < \frac{3}{2}L$$

Hence for x in the interval $(c - \delta, c + \delta)$, $x \neq c$,

$$g(x) > \frac{1}{2}L > 0.$$

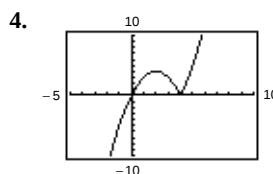
Section 1.3 Evaluating Limits Analytically



(a) $\lim_{x \rightarrow 4} g(x) = 2.4$

(b) $\lim_{x \rightarrow 0} g(x) = 4$

$$g(x) = \frac{12(\sqrt{x} - 3)}{x - 9}$$



(a) $\lim_{t \rightarrow 4} f(t) = 0$

(b) $\lim_{t \rightarrow -1} f(t) = -5$

$$f(t) = t|t - 4|$$

6. $\lim_{x \rightarrow -2} x^3 = (-2)^3 = -8$

8. $\lim_{x \rightarrow -3} (3x + 2) = 3(-3) + 2 = -7$

10. $\lim_{x \rightarrow 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

12. $\lim_{x \rightarrow 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

14. $\lim_{x \rightarrow -3} \frac{2}{x + 2} = \frac{2}{-3 + 2} = -2$

16. $\lim_{x \rightarrow 3} \frac{2x - 3}{x + 5} = \frac{2(3) - 3}{3 + 5} = \frac{3}{8}$

18. $\lim_{x \rightarrow 3} \frac{\sqrt{x + 1}}{x - 4} = \frac{\sqrt{3 + 1}}{3 - 4} = -2$

20. $\lim_{x \rightarrow 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

22. $\lim_{x \rightarrow 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

24. (a) $\lim_{x \rightarrow -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x \rightarrow 4} g(x) = 4^2 = 16$

(c) $\lim_{x \rightarrow -3} g(f(x)) = g(4) = 16$

26. (a) $\lim_{x \rightarrow 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x \rightarrow 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim_{x \rightarrow 4} g(f(x)) = g(21) = 3$

28. $\lim_{x \rightarrow \pi} \tan x = \tan \pi = 0$

30. $\lim_{x \rightarrow 1} \sin \frac{\pi x}{2} = \sin \frac{\pi}{2} = 1$

32. $\lim_{x \rightarrow \pi} \cos 3x = \cos 3\pi = -1$

34. $\lim_{x \rightarrow 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

36. $\lim_{x \rightarrow 7} \sec\left(\frac{\pi x}{6}\right) = \sec \frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

$$38. (a) \lim_{x \rightarrow c} [4f(x)] = 4 \lim_{x \rightarrow c} f(x) = 4\left(\frac{3}{2}\right) = 6$$

$$(b) \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = \frac{3}{2} + \frac{1}{2} = 2$$

$$(c) \lim_{x \rightarrow c} [f(x)g(x)] = \left[\lim_{x \rightarrow c} f(x)\right]\left[\lim_{x \rightarrow c} g(x)\right] = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$(d) \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{3/2}{1/2} = 3$$

$$40. (a) \lim_{x \rightarrow c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \rightarrow c} f(x)} = \sqrt[3]{27} = 3$$

$$(b) \lim_{x \rightarrow c} \frac{f(x)}{18} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} 18} = \frac{27}{18} = \frac{3}{2}$$

$$(c) \lim_{x \rightarrow c} [f(x)]^2 = \left[\lim_{x \rightarrow c} f(x)\right]^2 = (27)^2 = 729$$

$$(d) \lim_{x \rightarrow c} [f(x)]^{2/3} = \left[\lim_{x \rightarrow c} f(x)\right]^{2/3} = (27)^{2/3} = 9$$

$$42. f(x) = x - 3 \text{ and } h(x) = \frac{x^2 - 3x}{x} \text{ agree except at } x = 0.$$

$$(a) \lim_{x \rightarrow -2} h(x) = \lim_{x \rightarrow -2} f(x) = -5$$

$$(b) \lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} f(x) = -3$$

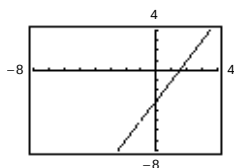
$$44. g(x) = \frac{1}{x-1} \text{ and } f(x) = \frac{x}{x^2 - x} \text{ agree except at } x = 0.$$

$$(a) \lim_{x \rightarrow 1} f(x) \text{ does not exist.}$$

$$(b) \lim_{x \rightarrow 0} f(x) = -1$$

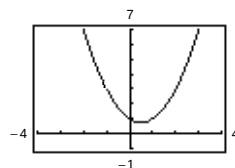
$$46. f(x) = \frac{2x^2 - x - 3}{x + 1} \text{ and } g(x) = 2x - 3 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = -5$$



$$48. f(x) = \frac{x^3 + 1}{x + 1} \text{ and } g(x) = x^2 - x + 1 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = 3$$



$$50. \lim_{x \rightarrow 2} \frac{2 - x}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{-(x - 2)}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{-1}{x + 2} = -\frac{1}{4}$$

$$52. \lim_{x \rightarrow 4} \frac{x^2 - 5x + 4}{x^2 - 2x - 8} = \lim_{x \rightarrow 4} \frac{(x - 4)(x - 1)}{(x - 4)(x + 2)} = \lim_{x \rightarrow 4} \frac{(x - 1)}{(x + 2)} = \frac{3}{6} = \frac{1}{2}$$

$$54. \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \cdot \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} = \lim_{x \rightarrow 0} \frac{2+x-2}{(\sqrt{2+x} + \sqrt{2})x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$56. \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x-3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x-3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{4}$$

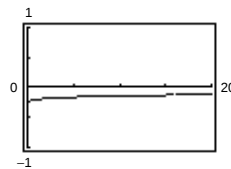
$$58. \lim_{x \rightarrow 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x \rightarrow 0} \frac{\frac{4 - (x+4)}{4(x+4)}}{x} = \lim_{x \rightarrow 0} \frac{-1}{4(x+4)} = -\frac{1}{16}$$

$$60. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

$$\begin{aligned}
 62. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2
 \end{aligned}$$

$$64. f(x) = \frac{4 - \sqrt{x}}{x - 16}$$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	-.1252	-.125	-.125	?	-.125	-.125	-.1248

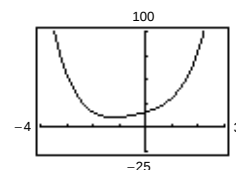


It appears that the limit is -0.125 .

$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{x - 16} &= \lim_{x \rightarrow 16} \frac{(4 - \sqrt{x})}{(\sqrt{x} + 4)(\sqrt{x} - 4)} \\
 &= \lim_{x \rightarrow 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.
 \end{aligned}$$

$$66. \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} = 80$$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41



$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} \\
 &= \lim_{x \rightarrow 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.
 \end{aligned}$$

(Hint: Use long division to factor $x^5 - 32$.)

$$68. \lim_{x \rightarrow 0} \frac{3(1 - \cos x)}{x} = \lim_{x \rightarrow 0} \left[3 \left(\frac{1 - \cos x}{x} \right) \right] = (3)(0) = 0$$

$$70. \lim_{\theta \rightarrow 0} \frac{\cos \theta \tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\begin{aligned}
 72. \lim_{x \rightarrow 0} \frac{\tan^2 x}{x} &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos^2 x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x} \right] \\
 &= (1)(0) = 0
 \end{aligned}$$

$$74. \lim_{\phi \rightarrow \pi} \phi \sec \phi = \pi(-1) = -\pi$$

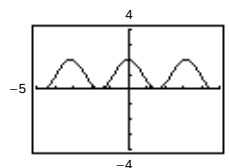
$$\begin{aligned}
 76. \lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x} &= \lim_{x \rightarrow \pi/4} \frac{\cos x - \sin x}{\sin x \cos x - \cos^2 x} \\
 &= \lim_{x \rightarrow \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)} \\
 &= \lim_{x \rightarrow \pi/4} \frac{-1}{\cos x} \\
 &= \lim_{x \rightarrow \pi/4} (-\sec x) \\
 &= -\sqrt{2}
 \end{aligned}$$

$$78. \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \left[2 \left(\frac{\sin 2x}{2x} \right) \left(\frac{1}{3} \right) \left(\frac{3x}{\sin 3x} \right) \right] = 2(1) \left(\frac{1}{3} \right) (1) = \frac{2}{3}$$

80. $f(h) = (1 + \cos 2h)$

h	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(h)$	1.98	1.9998	2	?	2	1.9998	1.98

Analytically, $\lim_{h \rightarrow 0} (1 + \cos 2h) = 1 + \cos(0) = 1 + 1 = 2$.

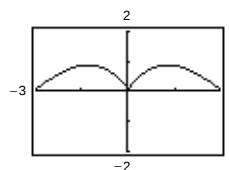


The limit appear to equal 2.

82. $f(x) = \frac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

Analytically, $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt[3]{x}} = \lim_{x \rightarrow 0} \sqrt[3]{x^2} \left(\frac{\sin x}{x} \right) = (0)(1) = 0$.



The limit appear to equal 0.

$$\begin{aligned}
 84. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
 &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

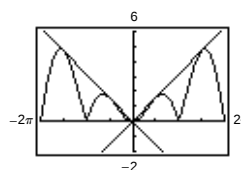
$$\begin{aligned}
 86. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2x + h - 4)}{h} = \lim_{h \rightarrow 0} (2x + h - 4) = 2x - 4
 \end{aligned}$$

88. $\lim_{x \rightarrow a} [b - |x - a|] \leq \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} [b + |x - a|]$

$$b \leq \lim_{x \rightarrow a} f(x) \leq b$$

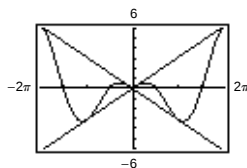
Therefore, $\lim_{x \rightarrow a} f(x) = b$.

90. $f(x) = |x \sin x|$



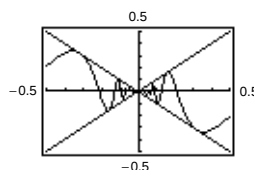
$$\lim_{x \rightarrow 0} |x \sin x| = 0$$

92. $f(x) = |x| \cos x$



$$\lim_{x \rightarrow 0} |x| \cos x = 0$$

94. $h(x) = x \cos \frac{1}{x}$

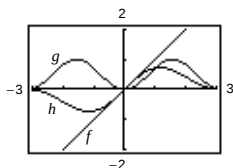


$$\lim_{x \rightarrow 0} \left(x \cos \frac{1}{x} \right) = 0$$

96. $f(x) = \frac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

98. If a function f is squeezed between two functions h and g , $h(x) \leq f(x) \leq g(x)$, and h and g have the same limit L as $x \rightarrow c$, then $\lim_{x \rightarrow c} f(x)$ exists and equals L .

100. $f(x) = x$, $g(x) = \sin^2 x$, $h(x) = \frac{\sin^2 x}{x}$



When you are “close to” 0 the magnitude of g is “smaller” than the magnitude of f and the magnitude of g is approaching zero “faster” than the magnitude of f . Thus, $|g|/|f| \approx 0$ when x is “close to” 0

102. $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\frac{1000}{16}} = \frac{5\sqrt{10}}{2}$ seconds

$$\begin{aligned} \lim_{t \rightarrow 5\sqrt{10}/2} \frac{s\left(\frac{5\sqrt{10}}{2}\right) - s(t)}{\frac{5\sqrt{10}}{2} - t} &= \lim_{t \rightarrow 5\sqrt{10}/2} \frac{0 - (-16t^2 + 1000)}{\frac{5\sqrt{10}}{2} - t} \\ &= \lim_{t \rightarrow 5\sqrt{10}/2} \frac{16\left(t^2 - \frac{125}{2}\right)}{\frac{5\sqrt{10}}{2} - t} = \lim_{t \rightarrow 5\sqrt{10}/2} \frac{16\left(t + \frac{5\sqrt{10}}{2}\right)\left(t - \frac{5\sqrt{10}}{2}\right)}{-\left(t - \frac{5\sqrt{10}}{2}\right)} \\ &= \lim_{t \rightarrow 5\sqrt{10}/2} -16\left(t + \frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec} \end{aligned}$$

104. $-4.9t^2 + 150 = 0$ when $t = \sqrt{\frac{150}{4.9}} = \sqrt{\frac{1500}{49}} \approx 5.53$ seconds.

The velocity at time $t = a$ is

$$\begin{aligned} \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow a} \frac{(-4.9a^2 + 150) - (-4.9t^2 + 150)}{a - t} = \lim_{t \rightarrow a} \frac{-4.9(a - t)(a + t)}{a - t} \\ &= \lim_{t \rightarrow a} -4.9(a + t) = -2a(4.9) = -9.8a \text{ m/sec.} \end{aligned}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

106. Suppose, on the contrary, that $\lim_{x \rightarrow c} g(x)$ exists. Then, since $\lim_{x \rightarrow c} f(x)$ exists, so would $\lim_{x \rightarrow c} [f(x) + g(x)]$, which is a contradiction. Hence, $\lim_{x \rightarrow c} g(x)$ does not exist.

108. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned} \lim_{x \rightarrow c} x^n &= \lim_{x \rightarrow c} (x x^{n-1}) = \left[\lim_{x \rightarrow c} x \right] \left[\lim_{x \rightarrow c} x^{n-1} \right] \\ &= c \left[\lim_{x \rightarrow c} (x x^{n-2}) \right] = c \left[\lim_{x \rightarrow c} x \right] \left[\lim_{x \rightarrow c} x^{n-2} \right] \\ &= c(c) \lim_{x \rightarrow c} (x x^{n-3}) = \dots = c^n. \end{aligned}$$

110. Given $\lim_{x \rightarrow c} f(x) = 0$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \epsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x \rightarrow c} |f(x)| = 0$.

112. (a) If $\lim_{x \rightarrow c} |f(x)| = 0$, then $\lim_{x \rightarrow c} [-|f(x)|] = 0$.

$$-|f(x)| \leq f(x) \leq |f(x)|$$

$$\lim_{x \rightarrow c} [-|f(x)|] \leq \lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} |f(x)|$$

$$0 \leq \lim_{x \rightarrow c} f(x) \leq 0$$

Therefore, $\lim_{x \rightarrow c} f(x) = 0$.

(b) Given $\lim_{x \rightarrow c} f(x) = L$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

Since $||f(x)| - |L|| \leq |f(x) - L| < \epsilon$ for $|x - c| < \delta$, then $\lim_{x \rightarrow c} |f(x)| = |L|$.

114. True. $\lim_{x \rightarrow 0} x^3 = 0^3 = 0$

116. False. Let $f(x) = \begin{cases} x & x \neq 1 \\ 3 & x = 1 \end{cases}$, $c = 1$

Then $\lim_{x \rightarrow 1} f(x) = 1$ but $f(1) \neq 1$.

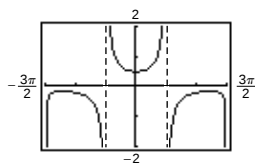
118. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \neq 0$. But $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$.

$$\begin{aligned} 120. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \\ &= \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} \right] \left[\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} \right] \\ &= (1)(0) = 0 \end{aligned}$$

122. $f(x) = \frac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \neq 0, \pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x \rightarrow 0} f(x) = \frac{1}{2}$

$$\begin{aligned} (d) \frac{\sec x - 1}{x^2} &= \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)} \\ &= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x} \left(\frac{\sin^2 x}{x^2} \right) \frac{1}{\sec x + 1} \end{aligned}$$

$$\begin{aligned} \text{Hence, } \lim_{x \rightarrow 0} \frac{\sec x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\cos^2 x} \left(\frac{\sin^2 x}{x^2} \right) \frac{1}{\sec x + 1} \\ &= 1(1) \left(\frac{1}{2} \right) = \frac{1}{2}. \end{aligned}$$

124. The calculator was set in degree mode, instead of radian mode.

Section 1.4 Continuity and One-Sided Limits

2. (a) $\lim_{x \rightarrow -2^+} f(x) = -2$

(b) $\lim_{x \rightarrow -2^-} f(x) = -2$

(c) $\lim_{x \rightarrow -2} f(x) = -2$

The function is continuous at $x = -2$.

4. (a) $\lim_{x \rightarrow -2^+} f(x) = 2$

(b) $\lim_{x \rightarrow -2^-} f(x) = 2$

(c) $\lim_{x \rightarrow -2} f(x) = 2$

The function is NOT continuous at $x = -2$.

6. (a) $\lim_{x \rightarrow -1^+} f(x) = 0$

(b) $\lim_{x \rightarrow -1^-} f(x) = 2$

(c) $\lim_{x \rightarrow -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

8. $\lim_{x \rightarrow 2^+} \frac{2-x}{x^2-4} = \lim_{x \rightarrow 2^+} -\frac{1}{x+2} = -\frac{1}{4}$

$$\begin{aligned} 10. \lim_{x \rightarrow 4^-} \frac{\sqrt{x}-2}{x-4} &= \lim_{x \rightarrow 4^-} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\ &= \lim_{x \rightarrow 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4} \end{aligned}$$

12. $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^+} \frac{x-2}{x-2} = 1$

$$\begin{aligned} 14. \lim_{\Delta x \rightarrow 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^+} (2x + \Delta x + 1) \\ &= 2x + 0 + 1 = 2x + 1 \end{aligned}$$

$$\begin{aligned} 16. \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (-x^2 + 4x - 2) = 2 \\ \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (x^2 - 4x + 6) = 2 \\ \lim_{x \rightarrow 2} f(x) &= 2 \end{aligned}$$

18. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1-x) = 0$

20. $\lim_{x \rightarrow \pi/2} \sec x$ does not exist since

$$\lim_{x \rightarrow (\pi/2)^+} \sec x \text{ and } \lim_{x \rightarrow (\pi/2)^-} \sec x \text{ do not exist.}$$

22. $\lim_{x \rightarrow 2^-} (2x - \lfloor x \rfloor) = 2(2) - 2 = 2$

24. $\lim_{x \rightarrow 1} \left(1 - \left\lfloor -\frac{x}{2} \right\rfloor \right) = 1 - (-1) = 2$

26. $f(x) = \frac{x^2-1}{x+1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

$$28. f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases} \text{ has discontinuity at } x = 1 \text{ since } f(1) = 2 \neq \lim_{x \rightarrow 1} f(x) = 1.$$

30. $f(t) = 3 - \sqrt{9-t^2}$ is continuous on $[-3, 3]$.

32. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

34. $f(x) = \frac{1}{x^2 - 1}$ is continuous for all real x .

36. $f(x) = \cos \frac{\pi x}{2}$ is continuous for all real x .

38. $f(x) = \frac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow -1} f(x)$ do not exist.

40. $f(x) = \frac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim_{x \rightarrow -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{1}{x + 3} = \frac{1}{6}.$$

42. $f(x) = \frac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since

$\lim_{x \rightarrow -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{1}{x + 2} = \frac{1}{3}.$$

44. $f(x) = \frac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim_{x \rightarrow 3} f(x)$ does not exist.

46. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left. \begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (-2x + 3) = 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 = 1 \end{aligned} \right\} \lim_{x \rightarrow 1} f(x) = 1$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x .

48. $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (-2x) = -4 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 - 4x + 1) = -3 \end{aligned} \right\} \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$

Therefore, f has a nonremovable discontinuity at $x = 2$.

50. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| \leq 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc \frac{\pi x}{6}, & 1 \leq x \leq 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has **possible** discontinuities at $x = 1$, $x = 5$.

1. $f(1) = \csc \frac{\pi}{6} = 2$ $f(5) = \csc \frac{5\pi}{6} = 2$

2. $\lim_{x \rightarrow 1} f(x) = 2$ $\lim_{x \rightarrow 5} f(x) = 2$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$ $f(5) = \lim_{x \rightarrow 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x .

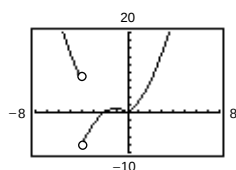
52. $f(x) = \tan \frac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

54. $f(x) = 3 - \llbracket x \rrbracket$ has nonremovable discontinuities at each integer k .

56. $\lim_{x \rightarrow 0^-} f(x) = 0$

$\lim_{x \rightarrow 0^+} f(x) = 0$

f is not continuous at $x = -4$



58. $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{4 \sin x}{x} = 4$

$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0^+} (a - 2x) = a$

Let $a = 4$.

60. $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a}$

$= \lim_{x \rightarrow a} (x + a) = 2a$

Find a such that $2a = 8 \Rightarrow a = 4$.

62. $f(g(x)) = \frac{1}{\sqrt{x-1}}$

Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.

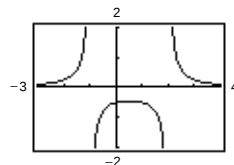
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

64. $f(g(x)) = \sin x^2$

Continuous for all real x

66. $h(x) = \frac{1}{(x+1)(x-2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.



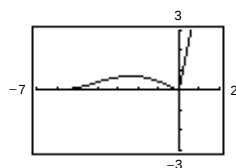
68. $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{(\cos x - 1)}{x} = 0$

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (5x) = 0$

Therefore, $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)



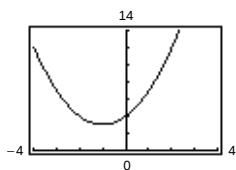
70. $f(x) = x\sqrt{x+3}$

Continuous on $[-3, \infty]$

72. $f(x) = \frac{x+1}{\sqrt{x}}$

Continuous on $(0, \infty)$

74. $f(x) = \frac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

78. $f(x) = \frac{-4}{x} + \tan \frac{\pi x}{8}$ is continuous on $[1, 3]$.

$$f(1) = -4 + \tan \frac{\pi}{8} < 0 \text{ and } f(3) = -\frac{4}{3} + \tan \frac{3\pi}{8} > 0.$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 1 and 3.

82. $h(\theta) = 1 + \theta - 3 \tan \theta$

h is continuous on $[0, 1]$.

$$h(0) = 1 > 0 \text{ and } h(1) \approx -2.67 < 0.$$

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

86. $f(x) = \frac{x^2 + x}{x - 1}$

f is continuous on $[\frac{5}{2}, 4]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$$f\left(\frac{5}{2}\right) = \frac{35}{6} \text{ and } f(4) = \frac{20}{3}$$

$$\frac{35}{6} < 6 < \frac{20}{3}$$

76. $f(x) = x^3 + 3x - 2$ is continuous on $[0, 1]$.

$$f(0) = -2 \text{ and } f(1) = 2$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

80. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$$f(0) = -2 \text{ and } f(1) = 2$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

84. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$$f(0) = 8 \text{ and } f(3) = -1$$

$$-1 < 0 < 8$$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$x = 2 \text{ or } x = 4$$

$$c = 2 \text{ (} x = 4 \text{ is not in the interval.)}$$

Thus, $f(2) = 0$.

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$

$$x^2 + x = 6x - 6$$

$$x^2 - 5x + 6 = 0$$

$$(x - 2)(x - 3) = 0$$

$$x = 2 \text{ or } x = 3$$

$$c = 3 \text{ (} x = 2 \text{ is not in the interval.)}$$

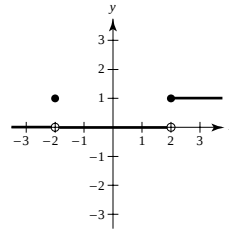
Thus, $f(3) = 6$.

88. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$. Answers will vary.

$$(a) f(x) = \frac{|x - 2|}{x - 2}$$

$$(b) f(x) = \frac{\sin(x + 2)}{x + 2}$$

$$(c) f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$$



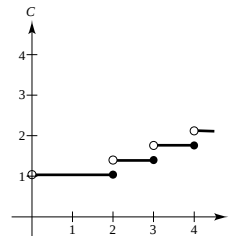
90. If f and g are continuous for all real x , then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x , but f/g is not continuous at $x = \pm 1$.

$$92. C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 + 0.36\lfloor t - 1 \rfloor, & t > 2, t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2, t \text{ is an integer} \end{cases}$$

Nonremovable discontinuity at each integer greater than 2.

You can also write C as

$$C = \begin{cases} 1.04, & 0 < t \leq 2 \\ 1.04 - 0.36\lfloor 2 - t \rfloor, & t > 2 \end{cases}$$



94. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t , where $0 \leq t \leq 10$, the position functions for the run up and the run down are equal.

96. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

98. If $x = 0$, then $f(0) = 0$ and $\lim_{x \rightarrow 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t \rightarrow x} f(t) = 0$ for x rational, whereas

$\lim_{t \rightarrow x} f(t) = \lim_{t \rightarrow x} kt = kx \neq 0$ for x irrational. Hence, f is not continuous for all $x \neq 0$.

100. True

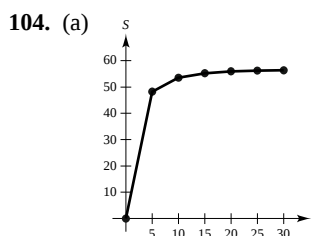
1. $f(c) = L$ is defined.

2. $\lim_{x \rightarrow c} f(x) = L$ exists.

3. $f(c) = \lim_{x \rightarrow c} f(x)$

All of the conditions for continuity are met.

102. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n , respectively. It can have, at most, n discontinuities.



- (b) There appears to be a limiting speed and a possible cause is air resistance.

106. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

108. 1. $f(c)$ is defined.

2. $\lim_{x \rightarrow c} f(x) = \lim_{\Delta x \rightarrow 0} f(c + \Delta x) = f(c)$ exists.
[Let $x = c + \Delta x$. As $x \rightarrow c$, $\Delta x \rightarrow 0$]

3. $\lim_{x \rightarrow c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

110. Define $f(x) = f_2(x) - f_1(x)$. Since f_1 and f_2 are continuous on $[a, b]$, so is f .

$$f(a) = f_2(a) - f_1(a) > 0 \quad \text{and} \quad f(b) = f_2(b) - f_1(b) < 0.$$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$$

Section 1.5 Infinite Limits

2. $\lim_{x \rightarrow -2^+} \frac{1}{x+2} = \infty$

$$\lim_{x \rightarrow -2^-} \frac{1}{x+2} = -\infty$$

4. $\lim_{x \rightarrow -2^+} \sec \frac{\pi x}{4} = \infty$

$$\lim_{x \rightarrow -2^-} \sec \frac{\pi x}{4} = -\infty$$

6. $f(x) = \frac{x}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$$\lim_{x \rightarrow -3^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -3^+} f(x) = \infty$$

8. $f(x) = \sec \frac{\pi x}{6}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-3.864	-19.11	-191.0	-1910	1910	191.0	19.11	3.864

$$\lim_{x \rightarrow -3^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -3^+} f(x) = \infty$$

10. $\lim_{x \rightarrow 2^+} \frac{4}{(x-2)^3} = \infty$

$$\lim_{x \rightarrow 2^-} \frac{4}{(x-2)^3} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

12. $\lim_{x \rightarrow 0^-} \frac{2+x}{x^2(1-x)} = \lim_{x \rightarrow 0^+} \frac{2+x}{x^2(1-x)} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

$$\lim_{x \rightarrow 1^-} \frac{2+x}{x^2(1-x)} = \infty$$

$$\lim_{x \rightarrow 1^+} \frac{2+x}{x^2(1-x)} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote.

14. No vertical asymptote since the denominator is never zero.

16. $\lim_{s \rightarrow -5^-} h(s) = -\infty$ and $\lim_{s \rightarrow -5^+} h(s) = \infty$.

Therefore, $s = -5$ is a vertical asymptote.

$$\lim_{s \rightarrow 5^-} h(s) = -\infty \text{ and } \lim_{s \rightarrow 5^+} h(s) = \infty.$$

Therefore, $s = 5$ is a vertical asymptote.

18. $f(x) = \sec \pi x = \frac{1}{\cos \pi x}$ has vertical asymptotes at

$$x = \frac{2n+1}{2}, n \text{ any integer.}$$

20. $g(x) = \frac{(1/2)x^3 - x^2 - 4x}{3x^2 - 6x - 24} = \frac{1}{6} \frac{x(x^2 - 2x - 8)}{x^2 - 2x - 8}$
 $= \frac{1}{6}x,$

$$x \neq -2, 4$$

No vertical asymptotes. The graph has holes at $x = -2$ and $x = 4$.

22. $f(x) = \frac{4(x^2 + x - 6)}{x(x^3 - 2x^2 - 9x + 18)} = \frac{4(x+3)(x-2)}{x(x-2)(x^2-9)} = \frac{4}{x(x-3)}, x \neq -3, 2$

Vertical asymptotes at $x = 0$ and $x = 3$. The graph has holes at $x = -3$ and $x = 2$.

24. $h(x) = \frac{x^2 - 4}{x^3 + 2x^2 + x + 2} = \frac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$$\lim_{x \rightarrow -2} h(x) = \lim_{x \rightarrow -2} \frac{x-2}{x^2+1} = -\frac{4}{5}.$$

26. $h(t) = \frac{t(t-2)}{(t-2)(t+2)(t^2+4)} = \frac{t}{(t+2)(t^2+4)}, t \neq 2$

Vertical asymptote at $t = -2$. The graph has a hole at $t = 2$.

28. $g(\theta) = \frac{\tan \theta}{\theta} = \frac{\sin \theta}{\theta \cos \theta}$ has vertical asymptotes at

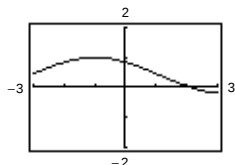
$$\theta = \frac{(2n+1)\pi}{2} = \frac{\pi}{2} + n\pi, n \text{ any integer.}$$

There is no vertical asymptote at $\theta = 0$ since

$$\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.$$

32. $\lim_{x \rightarrow -1} \frac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$



36. $\lim_{x \rightarrow 4^-} \frac{x^2}{x^2 + 16} = \frac{1}{2}$

40. $\lim_{x \rightarrow 3} \frac{x-2}{x^2} = \frac{1}{9}$

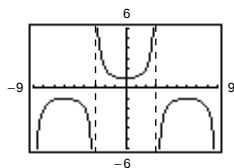
44. $\lim_{x \rightarrow (\pi/2)^+} \frac{-2}{\cos x} = \infty$

48. $\lim_{x \rightarrow (1/2)^-} x^2 \tan \pi x = \infty$ and $\lim_{x \rightarrow (1/2)^+} x^2 \tan \pi x = -\infty$.

Therefore, $\lim_{x \rightarrow (1/2)} x^2 \tan \pi x$ does not exist.

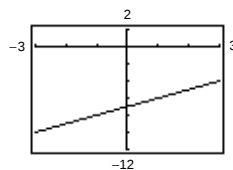
52. $f(x) = \sec \frac{\pi x}{6}$

$$\lim_{x \rightarrow 3^+} f(x) = -\infty$$



56. No. For example, $f(x) = \frac{1}{x^2 + 1}$ has no vertical asymptote.

30. $\lim_{x \rightarrow -1} \frac{x^2 - 6x - 7}{x + 1} = \lim_{x \rightarrow -1} (x - 7) = -8$



Removable discontinuity at $x = -1$

34. $\lim_{x \rightarrow 1^+} \frac{2+x}{1-x} = -\infty$

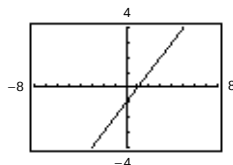
38. $\lim_{x \rightarrow -(1/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3} = \lim_{x \rightarrow -(1/2)^+} \frac{3x - 1}{2x - 3} = \frac{5}{8}$

42. $\lim_{x \rightarrow 0^-} \left(x^2 - \frac{1}{x} \right) = \infty$

46. $\lim_{x \rightarrow 0} \frac{(x+2)}{\cot x} = \lim_{x \rightarrow 0} [(x+2)\tan x] = 0$

50. $f(x) = \frac{x^3 - 1}{x^2 + x + 1}$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} (x - 1) = 0$$



54. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c .

58. $P = \frac{k}{V}$

$$\lim_{V \rightarrow 0^+} \frac{k}{V} = k(\infty) = \infty \quad (\text{In this case we know that } k > 0.)$$

$$60. (a) r = 50\pi \sec^2 \frac{\pi}{6} = \frac{200\pi}{3} \text{ ft/sec}$$

$$(b) r = 50\pi \sec^2 \frac{\pi}{3} = 200\pi \text{ ft/sec}$$

$$(c) \lim_{\theta \rightarrow (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$$

$$64. (a) \text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

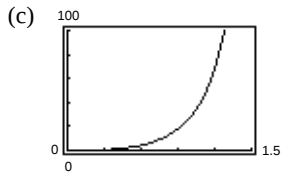
$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

$$66. (a) A = \frac{1}{2}bh - \frac{1}{2}r^2\theta = \frac{1}{2}(10)(10 \tan \theta) - \frac{1}{2}(10)^2\theta$$

$$= 50 \tan \theta - 50\theta$$

Domain: $\left(0, \frac{\pi}{2}\right)$



68. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

$$72. \text{ Let } f(x) = \frac{1}{x^2} \text{ and } g(x) = \frac{1}{x^4}, \text{ and } c = 0.$$

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty \text{ and } \lim_{x \rightarrow 0} \frac{1}{x^4} = \infty, \text{ but}$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{x^4} \right) = \lim_{x \rightarrow 0} \left(\frac{x^2 - 1}{x^4} \right) = -\infty \neq 0.$$

$$62. m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

$$\lim_{v \rightarrow c} m = \lim_{v \rightarrow c} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

$$(c) \lim_{x \rightarrow 25^+} \frac{25x}{x - 25} = \infty$$

As x gets close to 25 mph, y becomes larger and larger.

(b)

ϕ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

$$(d) \lim_{\theta \rightarrow \pi/2^-} A = \infty$$

70. True

$$74. \text{ Given } \lim_{x \rightarrow c} f(x) = \infty, \text{ let } g(x) = 1. \text{ then } \lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$$

by Theorem 1.15.

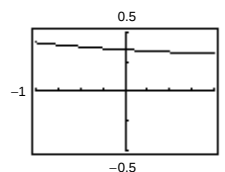
Review Exercises for Chapter 1

2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

4.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$$\lim_{x \rightarrow 0} f(x) \approx 0.2$$



6. $g(x) = \frac{3x}{x-2}$

(a) $\lim_{x \rightarrow 2} g(x)$ does not exist.

(b) $\lim_{x \rightarrow 0} g(x) = 0$

8. $\lim_{x \rightarrow 9} \sqrt{x} = \sqrt{9} = 3$.

Let $\epsilon > 0$ be given. We need

$$|\sqrt{x} - 3| < \epsilon \Rightarrow |\sqrt{x} + 3| |\sqrt{x} - 3| < \epsilon |\sqrt{x} + 3|$$

$$|x - 9| < \epsilon |\sqrt{x} + 3|$$

Assuming $4 < x < 16$, you can choose $\delta = 5\epsilon$.

Hence, for $0 < |x - 9| < \delta = 5\epsilon$, you have

$$|x - 9| < 5\epsilon < |\sqrt{x} + 3| \epsilon$$

$$|\sqrt{x} - 3| < \epsilon$$

$$|f(x) - L| < \epsilon$$

10. $\lim_{x \rightarrow 5} 9 = 9$. Let $\epsilon > 0$ be given. δ can be any positive number. Hence, for $0 < |x - 5| < \delta$, you have

$$|9 - 9| < \epsilon$$

$$|f(x) - L| < \epsilon$$

12. $\lim_{y \rightarrow 4} 3|y - 1| = 3|4 - 1| = 9$

14. $\lim_{t \rightarrow 3} \frac{t^2 - 9}{t - 3} = \lim_{t \rightarrow 3} (t + 3) = 6$

16. $\lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{4+x} + 2} = \frac{1}{4}$$

18. $\lim_{s \rightarrow 0} \frac{(1/\sqrt{1+s}) - 1}{s} = \lim_{s \rightarrow 0} \left[\frac{(1/\sqrt{1+s}) - 1}{s} \cdot \frac{(1/\sqrt{1+s}) + 1}{(1/\sqrt{1+s}) + 1} \right]$

$$= \lim_{s \rightarrow 0} \frac{[1/(1+s)] - 1}{s[(1/\sqrt{1+s}) + 1]} = \lim_{s \rightarrow 0} \frac{-1}{(1+s)[(1/\sqrt{1+s}) + 1]} = -\frac{1}{2}$$

20. $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x^3 + 8} = \lim_{x \rightarrow -2} \frac{(x+2)(x-2)}{(x+2)(x^2 - 2x + 4)}$

$$= \lim_{x \rightarrow -2} \frac{x-2}{x^2 - 2x + 4}$$

$$= -\frac{4}{12} = -\frac{1}{3}$$

22. $\lim_{x \rightarrow (\pi/4)} \frac{4x}{\tan x} = \frac{4(\pi/4)}{1} = \pi$

$$\begin{aligned}
 24. \lim_{\Delta x \rightarrow 0} \frac{\cos(\pi + \Delta x) + 1}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\cos \pi \cos \Delta x - \sin \pi \sin \Delta x + 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left[-\frac{(\cos \Delta x - 1)}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} \left[\sin \pi \frac{\sin \Delta x}{\Delta x} \right] \\
 &= -0 - (0)(1) = 0
 \end{aligned}$$

$$26. \lim_{x \rightarrow c} [f(x) + 2g(x)] = -\frac{3}{4} + 2\left(\frac{2}{3}\right) = \frac{7}{12}$$

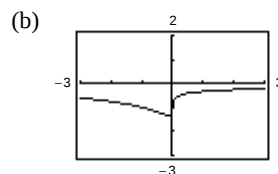
$$28. f(x) = \frac{1 - \sqrt[3]{x}}{x - 1}$$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	-0.3228	-0.3322	-0.3332	-0.3333

$$\lim_{x \rightarrow 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} \approx -0.333 \quad (\text{Actual limit is } -\frac{1}{3}.)$$

$$\begin{aligned}
 (c) \lim_{x \rightarrow 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{1 - \sqrt[3]{x}}{x - 1} \cdot \frac{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2} \\
 &= \lim_{x \rightarrow 1^+} \frac{1 - x}{(x - 1)[1 + \sqrt[3]{x} + (\sqrt[3]{x})^2]} \\
 &= \lim_{x \rightarrow 1^+} \frac{-1}{1 + \sqrt[3]{x} + (\sqrt[3]{x})^2} \\
 &= -\frac{1}{3}
 \end{aligned}$$



$$30. s(t) = 0 \Rightarrow -4.9t^2 + 200 = 0 \Rightarrow t^2 \approx 40.816 \Rightarrow t \approx 6.39 \text{ sec}$$

When $t = 6.39$, the velocity is approximately

$$\begin{aligned}
 \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow a} -4.9(a + t) \\
 &= \lim_{t \rightarrow 6.39} -4.9(6.39 + 6.39) = -62.6 \text{ m/sec.}
 \end{aligned}$$

$$32. \lim_{x \rightarrow 4} \llbracket x - 1 \rrbracket \text{ does not exist. The graph jumps from 2 to 3 at } x = 4.$$

$$34. \lim_{x \rightarrow 1^+} g(x) = 1 + 1 = 2.$$

$$36. \lim_{s \rightarrow -2} f(s) = 2$$

$$38. f(x) = \begin{cases} \frac{3x^2 - x - 2}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$

$$\begin{aligned}
 \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{3x^2 - x - 2}{x - 1} \\
 &= \lim_{x \rightarrow 1} (3x + 2) = 5 \neq 0
 \end{aligned}$$

Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$

$$40. f(x) = \begin{cases} 5 - x, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} (5 - x) = 3$$

$$\lim_{x \rightarrow 2^+} (2x - 3) = 1$$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

$$44. f(x) = \frac{x+1}{2x+2}$$

$$\lim_{x \rightarrow -1} \frac{x+1}{2(x+1)} = \frac{1}{2}$$

Removable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

$$48. \lim_{x \rightarrow 1^+} (x+1) = 2$$

$$\lim_{x \rightarrow 3^-} (x+1) = 4$$

Find b and c so that $\lim_{x \rightarrow 1^-} (x^2 + bx + c) = 2$ and $\lim_{x \rightarrow 3^+} (x^2 + bx + c) = 4$.

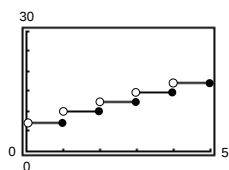
Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

$$50. C = 9.80 + 2.50[-\lfloor -x \rfloor - 1], \quad x > 0$$

$$= 9.80 - 2.50[\lfloor -x \rfloor + 1]$$

C has a nonremovable discontinuity at each integer.



$$54. h(x) = \frac{4x}{4-x^2}$$

Vertical asymptotes at $x = 2$ and $x = -2$

$$58. \lim_{x \rightarrow (1/2)^+} \frac{x}{2x-1} = \infty$$

$$62. \lim_{x \rightarrow -1^+} \frac{x^2 - 2x + 1}{x+1} = \infty$$

$$66. \lim_{x \rightarrow 0^+} \frac{\sec x}{x} = \infty$$

$$42. f(x) = \sqrt{\frac{x+1}{x}} = \sqrt{1 + \frac{1}{x}}$$

$$\lim_{x \rightarrow 0^+} \sqrt{1 + \frac{1}{x}} = \infty$$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$

Continuous on $(-\infty, -1] \cup (0, \infty)$

$$46. f(x) = \tan 2x$$

Nonremovable discontinuities when

$$x = \frac{(2n+1)\pi}{4}$$

Continuous on

$$\left(\frac{(2n-1)\pi}{4}, \frac{(2n+1)\pi}{4} \right)$$

for all integers n .

$$52. f(x) = \sqrt{(x-1)x}$$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

$$(b) \lim_{x \rightarrow 0^-} f(x) = 0$$

$$(c) \lim_{x \rightarrow 1^+} f(x) = 0$$

$$56. f(x) = \csc \pi x$$

Vertical asymptote at every integer k

$$60. \lim_{x \rightarrow -1^-} \frac{x+1}{x^4-1} = \lim_{x \rightarrow -1^-} \frac{1}{(x^2+1)(x-1)} = -\frac{1}{4}$$

$$64. \lim_{x \rightarrow 2^-} \frac{1}{\sqrt[3]{x^2-4}} = -\infty$$

$$68. \lim_{x \rightarrow 0^-} \frac{\cos^2 x}{x} = -\infty$$

$$70. f(x) = \frac{\tan 2x}{x}$$

(a)

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{x} = 2$$

(b) Yes, define

$$f(x) = \begin{cases} \frac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Now $f(x)$ is continuous at $x = 0$.

Problem Solving for Chapter 1

$$2. (a) \text{ Area } \triangle PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$$

$$\text{Area } \triangle PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$$

$$(b) a(x) = \frac{\text{Area } \triangle PBO}{\text{Area } \triangle PAO} = \frac{x^2/2}{x/2} = x$$

x	4	2	1	0.1	0.01
Area $\triangle PAO$	2	1	1/2	1/20	1/200
Area $\triangle PBO$	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

$$(c) \lim_{x \rightarrow 0^+} a(x) = \lim_{x \rightarrow 0^+} x = 0$$

$$6. \frac{\sqrt{a+bx} - \sqrt{3}}{x} = \frac{\sqrt{a+bx} - \sqrt{3}}{x} \cdot \frac{\sqrt{a+bx} + \sqrt{3}}{\sqrt{a+bx} + \sqrt{3}}$$

$$= \frac{(a+bx) - 3}{x(\sqrt{a+bx} + \sqrt{3})}$$

Letting $a = 3$ simplifies the numerator.

Thus,

$$\lim_{x \rightarrow 0} \frac{\sqrt{3+bx} - \sqrt{3}}{x} = \lim_{x \rightarrow 0} \frac{bx}{x(\sqrt{3+bx} + \sqrt{3})}$$

$$= \lim_{x \rightarrow 0} \frac{b}{\sqrt{3+bx} + \sqrt{3}}$$

Setting $\frac{b}{\sqrt{3} + \sqrt{3}} = \sqrt{3}$, you obtain $b = 6$.

Thus, $a = 3$ and $b = 6$.

$$4. (a) \text{ Slope} = \frac{4-0}{3-0} = \frac{4}{3}$$

$$(b) \text{ Slope} = -\frac{3}{4} \quad \text{Tangent line: } y - 4 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

$$(c) \text{ Let } Q = (x, y) = (x, \sqrt{25-x^2})$$

$$m_x = \frac{\sqrt{25-x^2} - 4}{x - 3}$$

$$(d) \lim_{x \rightarrow 3} m_x = \lim_{x \rightarrow 3} \frac{\sqrt{25-x^2} - 4}{x - 3} \cdot \frac{\sqrt{25-x^2} + 4}{\sqrt{25-x^2} + 4}$$

$$= \lim_{x \rightarrow 3} \frac{25 - x^2 - 16}{(x - 3)(\sqrt{25-x^2} + 4)}$$

$$= \lim_{x \rightarrow 3} \frac{(3-x)(3+x)}{(x-3)(\sqrt{25-x^2} + 4)}$$

$$= \lim_{x \rightarrow 3} \frac{-(3+x)}{\sqrt{25-x^2} + 4} = \frac{-6}{4+4} = -\frac{3}{4}$$

This is the slope of the tangent line at P .

$$8. \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (a^2 - 2) = a^2 - 2$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{ax}{\tan x} = a \left(\text{because } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right)$$

Thus,

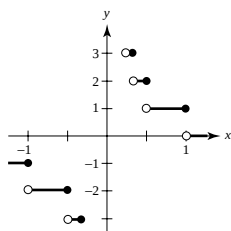
$$a^2 - 2 = a$$

$$a^2 - a - 2 = 0$$

$$(a-2)(a+1) = 0$$

$$a = -1, 2$$

10.



$$\begin{aligned} \text{(a)} \quad f\left(\frac{1}{4}\right) &= \llbracket 4 \rrbracket = 4 \\ f(3) &= \llbracket \frac{1}{3} \rrbracket = 0 \\ f(1) &= \llbracket 1 \rrbracket = 1 \end{aligned}$$

$$\text{(b)} \quad \lim_{x \rightarrow 1^-} f(x) = 1$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= 0 \\ \lim_{x \rightarrow 0^-} f(x) &= -\infty \\ \lim_{x \rightarrow 0^+} f(x) &= \infty \end{aligned}$$

(c) f is continuous for all real numbers except $x = 0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \dots$

$$12. \text{ (a)} \quad v^2 = \frac{192,000}{r} + v_0^2 - 48$$

$$\frac{192,000}{r} = v^2 - v_0^2 + 48$$

$$r = \frac{192,000}{v - v_0^2 + 48}$$

$$\lim_{v \rightarrow 0} r = \frac{192,000}{48 - v_0^2}$$

$$\text{Let } v_0 = \sqrt{48} = 4\sqrt{3} \text{ feet/sec.}$$

$$\text{(b)} \quad v^2 = \frac{1920}{r} + v_0^2 - 2.17$$

$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$

$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$

$$\lim_{v \rightarrow 0} r = \frac{1920}{2.17 - v_0^2}$$

$$\text{Let } v_0 = \sqrt{2.17} \text{ mi/sec } (\approx 1.47 \text{ mi/sec}).$$

$$\text{(c)} \quad r = \frac{10,600}{v^2 - v_0^2 + 6.99}$$

$$\lim_{v \rightarrow 0} r = \frac{10,600}{6.99 - v_0^2}$$

$$\text{Let } v_0 = \sqrt{6.99} \approx 2.64 \text{ mi/sec.}$$

Since this is smaller than the escape velocity for earth, the mass is less.

14. Let $a \neq 0$ and let $\epsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta$, then $|f(x) - L| < \epsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have

$$|x| < \frac{\delta_1}{|a|}$$

$$|ax| < \delta_1$$

$$|f(ax) - L| < \epsilon.$$

As a counterexample, let $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$.

Then $\lim_{x \rightarrow 0} f(x) = 1 = L$,

but $\lim_{x \rightarrow 0} f(ax) = \lim_{x \rightarrow 0} f(0) = 2$.

CHAPTER 1

Limits and Their Properties

Section 1.1	A Preview of Calculus	27
Section 1.2	Finding Limits Graphically and Numerically	27
Section 1.3	Evaluating Limits Analytically	31
Section 1.4	Continuity and One-Sided Limits	37
Section 1.5	Infinite Limits	42
Review Exercises		47
Problem Solving		49

CHAPTER 1

Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

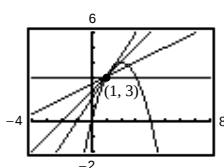
1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300 \text{ feet}$

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

7. Precalculus: Volume $= (2)(4)(3) = 24$ cubic units

9. (a)



(b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
 $\{0.2, 0.1, 0.01, 0.001\}$

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$
 $\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-x-2} \approx 0.3333 \quad (\text{Actual limit is } \frac{1}{3}.)$$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887 \quad (\text{Actual limit is } 1/(2\sqrt{3}).)$$

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$$\lim_{x \rightarrow 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625 \quad (\text{Actual limit is } -\frac{1}{16}.)$$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \approx 1.0000 \quad (\text{Actual limit is } 1.) \quad (\text{Make sure you use radian mode.})$$

9. $\lim_{x \rightarrow 3} (4 - x) = 1$

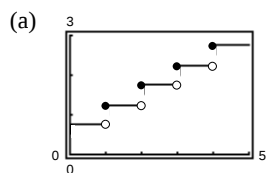
11. $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (4 - x) = 2$

13. $\lim_{x \rightarrow 5} \frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1 , whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1 .

15. $\lim_{x \rightarrow \pi/2} \tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

17. $\lim_{x \rightarrow 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0 .

19. $C(t) = 0.75 - 0.50\lfloor -(t-1) \rfloor$



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$$\lim_{t \rightarrow 3.5} C(t) = 2.25$$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t \rightarrow 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t = 3$.

21. You need to find δ such that $0 < |x-1| < \delta$ implies

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1. \text{ That is,}$$

$$-0.1 < \frac{1}{x} - 1 < 0.1$$

$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$

$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$

$$\frac{10}{9} > x > \frac{10}{11}$$

$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$

$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x-1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$

$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < \epsilon < 0.1.$$

$$23. \lim_{x \rightarrow 2} (3x + 2) = 8 = L$$

$$|(3x + 2) - 8| < 0.01$$

$$|3x - 6| < 0.01$$

$$3|x - 2| < 0.01$$

$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

Hence, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$

$$|3x - 6| < 0.01$$

$$|(3x + 2) - 8| < 0.01$$

$$|f(x) - L| < 0.01$$

$$27. \lim_{x \rightarrow 2} (x + 3) = 5$$

Given $\epsilon > 0$:

$$|(x + 3) - 5| < \epsilon$$

$$|x - 2| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x - 2| < \delta = \epsilon$, you have

$$|x - 2| < \epsilon$$

$$|(x + 3) - 5| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$31. \lim_{x \rightarrow 6} 3 = 3$$

Given $\epsilon > 0$:

$$|3 - 3| < \epsilon$$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|3 - 3| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$25. \lim_{x \rightarrow 2} (x^2 - 3) = 1 = L$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x + 2)(x - 2)| < 0.01$$

$$|x + 2| |x - 2| < 0.01$$

$$|x - 2| < \frac{0.01}{|x + 2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$

$$|x + 2| |x - 2| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|f(x) - L| < 0.01$$

$$29. \lim_{x \rightarrow -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$$

Given $\epsilon > 0$:

$$\left|\left(\frac{1}{2}x - 1\right) - (-3)\right| < \epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\frac{1}{2}|x - (-4)| < \epsilon$$

$$|x - (-4)| < 2\epsilon$$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x - (-4)| < \delta = 2\epsilon$, you have

$$|x - (-4)| < 2\epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\left|\left(\frac{1}{2}x - 1\right) + 3\right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$33. \lim_{x \rightarrow 0} \sqrt[3]{x} = 0$$

Given $\epsilon > 0$: $|\sqrt[3]{x} - 0| < \epsilon$

$$|\sqrt[3]{x}| < \epsilon$$

$$|x| < \epsilon^3 = \delta$$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x - 0| < \delta = \epsilon^3$, you have

$$|x| < \epsilon^3$$

$$|\sqrt[3]{x}| < \epsilon$$

$$|\sqrt[3]{x} - 0| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$35. \lim_{x \rightarrow -2} |x - 2| = |(-2) - 2| = 4$$

Given $\epsilon > 0$:

$$||x - 2| - 4| < \epsilon$$

$$|-(x - 2) - 4| < \epsilon \quad (x - 2 < 0)$$

$$|-x - 2| = |x + 2| = |x - (-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x - (-2)| < \delta = \epsilon$, you have

$$|x + 2| < \epsilon$$

$$|-(x + 2)| < \epsilon$$

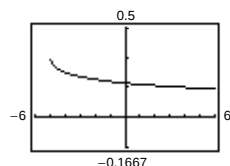
$$|-(x - 2) - 4| < \epsilon$$

$$||x - 2| - 4| < \epsilon \quad (\text{because } x - 2 < 0)$$

$$|f(x) - L| < \epsilon$$

$$39. f(x) = \frac{\sqrt{x+5} - 3}{x - 4}$$

$$\lim_{x \rightarrow 4} f(x) = \frac{1}{6}$$



The domain is $[-5, 4) \cup (4, \infty)$.

The graphing utility does not show the hole at $(4, \frac{1}{6})$.

$$37. \lim_{x \rightarrow 1} (x^2 + 1) = 2$$

Given $\epsilon > 0$:

$$|(x^2 + 1) - 2| < \epsilon$$

$$|x^2 - 1| < \epsilon$$

$$|(x + 1)(x - 1)| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{|x + 1|}$$

If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x - 1| < \delta = \frac{\epsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\epsilon < \frac{1}{|x + 1|}\epsilon$$

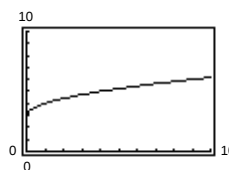
$$|x^2 - 1| < \epsilon$$

$$|(x^2 + 1) - 2| < \epsilon$$

$$|f(x) - 2| < \epsilon$$

$$41. f(x) = \frac{x - 9}{\sqrt{x} - 3}$$

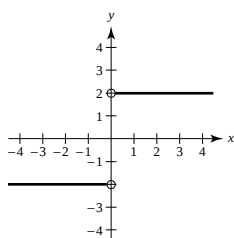
$$\lim_{x \rightarrow 9} f(x) = 6$$



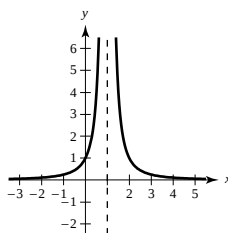
The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

$$43. \lim_{x \rightarrow 8} f(x) = 25 \text{ means that the values of } f \text{ approach 25 as } x \text{ gets closer and closer to 8.}$$

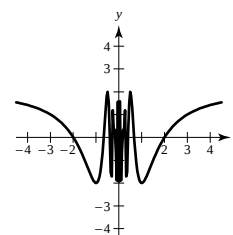
45. (i) The values of f approach different numbers as x approaches c from different sides of c :



(ii) The values of f increase without bound as x approaches c :

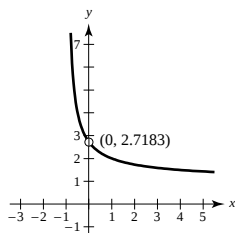


(iii) The values of f oscillate between two fixed numbers as x approaches c :



$$47. f(x) = (1 + x)^{1/x}$$

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e \approx 2.71828$$



x	$f(x)$	x	$f(x)$
-0.1	2.867972	0.1	2.593742
-0.01	2.731999	0.01	2.704814
-0.001	2.719642	0.001	2.716942
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718268
-0.000001	2.718283	0.000001	2.718280

49. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$.
From Exercise 7, we have

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

51. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$$f(4) = 10$$

$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 - 4x) = 0 \neq 10$$

53. Answers will vary.

55. If $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2 . Then for $|x - c| < \delta$, we have
- $$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

57. $\lim_{x \rightarrow c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

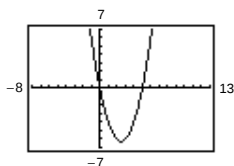
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x \rightarrow c} f(x) = L$.

Section 1.3 Evaluating Limits Analytically

1.

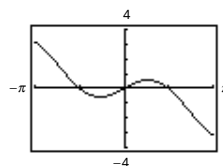


$$h(x) = x^2 - 5x$$

(a) $\lim_{x \rightarrow 5} h(x) = 0$

(b) $\lim_{x \rightarrow -1} h(x) = 6$

3.



$$f(x) = x \cos x$$

(a) $\lim_{x \rightarrow 0} f(x) = 0$

(b) $\lim_{x \rightarrow \pi/3} f(x) \approx 0.524$
 $\left(= \frac{\pi}{6} \right)$

5. $\lim_{x \rightarrow 2} x^4 = 2^4 = 16$

7. $\lim_{x \rightarrow 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x \rightarrow -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

11. $\lim_{x \rightarrow -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

13. $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$

15. $\lim_{x \rightarrow 1} \frac{x-3}{x^2+4} = \frac{1-3}{1^2+4} = \frac{-2}{5} = -\frac{2}{5}$

17. $\lim_{x \rightarrow 7} \frac{5x}{\sqrt{x+2}} = \frac{5(7)}{\sqrt{7+2}} = \frac{35}{\sqrt{9}} = \frac{35}{3}$

19. $\lim_{x \rightarrow 3} \sqrt{x+1} = \sqrt{3+1} = 2$

$$21. \lim_{x \rightarrow -4} (x + 3)^2 = (-4 + 3)^2 = 1$$

$$25. \begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow 1} f(x) = 4 - 1 = 3 \\ \text{(b)} \quad & \lim_{x \rightarrow 3} g(x) = \sqrt{3 + 1} = 2 \\ \text{(c)} \quad & \lim_{x \rightarrow 1} g(f(x)) = g(3) = 2 \end{aligned}$$

$$29. \lim_{x \rightarrow 2} \cos \frac{\pi x}{3} = \cos \frac{\pi 2}{3} = -\frac{1}{2}$$

$$33. \lim_{x \rightarrow 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$$

$$37. \begin{aligned} \text{(a)} \quad & \lim_{x \rightarrow c} [5g(x)] = 5 \lim_{x \rightarrow c} g(x) = 5(3) = 15 \\ \text{(b)} \quad & \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = 2 + 3 = 5 \\ \text{(c)} \quad & \lim_{x \rightarrow c} [f(x)g(x)] = \left[\lim_{x \rightarrow c} f(x) \right] \left[\lim_{x \rightarrow c} g(x) \right] = (2)(3) = 6 \\ \text{(d)} \quad & \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{2}{3} \end{aligned}$$

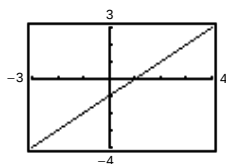
$$41. f(x) = -2x + 1 \text{ and } g(x) = \frac{-2x^2 + x}{x} \text{ agree except at } x = 0.$$

$$\text{(a)} \quad \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} f(x) = 1$$

$$\text{(b)} \quad \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 3$$

$$45. f(x) = \frac{x^2 - 1}{x + 1} \text{ and } g(x) = x - 1 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = -2$$



$$49. \begin{aligned} \lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25} &= \lim_{x \rightarrow 5} \frac{x - 5}{(x + 5)(x - 5)} \\ &= \lim_{x \rightarrow 5} \frac{1}{x + 5} = \frac{1}{10} \end{aligned}$$

$$23. \text{(a)} \quad \lim_{x \rightarrow 1} f(x) = 5 - 1 = 4$$

$$\text{(b)} \quad \lim_{x \rightarrow 4} g(x) = 4^3 = 64$$

$$\text{(c)} \quad \lim_{x \rightarrow 1} g(f(x)) = g(f(1)) = g(4) = 64$$

$$27. \lim_{x \rightarrow \pi/2} \sin x = \sin \frac{\pi}{2} = 1$$

$$31. \lim_{x \rightarrow 0} \sec 2x = \sec 0 = 1$$

$$35. \lim_{x \rightarrow 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$$

$$39. \text{(a)} \quad \lim_{x \rightarrow c} [f(x)]^3 = \left[\lim_{x \rightarrow c} f(x) \right]^3 = (4)^3 = 64$$

$$\text{(b)} \quad \lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow c} f(x)} = \sqrt{4} = 2$$

$$\text{(c)} \quad \lim_{x \rightarrow c} [3f(x)] = 3 \lim_{x \rightarrow c} f(x) = 3(4) = 12$$

$$\text{(d)} \quad \lim_{x \rightarrow c} [f(x)]^{3/2} = \left[\lim_{x \rightarrow c} f(x) \right]^{3/2} = (4)^{3/2} = 8$$

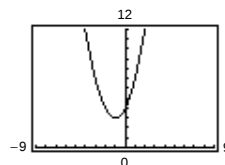
$$43. f(x) = x(x + 1) \text{ and } g(x) = \frac{x^3 - x}{x - 1} \text{ agree except at } x = 1.$$

$$\text{(a)} \quad \lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} f(x) = 2$$

$$\text{(b)} \quad \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 0$$

$$47. f(x) = \frac{x^3 - 8}{x - 2} \text{ and } g(x) = x^2 + 2x + 4 \text{ agree except at } x = 2.$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 12$$



$$51. \begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x^2 - 9} &= \lim_{x \rightarrow -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} \\ &= \lim_{x \rightarrow -3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6} \end{aligned}$$

$$\begin{aligned}
 53. \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}} \\
 &= \lim_{x \rightarrow 0} \frac{(x+5) - 5}{x(\sqrt{x+5} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}
 \end{aligned}$$

$$\begin{aligned}
 55. \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} &= \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} \\
 &= \lim_{x \rightarrow 4} \frac{(x+5) - 9}{(x-4)(\sqrt{x+5} + 3)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}
 \end{aligned}$$

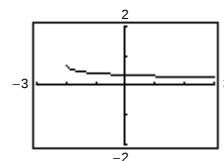
$$57. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{\frac{2 - (2+x)}{2(2+x)}}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$59. \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2 = 2$$

$$\begin{aligned}
 61. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2
 \end{aligned}$$

$$63. \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$$

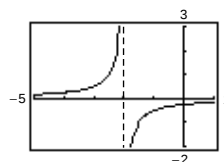
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349



$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} \\
 &= \lim_{x \rightarrow 0} \frac{x + 2 - 2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354
 \end{aligned}$$

$$65. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$67. \lim_{x \rightarrow 0} \frac{\sin x}{5x} = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right) \left(\frac{1}{5} \right) \right] = (1) \left(\frac{1}{5} \right) = \frac{1}{5}$$

$$69. \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{2x^2} = \lim_{x \rightarrow 0} \left[\frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right] \\ = \frac{1}{2}(1)(0) = 0$$

$$71. \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \sin x \right] = (1) \sin 0 = 0$$

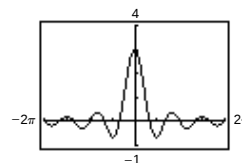
$$73. \lim_{h \rightarrow 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \rightarrow 0} \left[\frac{1 - \cos h}{h} (1 - \cos h) \right] \\ = (0)(0) = 0$$

$$75. \lim_{x \rightarrow \pi/2} \frac{\cos x}{\cot x} = \lim_{x \rightarrow \pi/2} \sin x = 1$$

$$77. \lim_{t \rightarrow 0} \frac{\sin 3t}{2t} = \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{3t} \right) \left(\frac{3}{2} \right) = (1) \left(\frac{3}{2} \right) = \frac{3}{2}$$

$$79. f(t) = \frac{\sin 3t}{t}$$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

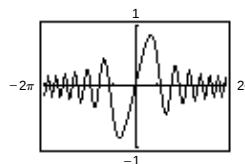


The limit appear to equal 3.

$$\text{Analytically, } \lim_{t \rightarrow 0} \frac{\sin 3t}{t} = \lim_{t \rightarrow 0} 3 \left(\frac{\sin 3t}{3t} \right) = 3(1) = 3.$$

$$81. f(x) = \frac{\sin x^2}{x}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} x \left(\frac{\sin x^2}{x^2} \right) = 0(1) = 0.$$

$$83. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2(x+h) + 3 - (2x+3)}{h} = \lim_{h \rightarrow 0} \frac{2x+2h+3-2x-3}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = 2$$

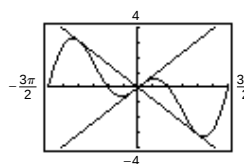
$$85. \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} = \lim_{h \rightarrow 0} \frac{4x - 4(x+h)}{(x+h)xh} = \lim_{h \rightarrow 0} \frac{-4}{(x+h)x} = \frac{-4}{x^2}$$

$$87. \lim_{x \rightarrow 0} (4 - x^2) \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} (4 + x^2)$$

$$4 \leq \lim_{x \rightarrow 0} f(x) \leq 4$$

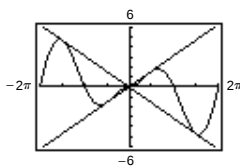
$$\text{Therefore, } \lim_{x \rightarrow 0} f(x) = 4.$$

$$89. f(x) = x \cos x$$



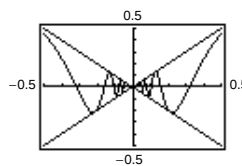
$$\lim_{x \rightarrow 0} (x \cos x) = 0$$

91. $f(x) = |x| \sin x$



$$\lim_{x \rightarrow 0} |x| \sin x = 0$$

93. $f(x) = x \sin \frac{1}{x}$



$$\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

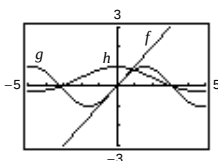
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

$$\text{for which } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$$

99. $f(x) = x$, $g(x) = \sin x$, $h(x) = \frac{\sin x}{x}$



When you are “close to” 0 the magnitude of f is approximately equal to the magnitude of g .
Thus, $|g|/|f| \approx 1$ when x is “close to” 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \rightarrow 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \rightarrow 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \rightarrow 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \rightarrow 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed = 160 ft/sec

103. $s(t) = -4.9t^2 + 150$

$$\begin{aligned} \lim_{t \rightarrow 3} \frac{s(3) - s(t)}{3 - t} &= \lim_{t \rightarrow 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \rightarrow 3} \frac{-4.9(9 - t^2)}{3 - t} \\ &= \lim_{x \rightarrow 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{x \rightarrow 3} -4.9(3 + t) = -29.4 \text{ m/sec} \end{aligned}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist.

$$\lim_{x \rightarrow 0} [f(x) + g(x)] = \lim_{x \rightarrow 0} \left[\frac{1}{x} + \left(-\frac{1}{x} \right) \right] = \lim_{x \rightarrow 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, whenever $0 < |x - c| < \delta$,

we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \rightarrow c} [bf(x)] = bL$.

111. $-M|f(x)| \leq f(x)g(x) \leq M|f(x)|$

$$\lim_{x \rightarrow c} (-M|f(x)|) \leq \lim_{x \rightarrow c} f(x)g(x) \leq \lim_{x \rightarrow c} (M|f(x)|)$$

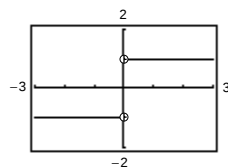
$$-M(0) \leq \lim_{x \rightarrow c} f(x)g(x) \leq M(0)$$

$$0 \leq \lim_{x \rightarrow c} f(x)g(x) \leq 0$$

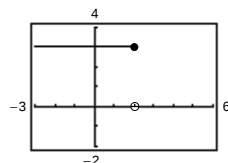
Therefore, $\lim_{x \rightarrow c} f(x)g(x) = 0$.

115. True.

113. False. As x approaches 0 from the left, $\frac{|x|}{x} = -1$.



117. False. The limit does not exist.



119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} |f(x)| = \lim_{x \rightarrow 0} 4 = 4.$$

$\lim_{x \rightarrow 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

121. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$ does not exist.

No matter how “close to” 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x \rightarrow 0} f(x)$ does not exist.

$$\lim_{x \rightarrow 0} g(x) = 0.$$

When x is “close to” 0, both parts of the function are “close to” 0.

123. (a) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x}$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= (1) \left(\frac{1}{2} \right) = \frac{1}{2}$$

(b) Thus, $\frac{1 - \cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2}x^2$

$$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2 \text{ for } x \approx 0.$$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x \rightarrow 3^+} f(x) = 1$

(b) $\lim_{x \rightarrow 3^-} f(x) = 1$

(c) $\lim_{x \rightarrow 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x \rightarrow 3^+} f(x) = 0$

(b) $\lim_{x \rightarrow 3^-} f(x) = 0$

(c) $\lim_{x \rightarrow 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x \rightarrow 4^+} f(x) = 2$

(b) $\lim_{x \rightarrow 4^-} f(x) = -2$

(c) $\lim_{x \rightarrow 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x \rightarrow 5^+} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5^+} \frac{1}{x+5} = \frac{1}{10}$

9. $\lim_{x \rightarrow -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ grows without bound as $x \rightarrow -3^-$.

11. $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$.

13.
$$\begin{aligned} \lim_{\Delta x \rightarrow 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} &= \lim_{\Delta x \rightarrow 0^-} \frac{x - (x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{-1}{x(x+\Delta x)} \\ &= \frac{-1}{x(x+0)} = -\frac{1}{x^2} \end{aligned}$$

15. $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+2}{2} = \frac{5}{2}$

17. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x+1) = 2$

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^3+1) = 2$

$\lim_{x \rightarrow 1} f(x) = 2$

19. $\lim_{x \rightarrow \pi} \cot x$ does not exist since

$\lim_{x \rightarrow \pi^+} \cot x$ and $\lim_{x \rightarrow \pi^-} \cot x$ do not exist.

21. $\lim_{x \rightarrow 4^-} (3\llbracket x \rrbracket - 5) = 3(3) - 5 = 4$

$(\llbracket x \rrbracket = 3 \text{ for } 3 < x < 4)$

23. $\lim_{x \rightarrow 3} (2 - \llbracket -x \rrbracket)$ does not exist

because

$\lim_{x \rightarrow 3^-} (2 - \llbracket -x \rrbracket) = 2 - (-3) = 5$

and

$\lim_{x \rightarrow 3^+} (2 - \llbracket -x \rrbracket) = 2 - (-4) = 6$.

25. $f(x) = \frac{1}{x^2-4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \frac{\llbracket x \rrbracket}{2} + x$

has discontinuities at each integer k since $\lim_{x \rightarrow k^-} f(x) \neq \lim_{x \rightarrow k^+} f(x)$.

29. $g(x) = \sqrt{25-x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x \rightarrow 0^-} f(x) = 3 = \lim_{x \rightarrow 0^+} f(x)$.
 f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x .

35. $f(x) = 3x - \cos x$ is continuous for all real x .

37. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \neq 0, x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

39. $f(x) = \frac{x}{x^2 + 1}$ is continuous for all real x .

41. $f(x) = \frac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \rightarrow 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \frac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x \rightarrow -2} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left. \begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} x = 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 = 1 \end{aligned} \right\} \lim_{x \rightarrow 1} f(x) = 1$

3. $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x .

47. $f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = \frac{2}{2} + 1 = 2$

2. $\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \left(\frac{x}{2} + 1 \right) = 2 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (3 - x) = 1 \end{aligned} \right\} \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} = \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$ has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $f(1) = 1$

2. $\lim_{x \rightarrow -1} f(x) = -1$ $\lim_{x \rightarrow 1} f(x) = 1$

3. $f(-1) = \lim_{x \rightarrow -1} f(x)$ $f(1) = \lim_{x \rightarrow 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x .

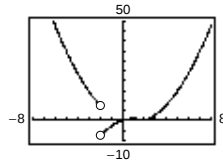
51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

53. $f(x) = \llbracket x - 1 \rrbracket$ has nonremovable discontinuities at each integer k .

55. $\lim_{x \rightarrow 0^+} f(x) = 0$

$\lim_{x \rightarrow 0^-} f(x) = 0$

f is not continuous at $x = -2$.



57. $f(2) = 8$

Find a so that $\lim_{x \rightarrow 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x \rightarrow -1^+} (ax + b) = -a + b = 2$ and $\lim_{x \rightarrow 3^-} (ax + b) = 3a + b = -2$.

$$a - b = -2$$

$$\begin{array}{r} (+) 3a + b = -2 \\ \hline 4a = -4 \end{array}$$

$$4a = -4$$

$$a = -1$$

$$b = 2 + (-1) = 1$$

$$f(x) = \begin{cases} 2, & x \leq -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

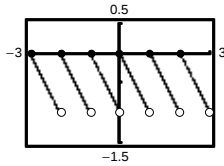
Continuous for all real x .

63. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

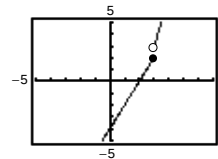
65. $y = \llbracket x \rrbracket - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \leq 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



69. $f(x) = \frac{x}{x^2 + 1}$

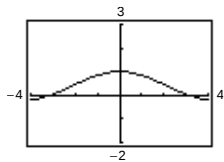
Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \frac{\pi x}{4}$

Continuous on:

$\dots, (-6, -2), (-2, 2), (2, 6), (6, 10), \dots$

73. $f(x) = \frac{\sin x}{x}$



75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

- 77.
- $f(x) = x^2 - 2 - \cos x$
- is continuous on
- $[0, \pi]$
- .

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π .

- 81.
- $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$$g(0) = 2 > 0 \text{ and } g(1) \approx -1.9 < 0.$$

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

- 85.
- $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$$f(0) = -2 \text{ and } f(3) = 19$$

$$-2 < 4 < 19$$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$

$$x^3 - x^2 + x - 6 = 0$$

$$(x - 2)(x^2 + x + 3) = 0$$

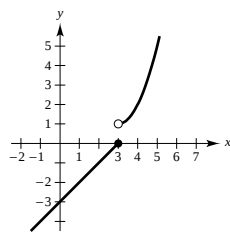
$$x = 2$$

$$(x^2 + x + 3 \text{ has no real solution.})$$

$$c = 2$$

Thus, $f(2) = 4$.

- 89.



The function is not continuous at $x = 3$ because

$$\lim_{x \rightarrow 3^+} f(x) = 1 \neq 0 = \lim_{x \rightarrow 3^-} f(x).$$

- 79.
- $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$$f(0) = -1 \text{ and } f(1) = 1$$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

- 83.
- $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$$f(0) = -1 \text{ and } f(5) = 29$$

$$-1 < 11 < 29$$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x = -4 \text{ or } x = 3$$

$$c = 3 \text{ (} x = -4 \text{ is not in the interval.)}$$

Thus, $f(3) = 11$.

87. (a) The limit does not exist at
- $x = c$
- .

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

91. The functions agree for integer values of
- x
- :

$$\left. \begin{aligned} g(x) &= 3 - \lfloor -x \rfloor = 3 - (-x) = 3 + x \\ f(x) &= 3 + \lfloor x \rfloor = 3 + x \end{aligned} \right\} \text{ for } x \text{ an integer}$$

However, for non-integer values of x , the functions differ by 1.

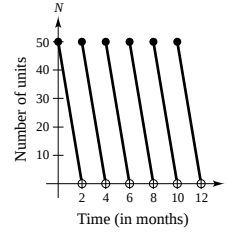
$$f(x) = 3 + \lfloor x \rfloor = g(x) - 1 = 2 - \lfloor -x \rfloor.$$

$$\text{For example, } f\left(\frac{1}{2}\right) = 3 + 0 = 3, g\left(\frac{1}{2}\right) = 3 - (-1) = 4.$$

93. $N(t) = 25\left(2\left\lceil\frac{t+2}{2}\right\rceil - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



95. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r .

$$V(1) = \frac{4}{3}\pi \approx 4.19$$

$$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

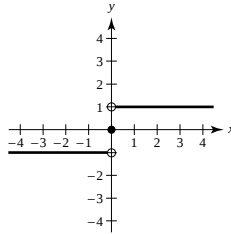
97. Let c be any real number. Then $\lim_{x \rightarrow c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c . Therefore, f is not continuous at c .

99. $\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\lim_{x \rightarrow 0^-} \text{sgn}(x) = -1$

(b) $\lim_{x \rightarrow 0^+} \text{sgn}(x) = 1$

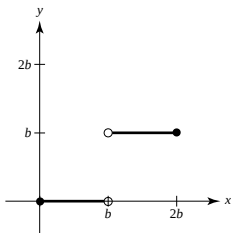
(c) $\lim_{x \rightarrow 0} \text{sgn}(x)$ does not exist.



101. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

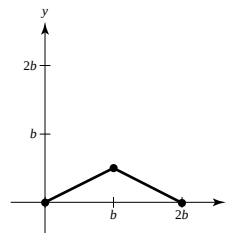
103. False; $f(1)$ is not defined and $\lim_{x \rightarrow 1} f(x)$ does not exist.

105. (a) $f(x) = \begin{cases} 0 & 0 \leq x < b \\ b & b < x \leq 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq b \\ b - \frac{x}{2} & b < x \leq 2b \end{cases}$



Continuous on $[0, 2b]$.

107. $f(x) = \frac{\sqrt{x+c^2}-c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, [-c^2, 0) \cup (0, \infty)$

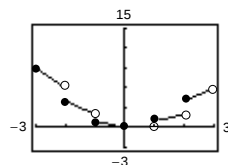
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+c^2}-c}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+c^2}-c}{x} \cdot \frac{\sqrt{x+c^2}+c}{\sqrt{x+c^2}+c} \\ &= \lim_{x \rightarrow 0} \frac{(x+c^2)-c^2}{x[\sqrt{x+c^2}+c]} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+c^2}+c} = \frac{1}{2c}\end{aligned}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

109. $h(x) = x\llbracket x \rrbracket$

h has nonremovable discontinuities at

$$x = \pm 1, \pm 2, \pm 3, \dots$$



Section 1.5 Infinite Limits

1. $\lim_{x \rightarrow -2^+} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$

3. $\lim_{x \rightarrow -2^+} \tan \frac{\pi x}{4} = -\infty$

$\lim_{x \rightarrow -2^-} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$

$\lim_{x \rightarrow -2^-} \tan \frac{\pi x}{4} = \infty$

5. $f(x) = \frac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\lim_{x \rightarrow -3^-} f(x) = \infty$

$\lim_{x \rightarrow -3^+} f(x) = -\infty$

7. $f(x) = \frac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\lim_{x \rightarrow -3^-} f(x) = \infty$

$\lim_{x \rightarrow -3^+} f(x) = -\infty$

$$9. \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty = \lim_{x \rightarrow 0^-} \frac{1}{x^2}$$

Therefore, $x = 0$ is a vertical asymptote.

$$13. \lim_{x \rightarrow -2^-} \frac{x^2}{x^2 - 4} = \infty \text{ and } \lim_{x \rightarrow -2^+} \frac{x^2}{x^2 - 4} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 2^-} \frac{x^2}{x^2 - 4} = -\infty \text{ and } \lim_{x \rightarrow 2^+} \frac{x^2}{x^2 - 4} = \infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$17. f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x} \text{ has vertical asymptotes at}$$

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2}, n \text{ any integer.}$$

$$21. \lim_{x \rightarrow -2^+} \frac{x}{(x+2)(x-1)} = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x}{(x+2)(x-1)} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 1^+} \frac{x}{(x+2)(x-1)} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x}{(x+2)(x-1)} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote.

$$25. f(x) = \frac{(x-5)(x+3)}{(x-5)(x^2+1)} = \frac{x+3}{x^2+1}, x \neq 5$$

No vertical asymptotes. The graph has a hole at $x = 5$.

$$11. \lim_{x \rightarrow 2^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$\lim_{x \rightarrow -1^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$$

Therefore, $x = -1$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

$$19. \lim_{t \rightarrow 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t \rightarrow 0^-} \left(1 - \frac{4}{t^2}\right)$$

Therefore, $t = 0$ is a vertical asymptote.

$$23. f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x+1)(x^2 - x + 1)}{x + 1}$$

has no vertical asymptote since

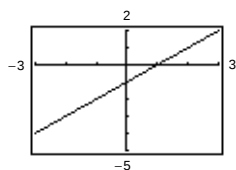
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x^2 - x + 1) = 3$$

$$27. s(t) = \frac{t}{\sin t} \text{ has vertical asymptotes at } t = n\pi, n$$

a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1.$$

$$29. \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$$



Removable discontinuity at $x = -1$

$$33. \lim_{x \rightarrow 2^+} \frac{x - 3}{x - 2} = -\infty$$

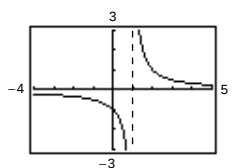
$$37. \lim_{x \rightarrow -3} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \rightarrow -3} \frac{x - 1}{x - 2} = \frac{4}{5}$$

$$41. \lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right) = -\infty$$

$$45. \lim_{x \rightarrow \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \rightarrow \pi} (\sqrt{x} \sin x) = 0$$

$$49. f(x) = \frac{x^2 + x + 1}{x^3 - 1}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty$$

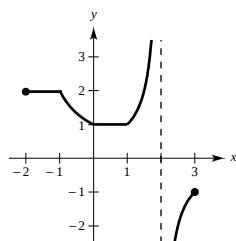


53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x \rightarrow c} f(x) = \infty$$

says how the limit fails to exist.

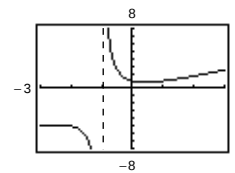
57.



$$31. \lim_{x \rightarrow -1^+} \frac{x^2 + 1}{x + 1} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 + 1}{x + 1} = -\infty$$

Vertical asymptote at $x = -1$



$$35. \lim_{x \rightarrow 3^+} \frac{x^2}{(x - 3)(x + 3)} = \infty$$

$$39. \lim_{x \rightarrow 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{x^2 + 1} = \frac{1}{2}$$

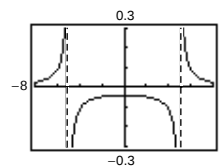
$$43. \lim_{x \rightarrow 0^+} \frac{2}{\sin x} = \infty$$

$$47. \lim_{x \rightarrow (1/2)^-} x \sec(\pi x) = \infty \text{ and } \lim_{x \rightarrow (1/2)^+} x \sec(\pi x) = -\infty.$$

Therefore, $\lim_{x \rightarrow (1/2)} x \sec(\pi x)$ does not exist.

$$51. f(x) = \frac{1}{x^2 - 25}$$

$$\lim_{x \rightarrow 5^-} f(x) = -\infty$$



$$55. \text{One answer is } f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}.$$

$$59. S = \frac{k}{1 - r}, 0 < |r| < 1. \text{ Assume } k \neq 0.$$

$$\lim_{r \rightarrow 1} S = \lim_{r \rightarrow 1} \frac{k}{1 - r} = \infty \quad (\text{or } -\infty \text{ if } k < 0)$$

61. $C = \frac{528x}{100 - x}, 0 \leq x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\lim_{x \rightarrow 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

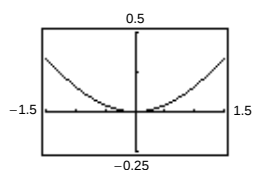
63. (a) $r = \frac{2(7)}{\sqrt{625 - 49}} = \frac{7}{12}$ ft/sec

(b) $r = \frac{2(15)}{\sqrt{625 - 225}} = \frac{3}{2}$ ft/sec

(c) $\lim_{x \rightarrow 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

65. (a)

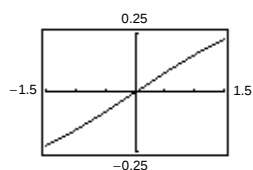
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x} = 0$$

(b)

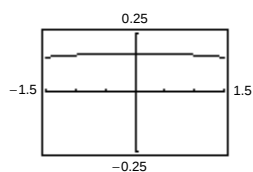
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

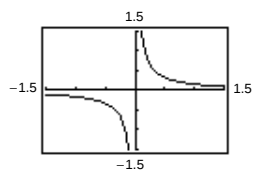
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^3} = 0.1167 \text{ (1/6)}$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^4} = \infty$$

For $n \geq 3$, $\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^n} = \infty$.

67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.
- (c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections.
The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

$$\text{Total length} = 60 \cot \phi + 30(\pi + 2\phi)$$

$$\text{Domain: } \left(0, \frac{\pi}{2}\right)$$

69. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

73. Given $\lim_{x \rightarrow c} f(x) = \infty$ and $\lim_{x \rightarrow c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \rightarrow c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \rightarrow c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have

$$\left| \frac{g(x)}{f(x)} \right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

$$\text{Therefore, } \lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0.$$

75. Given $\lim_{x \rightarrow c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \rightarrow c} f(x)$ exists and equals L . Then,

$$\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{\lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} f(x)} = \frac{1}{L} = 0.$$

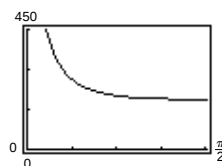
This is not possible. Thus, $\lim_{x \rightarrow c} f(x)$ does not exist.

- (b) The direction of rotation is reversed.

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



$$(f) \lim_{\phi \rightarrow (\pi/2)^-} L = 60\pi \approx 188.5$$

(All the belts are around pulleys.)

$$(g) \lim_{\phi \rightarrow 0^+} L = \infty$$

71. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

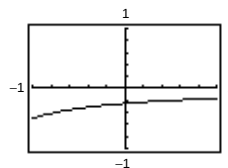
Review Exercises for Chapter 1

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3.
Or, the length is slightly longer than the distance between the two points, 8.25.

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.26	-0.25	-0.250	-0.2499	-0.249	-0.24

$$\lim_{x \rightarrow 0} f(x) \approx -0.25$$



5. $h(x) = \frac{x^2 - 2x}{x}$

(a) $\lim_{x \rightarrow 0} h(x) = -2$

(b) $\lim_{x \rightarrow -1} h(x) = -3$

7. $\lim_{x \rightarrow 1} (3 - x) = 3 - 1 = 2$

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for

$$0 < |x - 1| < \delta = \epsilon, \text{ you have}$$

$$|x - 1| < \epsilon$$

$$|1 - x| < \epsilon$$

$$|(3 - x) - 2| < \epsilon$$

$$|f(x) - L| < \epsilon$$

9. $\lim_{x \rightarrow 2} (x^2 - 3) = 1$

Let $\epsilon > 0$ be given. We need $|x^2 - 3 - 1| < \epsilon \Rightarrow |x^2 - 4| = |(x - 2)(x + 2)| < \epsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|} \epsilon$.

Assuming, $1 < x < 3$, you can choose $\delta = \epsilon/5$. Hence, for $0 < |x - 2| < \delta = \epsilon/5$ you have

$$|x - 2| < \frac{\epsilon}{5} < \frac{1}{|x + 2|} \epsilon$$

$$|x - 2||x + 2| < \epsilon$$

$$|x^2 - 4| < \epsilon$$

$$|(x^2 - 3) - 1| < \epsilon$$

$$|f(x) - L| < \epsilon$$

11. $\lim_{t \rightarrow 4} \sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} \approx 2.45$

13. $\lim_{t \rightarrow -2} \frac{t + 2}{t^2 - 4} = \lim_{t \rightarrow -2} \frac{1}{t - 2} = -\frac{1}{4}$

15. $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$

17. $\lim_{x \rightarrow 0} \frac{[1/(x + 1)] - 1}{x} = \lim_{x \rightarrow 0} \frac{1 - (x + 1)}{x(x + 1)} = \lim_{x \rightarrow 0} \frac{-1}{x + 1} = -1$

19. $\lim_{x \rightarrow -5} \frac{x^3 + 125}{x + 5} = \lim_{x \rightarrow -5} \frac{(x + 5)(x^2 - 5x + 25)}{x + 5} = \lim_{x \rightarrow -5} (x^2 - 5x + 25) = 75$

21. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \left(\frac{1 - \cos x}{x} \right) = (1)(0) = 0$

$$\begin{aligned}
 23. \lim_{\Delta x \rightarrow 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\sin(\pi/6) \cos \Delta x + \cos(\pi/6) \sin \Delta x - (1/2)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{2} \cdot \frac{(\cos \Delta x - 1)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\sqrt{3}}{2} \cdot \frac{\sin \Delta x}{\Delta x} \\
 &= 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}
 \end{aligned}$$

$$25. \lim_{x \rightarrow c} [f(x) \cdot g(x)] = \left(-\frac{3}{4}\right)\left(\frac{2}{3}\right) = -\frac{1}{2}$$

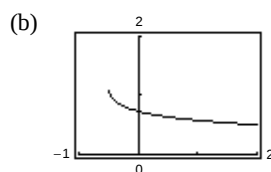
$$27. f(x) = \frac{\sqrt{2x+1} - \sqrt{3}}{x-1}$$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$$\lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \approx 0.577 \quad (\text{Actual limit is } \sqrt{3}/3)$$

$$\begin{aligned}
 (c) \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} &= \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{2x+1} + \sqrt{3}}{\sqrt{2x+1} + \sqrt{3}} \\
 &= \lim_{x \rightarrow 1^+} \frac{(2x+1) - 3}{(x-1)(\sqrt{2x+1} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 1^+} \frac{2}{\sqrt{2x+1} + \sqrt{3}} \\
 &= \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
 \end{aligned}$$



$$\begin{aligned}
 29. \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow 4} \frac{(-4.9(4)^2 + 200) - (-4.9t^2 + 200)}{4 - t} \\
 &= \lim_{t \rightarrow 4} \frac{4.9(t-4)(t+4)}{4 - t} \\
 &= \lim_{t \rightarrow 4} -4.9(t+4) = -39.2 \text{ m/sec}
 \end{aligned}$$

$$31. \lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} = \lim_{x \rightarrow 3^-} \frac{-(x-3)}{x-3} = -1$$

$$33. \lim_{x \rightarrow 2} f(x) = 0$$

$$\begin{aligned}
 35. \lim_{t \rightarrow 1} h(t) \text{ does not exist because } \lim_{t \rightarrow 1^-} h(t) &= 1 + 1 = 2 \text{ and} \\
 \lim_{t \rightarrow 1^+} h(t) &= \frac{1}{2}(1 + 1) = 1.
 \end{aligned}$$

$$37. f(x) = \llbracket x + 3 \rrbracket$$

$$\lim_{x \rightarrow k^+} \llbracket x + 3 \rrbracket = k + 3 \text{ where } k \text{ is an integer.}$$

$$\lim_{x \rightarrow k^-} \llbracket x + 3 \rrbracket = k + 2 \text{ where } k \text{ is an integer.}$$

Nonremovable discontinuity at each integer k

Continuous on $(k, k+1)$ for all integers k

$$39. f(x) = \frac{3x^2 - x - 2}{x-1} = \frac{(3x+2)(x-1)}{x-1}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (3x+2) = 5$$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

$$41. f(x) = \frac{1}{(x-2)^2}$$

$$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty$$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

$$43. f(x) = \frac{3}{x+1}$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \infty$$

Nonremovable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

$$45. f(x) = \csc \frac{\pi x}{2}$$

Nonremovable discontinuities at each even integer.
Continuous on

$$(2k, 2k + 2)$$

for all integers k .

49. f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

$$53. g(x) = 1 + \frac{2}{x}$$

Vertical asymptote at $x = 0$

$$57. \lim_{x \rightarrow -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty$$

$$61. \lim_{x \rightarrow 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty$$

$$65. \lim_{x \rightarrow 0^+} \frac{\sin 4x}{5x} = \lim_{x \rightarrow 0^+} \left[\frac{4}{5} \left(\frac{\sin 4x}{4x} \right) \right] = \frac{4}{5}$$

$$69. C = \frac{80,000p}{100 - p}, 0 \leq 0 < 100$$

$$(a) C(15) \approx \$14,117.65 \quad (b) C(50) = \$80,000$$

$$(c) C(90) = \$720,000 \quad (d) \lim_{p \rightarrow 100^-} \frac{80,000p}{100 - p} = \infty$$

$$47. f(2) = 5$$

Find c so that $\lim_{x \rightarrow 2^+} (cx + 6) = 5$.

$$c(2) + 6 = 5$$

$$2c = -1$$

$$c = -\frac{1}{2}$$

$$51. f(x) = \frac{x^2 - 4}{|x - 2|} = (x + 2) \left[\frac{x - 2}{|x - 2|} \right]$$

$$(a) \lim_{x \rightarrow 2^-} f(x) = -4$$

$$(b) \lim_{x \rightarrow 2^+} f(x) = 4$$

$$(c) \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$$

$$55. f(x) = \frac{8}{(x - 10)^2}$$

Vertical asymptote at $x = 10$

$$59. \lim_{x \rightarrow -1^+} \frac{x + 1}{x^3 + 1} = \lim_{x \rightarrow -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3}$$

$$63. \lim_{x \rightarrow 0^+} \left(x - \frac{1}{x^3} \right) = -\infty$$

$$67. \lim_{x \rightarrow 0^+} \frac{\csc 2x}{x} = \lim_{x \rightarrow 0^+} \frac{1}{x \sin 2x} = \infty$$

Problem Solving for Chapter 1

$$1. (a) \text{ Perimeter } \triangle PAO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1 \\ = \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$$

$$\text{Perimeter } \triangle PBO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1 \\ = \sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$$

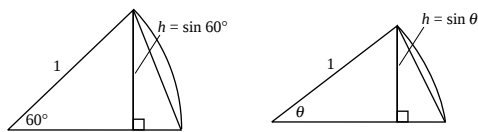
$$(c) \lim_{x \rightarrow 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1$$

$$(b) r(x) = \frac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$$

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$	33.02	9.08	3.41	2.10	2.01
Perimeter $\triangle PBO$	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

3. (a) There are 6 triangles, each with a central angle of $60^\circ = \pi/3$. Hence,

$$\begin{aligned}\text{Area hexagon} &= 6 \left[\frac{1}{2}bh \right] = 6 \left[\frac{1}{2}(1) \sin \frac{\pi}{3} \right] \\ &= \frac{3\sqrt{3}}{2} \approx 2.598.\end{aligned}$$



$$\text{Error: } \pi - \frac{3\sqrt{3}}{2} \approx 0.5435.$$

- (b) There are n triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$An = n \left[\frac{1}{2}bh \right] = n \left[\frac{1}{2}(1) \sin \frac{2\pi}{n} \right] = \frac{n \sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
An	2.598	3	3.106	3.133	3.139

- (d) As n gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$An = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)} \pi = \frac{\sin x}{x} \pi$$

which approaches $(1)\pi = \pi$.

5. (a) Slope $= -\frac{12}{5}$

(b) Slope of tangent line is $\frac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

(c) $Q = (x, y) = (x, \sqrt{169 - x^2})$

$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

(d) $\lim_{x \rightarrow 5} m_x = \lim_{x \rightarrow 5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}$

$$= \lim_{x \rightarrow 5} \frac{144 - (169 - x^2)}{(x - 5)(12 + \sqrt{169 - x^2})}$$

$$= \lim_{x \rightarrow 5} \frac{x^2 - 25}{(x - 5)(12 + \sqrt{169 - x^2})}$$

$$= \lim_{x \rightarrow 5} \frac{(x + 5)}{12 + \sqrt{169 - x^2}}$$

$$= \frac{10}{12 + 12} = \frac{5}{12}$$

This is the same slope as part (b).

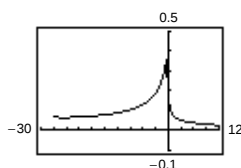
7. (a) $3 + x^{1/3} \geq 0$

$$x^{1/3} \geq -3$$

$$x \geq -27$$

Domain: $x \geq -27, x \neq 1$

(b)



(c) $\lim_{x \rightarrow -27^+} f(x) = \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1}$

$$= \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$$

(d) $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}$

$$= \lim_{x \rightarrow 1} \frac{3 + x^{1/3} - 4}{(x - 1)(\sqrt{3 + x^{1/3}} + 2)}$$

$$= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)(\sqrt{3 + x^{1/3}} + 2)}$$

$$= \lim_{x \rightarrow 1} \frac{1}{(x^{2/3} + x^{1/3} + 1)(\sqrt{3 + x^{1/3}} + 2)}$$

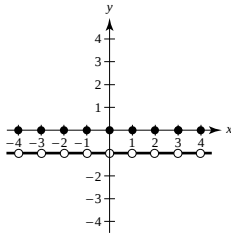
$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

9. (a) $\lim_{x \rightarrow 2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\lim_{x \rightarrow 2^-} f(x) = 3$: g_1, g_3, g_4

11.



$$(a) \quad f(1) = \llbracket 1 \rrbracket + \llbracket -1 \rrbracket = 1 + (-1) = 0$$

$$f(0) = 0$$

$$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$$

$$f(-2.7) = -3 + 2 = -1$$

$$(b) \quad \lim_{x \rightarrow 1^-} f(x) = -1$$

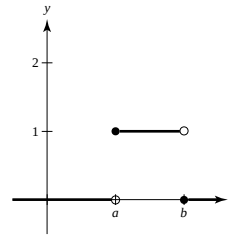
$$\lim_{x \rightarrow 1^+} f(x) = -1$$

$$\lim_{x \rightarrow 1/2} f(x) = -1$$

(c) f is continuous for all real numbers except

$$x = 0, \pm 1, \pm 2, \pm 3, \dots$$

13. (a)



$$(b) \quad (i) \quad \lim_{x \rightarrow a^+} P_{a,b}(x) = 1$$

$$(ii) \quad \lim_{x \rightarrow a^-} P_{a,b}(x) = 0$$

$$(iii) \quad \lim_{x \rightarrow b^+} P_{a,b}(x) = 0$$

$$(iv) \quad \lim_{x \rightarrow b^-} P_{a,b}(x) = 1$$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of u , and above the x -axis, is 1.

CHAPTER 2

Differentiation

Section 2.1	The Derivative and the Tangent Line Problem	. . . 53
Section 2.2	Basic Differentiation Rules and Rates of Change	. 60
Section 2.3	The Product and Quotient Rules and Higher-Order Derivatives	 67
Section 2.4	The Chain Rule	 73
Section 2.5	Implicit Differentiation	 79
Section 2.6	Related Rates	 85
	Review Exercises	 92
	Problem Solving	 98

CHAPTER 2

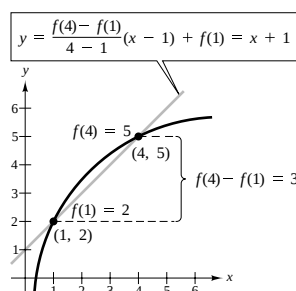
Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

1. (a) $m = 0$
(b) $m = -3$

3. (a), (b)



$$\begin{aligned} \text{(c) } y &= \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1) \\ &= \frac{3}{3}(x - 1) + 2 \\ &= 1(x - 1) + 2 \\ &= x + 1 \end{aligned}$$

5. $f(x) = 3 - 2x$ is a line. Slope $= -2$

$$\begin{aligned} 7. \text{ Slope at } (1, -3) &= \lim_{\Delta x \rightarrow 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} [2 + 2(\Delta x)] = 2 \end{aligned}$$

$$\begin{aligned} 9. \text{ Slope at } (0, 0) &= \lim_{\Delta t \rightarrow 0} \frac{f(0 + \Delta t) - f(0)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (3 - \Delta t) = 3 \end{aligned}$$

$$\begin{aligned} 11. f(x) &= 3 \\ f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3 - 3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 0 = 0 \end{aligned}$$

13. $f(x) = -5x$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} -5 = -5 \end{aligned}$$

15. $h(s) = 3 + \frac{2}{3}s$

$$\begin{aligned} h'(s) &= \lim_{\Delta s \rightarrow 0} \frac{h(s + \Delta s) - h(s)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3} \end{aligned}$$

17. $f(x) = 2x^2 + x - 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (4x + 2\Delta x + 1) = 4x + 1
 \end{aligned}$$

19. $f(x) = x^3 - 12x$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12
 \end{aligned}$$

21. $f(x) = \frac{1}{x - 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x - 1} - \frac{1}{x - 1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x - 1)(x - 1)} \\
 &= -\frac{1}{(x - 1)^2}
 \end{aligned}$$

23. $f(x) = \sqrt{x + 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x[\sqrt{x + \Delta x + 1} + \sqrt{x + 1}]} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \\
 &= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}
 \end{aligned}$$

25. (a) $f(x) = x^2 + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x
 \end{aligned}$$

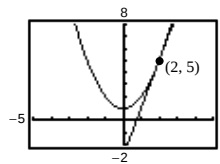
At $(2, 5)$, the slope of the tangent line is
 $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$

(b)



27. (a) $f(x) = x^3$

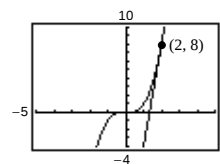
$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2
 \end{aligned}$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$.
 The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$

(b)



29. (a) $f(x) = \sqrt{x}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

At $(1, 1)$, the slope of the tangent line is

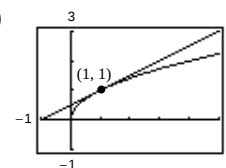
$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)



31. (a) $f(x) = \frac{4}{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) + \frac{4}{x + \Delta x} - \left(x + \frac{4}{x}\right)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)} \\ &= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2} \end{aligned}$$

At (4, 5), the slope of the tangent line is

$$m = 1 - \frac{4}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$y - 5 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x + 2$$

33. From Exercise 27 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

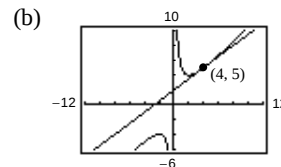
$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points (1, 1) and (-1, -1) the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \quad \quad \quad y = 3x + 2.$$



35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$

$$x = 1.$$

Therefore, at the point (1, 1) the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

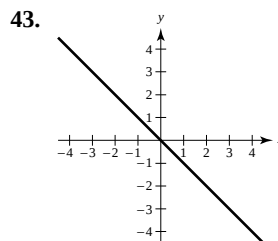
$$y = -\frac{1}{2}x + \frac{3}{2}.$$

37. $g(5) = 2$ because the tangent line passes through (5, 2)

$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

39. $f(x) = x \Rightarrow f'(x) = 1$ (b)

41. $f(x) = \sqrt{x} \Rightarrow f'(x)$ matches (a)
(decreasing slope as $x \rightarrow \infty$)



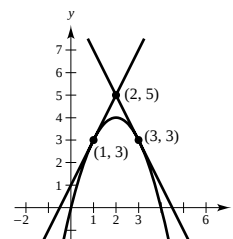
Answers will vary.

Sample answer: $y = -x$

45. (a) If $f'(c) = 3$ and f is odd, then $f'(-c) = f'(c) = 3$
(b) If $f'(c) = 3$ and f is even, then $f'(-c) = -f'(c) = -3$

47. Let (x_0, y_0) be a point of tangency on the graph of f . By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0 :

$$\begin{aligned}\frac{5 - y_0}{2 - x_0} &= 4 - 2x_0 \\ 5 - y_0 &= (2 - x_0)(4 - 2x_0) \\ 5 - (4x_0 - x_0^2) &= 8 - 8x_0 + 2x_0^2 \\ 0 &= x_0^2 - 4x_0 + 3 \\ 0 &= (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3\end{aligned}$$



Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2 . The equations of the tangent lines are

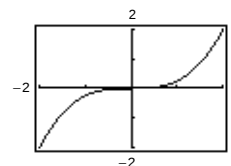
$$\begin{aligned}y - 5 &= 2(x - 2) & y - 5 &= -2(x - 2) \\ y &= 2x + 1 & y &= -2x + 9\end{aligned}$$

49. (a) $g'(0) = -3$
(b) $g'(3) = 0$
(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.
(d) Because $g'(-4) = \frac{7}{5}$, g is increasing (rising) at $x = -4$.
(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.
(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

51. $f(x) = \frac{1}{4}x^3$

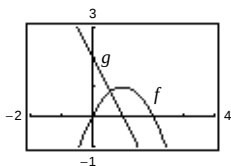
By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3



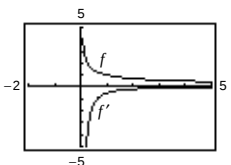
$$53. g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

$$57. f(x) = \frac{1}{\sqrt{x}} \text{ and } f'(x) = \frac{-1}{2x^{3/2}}.$$



As $x \rightarrow \infty$, f is nearly horizontal and thus $f' \approx 0$.

$$59. f(x) = 4 - (x - 3)^2$$

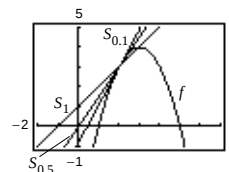
$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$$

$$= \frac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \frac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$$

$$(a) \Delta x = 1: S_{\Delta x} = (x - 2) + 3 = x + 1$$

$$\Delta x = 0.5: S_{\Delta x} = \left(\frac{3}{2}\right)(x - 2) + 3 = \frac{3}{2}x$$

$$\Delta x = 0.1: S_{\Delta x} = \left(\frac{19}{10}\right)(x - 2) + 3 = \frac{19}{10}x - \frac{4}{5}$$



(b) As $\Delta x \rightarrow 0$, the line approaches the tangent line to f at $(2, 3)$.

$$61. f(x) = x^2 - 1, c = 2$$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^2 - 1) - 3}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4$$

$$63. f(x) = x^3 + 2x^2 + 1, c = -2$$

$$f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \rightarrow -2} \frac{x^2(x + 2)}{x + 2} = \lim_{x \rightarrow -2} \frac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x \rightarrow -2} x^2 = 4$$

$$65. g(x) = \sqrt{|x|}, c = 0$$

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sqrt{|x|}}{x}. \text{ Does not exist.}$$

$$\text{As } x \rightarrow 0^-, \frac{\sqrt{|x|}}{x} = \frac{-1}{\sqrt{x}} \rightarrow -\infty$$

$$\text{As } x \rightarrow 0^+, \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} \rightarrow \infty$$

$$67. f(x) = (x - 6)^{2/3}, c = 6$$

$$f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6}$$

$$= \lim_{x \rightarrow 6} \frac{(x - 6)^{2/3} - 0}{x - 6}$$

$$= \lim_{x \rightarrow 6} \frac{1}{(x - 6)^{1/3}}$$

Does not exist.

69. $h(x) = (x + 5)$, $c = -5$

$$\begin{aligned} h'(-5) &= \lim_{x \rightarrow -5} \frac{h(x) - h(-5)}{x - (-5)} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5| - 0}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5|}{x + 5} \end{aligned}$$

Does not exist.

73. $f(x)$ is differentiable everywhere except at $x = -1$.
(Discontinuity)

77. $f(x)$ is differentiable on the interval $(1, \infty)$.
(At $x = 1$ the tangent line is vertical)

81. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

71. $f(x)$ is differentiable everywhere except at $x = -3$.
(Sharp turn in the graph.)

75. $f(x)$ is differentiable everywhere except at $x = 3$.
(Sharp turn in the graph)

79. $f(x)$ is differentiable everywhere except at $x = 0$.
(Discontinuity)

83. $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{(x - 1)^3 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^-} (x - 1)^2 = 0. \end{aligned}$$

The derivative from the right is

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{(x - 1)^2 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^+} (x - 1) = 0. \end{aligned}$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. ($f'(1) = 0$)

85. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is $\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \rightarrow 2^-} (x + 2) = 4$.

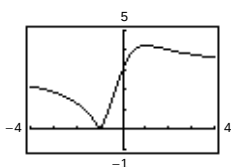
The derivative from the right is $\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \rightarrow 2^+} 4 = 4$.

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. ($f'(2) = 4$)

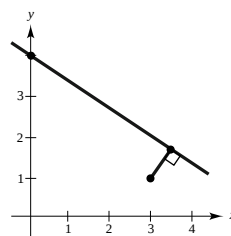
87. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is

$$\begin{aligned} d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\ &= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}. \end{aligned}$$

(b)



The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.



89. False. the slope is $\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$.

91. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1 . At $x = 0$, the derivative does not exist.

93. $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, we have $-|x| \leq x \sin(1/x) \leq |x|$, $x \neq 0$. Thus, $\lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} \left(\sin \frac{1}{x} \right).$$

Since this limit does not exist (it oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \leq x^2 \sin(1/x) \leq x^2$, $x \neq 0$. Thus, $\lim_{x \rightarrow 0} x^2 \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

Section 2.2 Basic Differentiation Rules and Rates of Change

1. (a) $y = x^{1/2}$
 $y' = \frac{1}{2}x^{-1/2}$
 $y'(1) = \frac{1}{2}$

(b) $y = x^{3/2}$
 $y' = \frac{3}{2}x^{1/2}$
 $y'(1) = \frac{3}{2}$

(c) $y = x^2$
 $y' = 2x$
 $y'(1) = 2$

(d) $y = x^3$
 $y' = 3x^2$
 $y'(1) = 3$

3. $y = 8$
 $y' = 0$

5. $y = x^6$
 $y' = 6x^5$

7. $y = \frac{1}{x^7} = x^{-7}$
 $y' = -7x^{-8} = -\frac{7}{x^8}$

9. $y = \sqrt[5]{x} = x^{1/5}$
 $y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$

11. $f(x) = x + 1$
 $f'(x) = 1$

13. $f(t) = -2t^2 + 3t - 6$
 $f'(t) = -4t + 3$

15. $g(x) = x^2 + 4x^3$
 $g'(x) = 2x + 12x^2$

17. $s(t) = t^3 - 2t + 4$
 $s'(t) = 3t^2 - 2$

19. $y = \frac{\pi}{2} \sin \theta - \cos \theta$
 $y' = \frac{\pi}{2} \cos \theta + \sin \theta$

21. $y = x^2 - \frac{1}{2} \cos x$
 $y' = 2x + \frac{1}{2} \sin x$

23. $y = \frac{1}{x} - 3 \sin x$
 $y' = -\frac{1}{x^2} - 3 \cos x$

- | <u>Function</u> | <u>Rewrite</u> | <u>Derivative</u> | <u>Simplify</u> |
|------------------------------|-------------------------|-----------------------------|----------------------------|
| 25. $y = \frac{5}{2x^2}$ | $y = \frac{5}{2}x^{-2}$ | $y' = -5x^{-3}$ | $y' = \frac{-5}{x^3}$ |
| 27. $y = \frac{3}{(2x)^3}$ | $y = \frac{3}{8}x^{-3}$ | $y' = \frac{-9}{8}x^{-4}$ | $y' = \frac{-9}{8x^4}$ |
| 29. $y = \frac{\sqrt{x}}{x}$ | $y = x^{-1/2}$ | $y' = -\frac{1}{2}x^{-3/2}$ | $y' = -\frac{1}{2x^{3/2}}$ |
-
31. $f(x) = \frac{3}{x^2} = 3x^{-2}, (1, 3)$

$f'(x) = -6x^{-3} = \frac{-6}{x^3}$

$f'(1) = -6$

33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)$

$f'(x) = \frac{21}{5}x^2$

$f'(0) = 0$

35. $y = (2x + 1)^2, (0, 1)$

$= 4x^2 + 4x + 1$

$y' = 8x + 4$

$y'(0) = 4$
-
37. $f(\theta) = 4 \sin \theta - \theta, (0, 0)$

$f'(\theta) = 4 \cos \theta - 1$

$f'(0) = 4(1) - 1 = 3$

39. $f(x) = x^2 + 5 - 3x^{-2}$

$f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}$

41. $g(t) = t^2 - \frac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \frac{12}{t^4}$
-
43. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

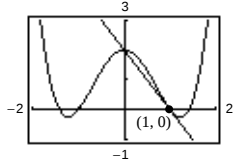
$f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}$

45. $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$
-
47. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \frac{1}{2}x^{-1/2} + 2x^{-2/3} = \frac{1}{2\sqrt{x}} + \frac{2}{x^{2/3}}$

49. $h(s) = s^{4/5} - s^{2/3}$

$h'(s) = \frac{4}{5}s^{-4/5} - \frac{2}{3}s^{-1/3} = \frac{4}{5s^{4/5}} - \frac{2}{3s^{1/3}}$
-
51. $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$
- $f'(x) = 3x^{-1/2} - 5 \sin x = \frac{3}{\sqrt{x}} - 5 \sin x$
-
53. (a) $y = x^4 - 3x^2 + 2$
- $y' = 4x^3 - 6x$
- At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2$.
- Tangent line: $y - 0 = -2(x - 1)$
- $2x + y - 2 = 0$
- (b)
- 

55. (a) $f(x) = \frac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

$f'(x) = \frac{-3}{2}x^{-7/4} = \frac{-3}{2x^{7/4}}$

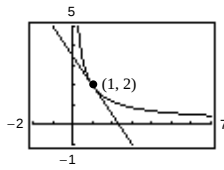
At $(1, 2)$, $f'(1) = \frac{-3}{2}$

Tangent line: $y - 2 = -\frac{3}{2}(x - 1)$

$y = -\frac{3}{2}x + \frac{7}{2}$

$3x + 2y - 7 = 0$

(b)



57. $y = x^4 - 8x^2 + 2$

$$y' = 4x^3 - 16x$$

$$= 4x(x^2 - 4)$$

$$= 4x(x - 2)(x + 2)$$

$$y' = 0 \Rightarrow x = 0, \pm 2$$

Horizontal tangents: $(0, 2)$, $(2, -14)$, $(-2, -14)$

61. $y = x + \sin x$, $0 \leq x < 2\pi$

$$y' = 1 + \cos x = 0$$

$$\cos x = -1 \Rightarrow x = \pi$$

At $x = \pi$, $y = \pi$.Horizontal tangent: (π, π)

65. $\frac{k}{x} = -\frac{3}{4}x + 3$ Equate functions

$$-\frac{k}{x^2} = -\frac{3}{4}$$
 Equate derivatives

$$\text{Hence, } k = \frac{3}{4}x^2 \text{ and } \frac{\frac{3}{4}x^2}{x} = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{4}x = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3.$$

67. (a) The slope appears to be steepest between A and B.

(b) The average rate of change between A and B is **greater** than the instantaneous rate of change at B.

59. $y = \frac{1}{x^2} = x^{-2}$

$$y' = -2x^{-3} = \frac{-2}{x^3} \text{ cannot equal zero.}$$

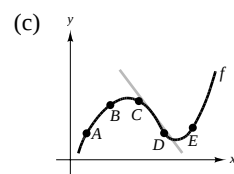
Therefore, there are no horizontal tangents.

63. $x^2 - kx = 4x - 9$ Equate functions

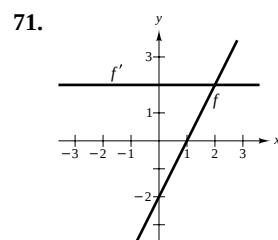
$$2x - k = 4$$
 Equate derivatives

Hence, $k = 2x - 4$ and

$$x^2 - (2x - 4)x = 4x - 9 \Rightarrow -x^2 = -9 \Rightarrow x = \pm 3.$$

For $x = 3$, $k = 2$ and for $x = -3$, $k = -10$.

69. $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

If f is linear then its derivative is a constant function.

$$f(x) = ax + b$$

$$f'(x) = a$$

73. Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \Rightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Rightarrow m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$

$$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2 \text{ and } y_1 = 4$$

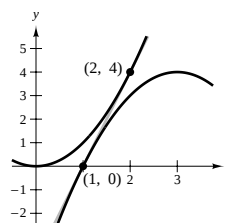
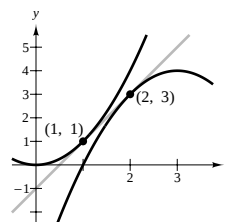
Thus, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$

$$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1 \text{ and } y_1 = 1$$

Thus, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$



75. $f(x) = \sqrt{x}$, $(-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{xy}$$

$$4 + x = 2\sqrt{x}\sqrt{y}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

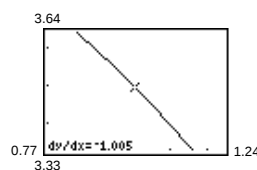
The point $(4, 2)$ is on the graph of f .

Tangent line: $y - 2 = \frac{0 - 2}{-4 - 4}(x - 4)$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

77. $f'(1) = -1$

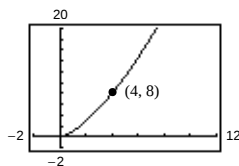


79. (a) One possible secant is between (3.9, 7.7019) and (4, 8):

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

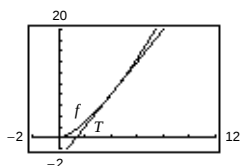


$$(b) f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$$

$$T(x) = 3(x - 4) + 8 = 3x - 4$$

$S(x)$ is an approximation of the tangent line $T(x)$.

- (c) As you move further away from (4, 8), the accuracy of the approximation T gets worse.



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

81. False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

83. False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

85. True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

87. $f(t) = 2t + 7, [1, 2]$

$$f'(t) = 2$$

Instantaneous rate of change is the constant 2.

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because f is a line of slope 2.)

89. $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \Rightarrow f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

91. (a) $s(t) = -16t^2 + 1362$

$$v(t) = -32t$$

(b) $\frac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48 \text{ ft/sec}$

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32 \text{ ft/sec}$.

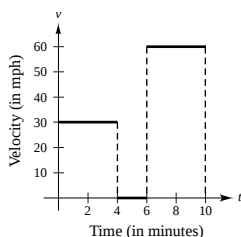
When $t = 2$: $v(2) = -64 \text{ ft/sec}$.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right) = -8\sqrt{1362} \approx -295.242 \text{ ft/sec}$

95.



(The velocity has been converted to miles per hour)

99. (a) Using a graphing utility, you obtain

$$R = 0.167v - 0.02.$$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(e) $\frac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

101. $A = s^2, \frac{dA}{ds} = 2s$

When $s = 4 \text{ m}$,

$$\frac{dA}{ds} = 8 \text{ square meters per meter change in } s.$$

93. $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + 120t$$

$$v(t) = -9.8t + 120$$

$$v(5) = -9.8(5) + 120 = 71 \text{ m/sec}$$

$$v(10) = -9.8(10) + 120 = 22 \text{ m/sec}$$

97. $v = 40 \text{ mph} = \frac{2}{3} \text{ mi/min}$

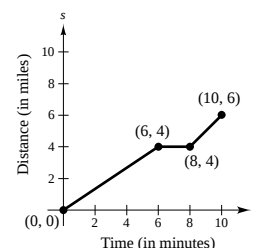
$$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4 \text{ mi}$$

$$v = 0 \text{ mph} = 0 \text{ mi/min}$$

$$(0 \text{ mi/min})(2 \text{ min}) = 0 \text{ mi}$$

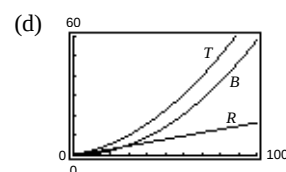
$$v = 60 \text{ mph} = 1 \text{ mi/min}$$

$$(1 \text{ mi/min})(2 \text{ min}) = 2 \text{ mi}$$



(b) Using a graphing utility, you obtain

$$B = 0.00586v^2 - 0.0239v + 0.46.$$



(f) For increasing speeds, the total stopping distance increases.

103. $C = \frac{1,008,000}{Q} + 6.3Q$

$$\frac{dC}{dQ} = -\frac{1,008,000}{Q^2} + 6.3$$

$$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$$

When $Q = 350$, $\frac{dC}{dQ} \approx -\$1.93$.

 105. (a) $f'(1.47)$ is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

 (b) $f'(1.47)$ is usually negative. As prices go up, sales go down.

107. $y = ax^2 + bx + c$

Since the parabola passes through (0, 1) and (1, 0), we have

$$(0, 1): 1 = a(0)^2 + b(0) + c \Rightarrow c = 1$$

$$(1, 0): 0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1.$$

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point (1, 0).

$$y' = 2ax + (-a - 1)$$

$$1 = 2a(1) + (-a - 1)$$

$$1 = a - 1$$

$$a = 2$$

$$b = -a - 1 = -3$$

Therefore, $y = 2x^2 - 3x + 1$.

109. $y = x^3 - 9x$

$$y' = 3x^2 - 9$$

Tangent lines through (1, -9):

$$y + 9 = (3x^2 - 9)(x - 1)$$

$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$

$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

The points of tangency are (0, 0) and $(\frac{3}{2}, -\frac{81}{8})$. At (0, 0) the slope is $y'(0) = -9$. At $(\frac{3}{2}, -\frac{81}{8})$ the slope is $y'(\frac{3}{2}) = -\frac{9}{4}$.

Tangent lines:

$$y - 0 = -9(x - 0) \quad \text{and} \quad y + \frac{81}{8} = -\frac{9}{4}(x - \frac{3}{2})$$

$$y = -9x \quad y = -\frac{9}{4}x - \frac{27}{4}$$

$$9x + y = 0 \quad 9x + 4y + 27 = 0$$

111. $f(x) = \begin{cases} ax^3, & x \leq 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} ax^3 = 8a \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 + b) = 4 + b \end{aligned} \right\} \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

$$b = 8a - 4 = -\frac{4}{3}$$

113. Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x (\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \sin x \left(\frac{\sin \Delta x}{\Delta x} \right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1. $g(x) = (x^2 + 1)(x^2 - 2x)$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

3. $h(t) = \sqrt[3]{t}(t^2 + 4) = t^{1/3}(t^2 + 4)$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4)\frac{1}{3}t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

5. $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \end{aligned}$$

7. $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

9. $h(x) = \frac{\sqrt[3]{x}}{x^3 + 1} = \frac{x^{1/3}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1)\frac{1}{3}x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2} \\ &= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2} \end{aligned}$$

11. $g(x) = \frac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \\ f'(0) &= -15 \end{aligned}$$

15. $f(x) = \frac{x^2 - 4}{x - 3}$

$$\begin{aligned} f'(x) &= \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2} \\ &= \frac{x^2 - 6x + 4}{(x - 3)^2} \end{aligned}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

17. $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
19. $y = \frac{x^2 + 2x}{3}$	$y = \frac{1}{3}x^2 + \frac{2}{3}x$	$y' = \frac{2}{3}x + \frac{2}{3}$	$y' = \frac{2x + 2}{3}$
21. $y = \frac{7}{3x^3}$	$y = \frac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\frac{7}{x^4}$
23. $y = \frac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \frac{2}{\sqrt{x}}$
25. $f(x) = \frac{3 - 2x - x^2}{x^2 - 1}$			
	$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$		
	$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$		
	$= \frac{2}{(x + 1)^2}, x \neq 1$		
27. $f(x) = x\left(1 - \frac{4}{x + 3}\right) = x - \frac{4x}{x + 3}$			
	$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$		
	$= \frac{x^2 + 6x - 3}{(x + 3)^2}$		
29. $f(x) = \frac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$			
	$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$		
	$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$		
31. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$			
	$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$		
33. $f(x) = \frac{2 - \frac{1}{x}}{x - 3} = \frac{2x - 1}{x(x - 3)} = \frac{2x - 1}{x^2 - 3x}$			
	$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$		
	$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$		
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$			
	$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$		
	$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$		
	$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$		
	$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$		
37. $f(x) = \frac{x^2 + c^2}{x^2 - c^2}$			
	$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$		
	$= \frac{-4xc^2}{(x^2 - c^2)^2}$		
39. $f(x) = t^2 \sin t$			
	$f'(t) = t^2 \cos t + 2t \sin t$		
	$= t(t \cos t + 2 \sin t)$		

$$41. f(t) = \frac{\cos t}{t}$$

$$f'(t) = \frac{-t \sin t - \cos t}{t^2} = -\frac{t \sin t + \cos t}{t^2}$$

$$45. g(t) = \sqrt[4]{t} + 8 \sec t = t^{1/4} + 8 \sec t$$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8 \sec t \tan t = \frac{1}{4t^{3/4}} + 8 \sec t \tan t$$

$$49. y = -\csc x - \sin x$$

$$y' = \csc x \cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x \cot^2 x$$

$$53. y = 2x \sin x + x^2 \cos x$$

$$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$$

$$= 4x \cos x + 2 \sin x - x^2 \sin x$$

$$57. g(\theta) = \frac{\theta}{1 - \sin \theta}$$

$$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2} \quad (\text{form of answer may vary})$$

$$59. y = \frac{1 + \csc x}{1 - \csc x}$$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1-2)^2} = -4\sqrt{3}$$

$$61. h(t) = \frac{\sec t}{t}$$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t \tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

$$43. f(x) = -x + \tan x$$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

$$47. y = \frac{3(1 - \sin x)}{2 \cos x} = \frac{3}{2}(\sec x - \tan x)$$

$$y' = \frac{3}{2}(\sec x \tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x \tan x - \tan^2 x - 1)$$

$$51. f(x) = x^2 \tan x$$

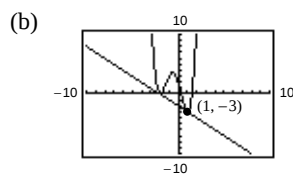
$$f'(x) = x^2 \sec^2 x + 2x \tan x$$

$$= x(x \sec^2 x + 2 \tan x)$$

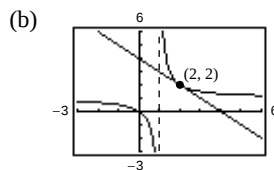
$$55. g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2} \quad (\text{form of answer may vary})$$

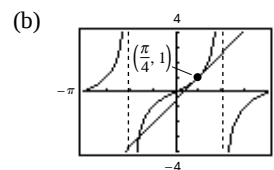
63. (a) $f(x) = (x^3 - 3x + 1)(x + 2)$, $(1, -3)$
 $f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$
 $= 4x^3 + 6x^2 - 6x - 5$
 $f'(1) = -1 = \text{slope at } (1, -3).$
Tangent line: $y + 3 = -1(x - 1) \Rightarrow y = -x - 2$



65. (a) $f(x) = \frac{x}{x-1}$, $(2, 2)$
 $f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$
 $f'(2) = \frac{-1}{(2-1)^2} = -1 = \text{slope at } (2, 2).$
Tangent line: $y - 2 = -1(x - 2) \Rightarrow y = -x + 4$



67. (a) $f(x) = \tan x$, $\left(\frac{\pi}{4}, 1\right)$
 $f'(x) = \sec^2 x$
 $f'\left(\frac{\pi}{4}\right) = 2 = \text{slope at } \left(\frac{\pi}{4}, 1\right).$
Tangent line:



$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$

69. $f(x) = \frac{x^2}{x-1}$
 $f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$
 $= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$
 $f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

71. $f'(x) = \frac{(x+2)3 - 3x(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$
 $g'(x) = \frac{(x+2)5 - (5x+4)(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$
 $g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$
 f and g differ by a constant.

73. $f(x) = x^n \sin x$
 $f'(x) = x^n \cos x + nx^{n-1} \sin x$
 $= x^{n-1}(x \cos x + n \sin x)$
When $n = 1$: $f'(x) = x \cos x + \sin x$.
When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.
When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.
When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.
For general n , $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

75. Area = $A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$
 $A'(t) = 2\left(\frac{3}{2}t^{1/2}\right) + \frac{1}{2}t^{-1/2}$
 $= 3t^{1/2} + \frac{1}{2}t^{-1/2}$
 $= \frac{6t + 1}{2\sqrt{t}} \text{ cm}^2/\text{sec}$

$$77. C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad 1 \leq x$$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

$$(a) \text{ When } x = 10: \frac{dC}{dx} = -\$38.13.$$

$$(b) \text{ When } x = 15: \frac{dC}{dx} = -\$10.37.$$

$$(c) \text{ When } x = 20: \frac{dC}{dx} = -\$3.80.$$

As the order size increases, the cost per item decreases.

$$79. P(t) = 500\left[1 + \frac{4t}{50 + t^2}\right]$$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right]$$

$$= 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right]$$

$$= 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$$P'(2) \approx 31.55 \text{ bacteria per hour}$$

$$81. (a) \quad \sec x = \frac{1}{\cos x}$$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$(b) \quad \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$(c) \quad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$83. f(x) = 4x^{3/2}$$

$$f'(x) = 6x^{1/2}$$

$$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$$

$$85. f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

$$87. f(x) = 3 \sin x$$

$$f'(x) = 3 \cos x$$

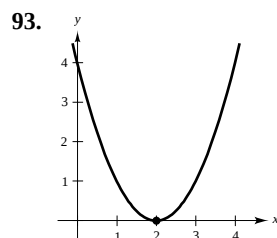
$$f''(x) = -3 \sin x$$

$$89. f'(x) = x^2$$

$$f''(x) = 2x$$

$$91. f'''(x) = 2\sqrt{x}$$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$



$$f(2) = 0$$

One such function is $f(x) = (x-2)^2$.

$$95. f(x) = 2g(x) + h(x)$$

$$f'(x) = 2g'(x) + h'(x)$$

$$f'(2) = 2g'(2) + h'(2)$$

$$= 2(-2) + 4$$

$$= 0$$

$$97. f(x) = \frac{g(x)}{h(x)}$$

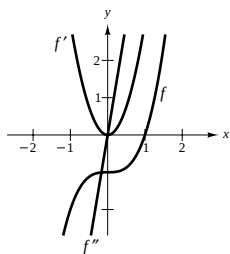
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

99.



It appears that f is cubic; so f' would be quadratic and f'' would be linear.

$$101. v(t) = 36 - t^2, \quad 0 \leq t \leq 6$$

$$a(t) = -2t$$

$$v(3) = 27 \text{ m/sec}$$

$$a(3) = -6 \text{ m/sec}$$

The speed of the object is decreasing.

$$103. v(t) = \frac{100t}{2t + 15}$$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2}$$

$$= \frac{1500}{(2t + 15)^2}$$

$$(a) \quad a(5) = \frac{1500}{[2(5) + 15]^2} = 2.4 \text{ ft/sec}^2$$

$$(b) \quad a(10) = \frac{1500}{[2(10) + 15]^2} \approx 1.2 \text{ ft/sec}^2$$

$$(c) \quad a(20) = \frac{1500}{[2(20) + 15]^2} \approx 0.5 \text{ ft/sec}^2$$

$$105. f(x) = g(x)h(x)$$

$$(a) \quad f'(x) = g(x)h'(x) + h(x)g'(x)$$

$$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$$

$$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$$

$$f'''(x) = g(x)h'''(x) + g'(x)h''(x) + 2g''(x)h'(x) + 2g'(x)h''(x) + h(x)g'''(x) + h'(x)g''(x)$$

$$= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$

$$f^{(4)}(x) = g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)$$

$$+ g^{(4)}(x)h(x) + g^{(4)}(x)h(x)$$

$$= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$$

$$(b) \quad f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{1[(n-1)(n-2) \cdots (2)(1)]} g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{(2)(1)[(n-2)(n-3) \cdots (2)(1)]} g''(x)h^{(n-2)}(x)$$

$$+ \frac{n(n-1)(n-2) \cdots (2)(1)}{(3)(2)(1)[(n-3)(n-4) \cdots (2)(1)]} g'''(x)h^{(n-3)}(x) + \cdots$$

$$+ \frac{n(n-1)(n-2) \cdots (2)(1)}{[(n-1)(n-2) \cdots (2)(1)](1)} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!} g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!} g''(x)h^{(n-2)}(x) + \cdots$$

$$+ \frac{n!}{(n-1)!1!} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

Note: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ (read “ n factorial.”)

$$107. f(x) = \cos x \quad f\left(\frac{\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$f'(x) = -\sin x \quad f'\left(\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

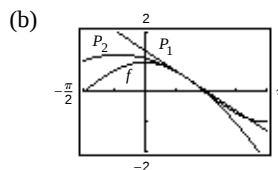
$$f''(x) = -\cos x \quad f''\left(\frac{\pi}{3}\right) = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

$$(a) P_1(x) = f'(a)(x - a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

$$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$$

$$= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

(c) P_2 is a better approximation.



(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

109. False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

111. True

$$\begin{aligned} h'(c) &= f(c)g'(c) + g(c)f'(c) \\ &= f(c)(0) + g(c)(0) \\ &= 0 \end{aligned}$$

113. True

$$115. f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

$f''(0)$ does not exist since the left and right derivatives are not equal.

Section 2.4 The Chain Rule

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

$$1. y = (6x - 5)^4$$

$$u = 6x - 5$$

$$y = u^4$$

$$3. y = \sqrt{x^2 - 1}$$

$$u = x^2 - 1$$

$$y = \sqrt{u}$$

$$5. y = \csc^3 x$$

$$u = \csc x$$

$$y = u^3$$

$$7. y = (2x - 7)^3$$

$$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$$

$$9. g(x) = 3(4 - 9x)^4$$

$$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$$

$$11. f(x) = (9 - x^2)^{2/3}$$

$$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}}$$

$$13. f(t) = (1 - t)^{1/2}$$

$$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$$

15. $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

19. $y = (x - 2)^{-1}$

$$y' = -1(2 - x)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

23. $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

27. $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$\begin{aligned} y' &= x \left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x) \right] + (1 - x^2)^{1/2}(1) \\ &= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2} \\ &= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)] \\ &= \frac{1 - 2x^2}{\sqrt{1 - x^2}} \end{aligned}$$

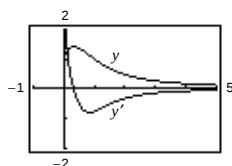
31. $g(x) = \left(\frac{x + 5}{x^2 + 2} \right)^2$

$$\begin{aligned} g'(x) &= 2 \left(\frac{x + 5}{x^2 + 2} \right) \left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2} \right) \\ &= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3} \end{aligned}$$

35. $y = \frac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



17. $y = 2(4 - x^2)^{1/4}$

$$\begin{aligned} y' &= 2 \left(\frac{1}{4} \right) (4 - x^2)^{-3/4} (-2x) \\ &= \frac{-x}{\sqrt[4]{(4 - x^2)^3}} \end{aligned}$$

21. $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

25. $f(x) = x^2(x - 2)^4$

$$\begin{aligned} f'(x) &= x^2[4(x - 2)^3(1)] + (x - 2)^4(2x) \\ &= 2x(x - 2)^3[2x + (x - 2)] \\ &= 2x(x - 2)^3(3x - 2) \end{aligned}$$

29. $y = \frac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$\begin{aligned} y' &= x \left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2}(1) \\ &= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2} \\ &= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)] \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

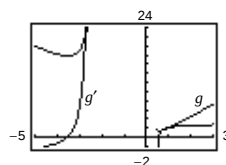
33. $f(v) = \left(\frac{1 - 2v}{1 + v} \right)^3$

$$\begin{aligned} f'(v) &= 3 \left(\frac{1 - 2v}{1 + v} \right)^2 \left(\frac{(1 + v)(-2) - (1 - 2v)}{(1 + v)^2} \right) \\ &= \frac{-9(1 - 2v)^2}{(1 + v)^4} \end{aligned}$$

37. $g(t) = \frac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

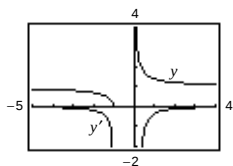
The zeros of g' correspond to the points on the graph of g where the tangent lines are horizontal.



$$39. y = \sqrt{\frac{x+1}{x}}$$

$$y' = -\frac{\sqrt{(x+1)/x}}{2x(x+1)}$$

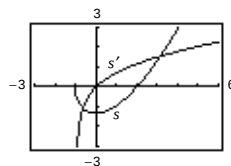
y' has no zeros.



$$41. s(t) = \frac{-2(2-t)\sqrt{1+t}}{3}$$

$$s'(t) = \frac{t}{\sqrt{1+t}}$$

The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.

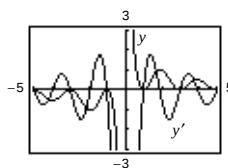


$$43. y = \frac{\cos \pi x + 1}{x}$$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.



$$45. (a) y = \sin x$$

$$y' = \cos x$$

$$y'(0) = 1$$

$$1 \text{ cycle in } [0, 2\pi]$$

$$(b) y = \sin 2x$$

$$y' = 2 \cos 2x$$

$$y'(0) = 2$$

$$2 \text{ cycles in } [0, 2\pi]$$

The slope of $\sin ax$ at the origin is a .

$$47. y = \cos 3x$$

$$\frac{dy}{dx} = -3 \sin 3x$$

$$49. g(x) = 3 \tan 4x$$

$$g'(x) = 12 \sec^2 4x$$

$$51. y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$$

$$y' = \cos(\pi x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$$

$$53. h(x) = \sin 2x \cos 2x$$

$$h'(x) = \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x)$$

$$= 2 \cos^2 2x - 2 \sin^2 2x$$

$$= 2 \cos 4x.$$

Alternate solution: $h(x) = \frac{1}{2} \sin 4x$

$$h'(x) = \frac{1}{2} \cos 4x(4) = 2 \cos 4x$$

$$55. f(x) = \frac{\cot x}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2 \cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x}$$

57. $y = 4 \sec^2 x$

$$y' = 8 \sec x \cdot \sec x \tan x = 8 \sec^2 x \tan x$$

61. $f(x) = 3 \sec^2(\pi t - 1)$

$$\begin{aligned} f'(t) &= 6 \sec(\pi t - 1) \sec(\pi t - 1) \tan(\pi t - 1)(\pi) \\ &= 6\pi \sec^2(\pi t - 1) \tan(\pi t - 1) = \frac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)} \end{aligned}$$

65. $y = \sin(\cos x)$

$$\begin{aligned} \frac{dy}{dx} &= \cos(\cos x) \cdot (-\sin x) \\ &= -\sin x \cos(\cos x) \end{aligned}$$

69. $f(x) = \frac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \left(-1, -\frac{3}{5}\right)$

$$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\frac{9x^2}{(x^3 - 4)^2}$$

$$f'(-1) = -\frac{9}{25}$$

73. $y = 37 - \sec^3(2x), (0, 36)$

$$y' = -3 \sec^2(2x)[2 \sec(2x) \tan(2x)]$$

$$= -6 \sec^3(2x) \tan(2x)$$

$$y'(0) = 0$$

75. (a) $f(x) = \sqrt{3x^2 - 2}, (3, 5)$

$$f'(x) = \frac{1}{2}(3x^2 - 2)^{-1/2}(6x)$$

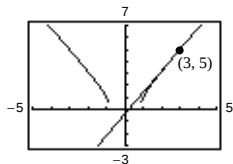
$$= \frac{3x}{\sqrt{3x^2 - 2}}$$

$$f'(3) = \frac{9}{5}$$

Tangent line:

$$y - 5 = \frac{9}{5}(x - 3) \Rightarrow 9x - 5y - 2 = 0$$

(b)



59. $f(\theta) = \frac{1}{4} \sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$$\begin{aligned} f'(\theta) &= 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2) \\ &= \sin 2\theta \cos 2\theta = \frac{1}{2} \sin 4\theta \end{aligned}$$

63. $y = \sqrt{x} + \frac{1}{4} \sin(2x)^2$

$$= \sqrt{x} + \frac{1}{4} \sin(4x^2)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{1}{4} \cos(4x^2)(8x)$$

$$= \frac{1}{2\sqrt{x}} + 2x \cos(2x)^2$$

67. $s(t) = (t^2 + 2t + 8)^{1/2}, (2, 4)$

$$s'(t) = \frac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$$

$$= \frac{t + 1}{\sqrt{t^2 + 2t + 8}}$$

$$s'(2) = \frac{3}{4}$$

71. $f(t) = \frac{3t + 2}{t - 1}, (0, -2)$

$$f'(t) = \frac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \frac{-5}{(t - 1)^2}$$

$$f'(0) = -5$$

77. (a) $f(x) = \sin 2x, (\pi, 0)$

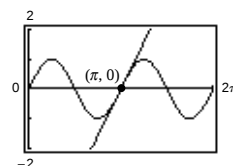
$$f'(x) = 2 \cos 2x$$

$$f'(\pi) = 2$$

Tangent line:

$$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$$

(b)



79. $f(x) = 2(x^2 - 1)^3$

$$f'(x) = 6(x^2 - 1)^2(2x)$$

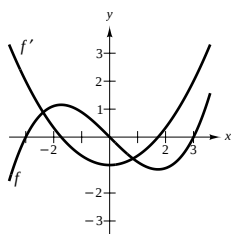
$$= 12x(x^4 - 2x^2 + 1)$$

$$= 12x^5 - 24x^3 + 12x$$

$$f''(x) = 60x^4 - 72x^2 + 12$$

$$= 12(5x^2 - 1)(x^2 - 1)$$

83.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

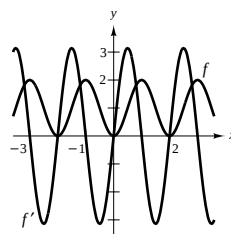
81. $f(x) = \sin x^2$

$$f'(x) = 2x \cos x^2$$

$$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2$$

$$= 2[\cos x^2 - 2x^2 \sin x^2]$$

85.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

87. $g(x) = f(3x)$

$$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$$

89. (a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$

$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(c) $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

91. (a) $f = 132,400(331 - v)^{-1}$

$$f' = (-1)(132,400)(331 - v)^{-2}(-1)$$

$$= \frac{132,400}{(331 - v)^2}$$

$$\text{When } v = 30, f' \approx 1.461.$$

(b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$

$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need $g'(3)$ to find $f'(5)$.

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2g'(x)$$

$$f'(5) = 3(-3)^2(6) = 162$$

(b) $f = 132,400(331 + v)^{-1}$

$$f' = (-1)(132,400)(331 + v)^{-2}(1)$$

$$= \frac{-132,400}{(331 + v)^2}$$

$$\text{When } v = 30, f' \approx -1.016.$$

93. $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \leq \cos 8t \leq 1$).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ radians per second.

95. $S = C(R^2 - r^2)$

$$\frac{dS}{dt} = C\left(2R \frac{dR}{dt} - 2r \frac{dr}{dt}\right)$$

Since r is constant, we have $dr/dt = 0$ and

$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$

$$= 4.224 \times 10^{-2} = 0.04224.$$

97. (a) $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161$

(b) $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function $\frac{dC}{dt}$ is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

99. $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$$f''(x) = -\beta^2 \sin \beta x$$

$$f'''(x) = -\beta^3 \cos \beta x$$

$$f^{(4)}(x) = \beta^4 \sin \beta x$$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2 (\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$$

101. (a) $r'(x) = f'(g(x))g'(x)$

$$r'(1) = f'(g(1))g'(1)$$

Note that $g(1) = 4$ and $f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$

Also, $g'(1) = 0$. Thus, $r'(1) = 0$

(b) $s'(x) = g'(f(x))f'(x)$

$$s'(4) = g'(f(4))f'(4)$$

Note that $f(4) = \frac{5}{2}$, $g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2}$ and

$$f'(4) = \frac{5}{4}.$$

Thus, $s'(4) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}.$

103. $g = \sqrt{x(x+n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x+n)}}$$

$$= \frac{[x + (x + n)]/2}{\sqrt{x(x+n)}}$$

$$= \frac{a}{g}$$

105. $g(x) = |2x - 3|$

$$g'(x) = 2\left(\frac{2x - 3}{|2x - 3|}\right), \quad x \neq \frac{3}{2}$$

107. $h(x) = |x|\cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|} \cos x, \quad x \neq 0$$

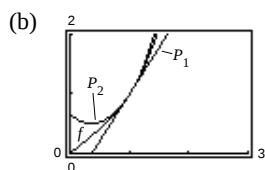
109. (a) $f(x) = \tan \frac{\pi x}{4}$ $f(1) = 1$

$$f'(x) = \frac{\pi}{4} \sec^2 \frac{\pi x}{4} \quad f'(1) = \frac{\pi}{4}(2) = \frac{\pi}{2}$$

$$f''(x) = \frac{\pi}{2} \sec^2 \frac{\pi x}{4} \cdot \tan \frac{\pi x}{4} \left(\frac{\pi}{4} \right) \quad f''(1) = \frac{\pi}{8}(2)(1) = \frac{\pi}{4}$$

$$P_1(x) = f'(1)(x - 1) + f(1) = \frac{\pi}{2}(x - 1) + 1.$$

$$P_2(x) = \frac{1}{2} \left(\frac{\pi}{4} \right) (x - 1)^2 + f'(1)(x - 1) + f(1) = \frac{\pi}{8}(x - 1)^2 + \frac{\pi}{2}(x - 1) + 1$$



(c) P_2 is a better approximation than P_1

(d) The accuracy worsens as you move away from $x = c = 1$.

111. False. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$.

113. True

Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

3. $x^{1/2} + y^{1/2} = 9$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$$

5. $x^3 - xy + y^2 = 4$

$$3x^2 - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 3x^2$$

$$y' = \frac{y - 3x^2}{2y - x}$$

7. $x^3y^3 - y - x = 0$

$$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$$

$$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$$

$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

9. $x^3 - 3x^2 + 2xy^2 = 12$

$$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$$

$$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$$

$$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$$

11. $\sin x + 2\cos 2y = 1$

$$\cos x - 4(\sin 2y)y' = 0$$

$$y' = \frac{\cos x}{4 \sin 2y}$$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

17. (a) $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm \sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{1}{2}(16 - x^2)^{-1/2}(-2x) \\ &= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm \sqrt{16 - x^2}} = \frac{-x}{y} \end{aligned}$$

19. (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm \frac{3}{4}\sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{3}{8}(16 - x^2)^{-1/2}(-2x) \\ &= \mp \frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y} \end{aligned}$$

21. $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

$$\text{At } (-4, -1): y' = -\frac{1}{4}$$

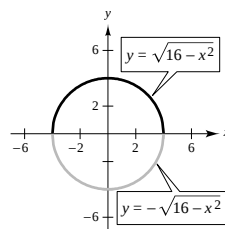
15. $y = \sin(xy)$

$$y' = [xy' + y] \cos(xy)$$

$$y' - x \cos(xy)y' = y \cos(xy)$$

$$y' = \frac{y \cos(xy)}{1 - x \cos(xy)}$$

(b)

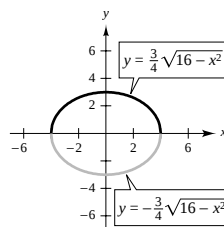


(d) Implicitly:

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

(b)



(d) Implicitly:

$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

23. $y^2 = \frac{x^2 - 4}{x^2 + 4}$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At $(2, 0)$, y' is undefined.

25. $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At (8, 1): $y' = -\frac{1}{2}$.

29. $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$\begin{aligned} y' &= \frac{-2xy}{x^2 + 4} \\ &= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4} \\ &= \frac{-16x}{(x^2 + 4)^2} \end{aligned}$$

At (2, 1): $y' = \frac{-32}{64} = -\frac{1}{2}$

(Or, you could just solve for y : $y = \frac{8}{x^2 + 4}$)

33. $\tan y = x$

$$y' \sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

37. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

27. $\tan(x + y) = x$

$$(1 + y') \sec^2(x + y) = 1$$

$$\begin{aligned} y' &= \frac{1 - \sec^2(x + y)}{\sec^2(x + y)} \\ &= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y) \\ &= -\frac{x^2}{x^2 + 1} \end{aligned}$$

At (0, 0): $y' = 0$.

31. $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At (1, 1): $y' = 0$.

35. $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

39. $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

41. $\sqrt{x} + \sqrt{y} = 4$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

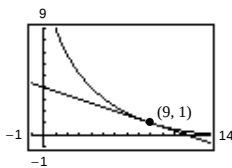
$$y' = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\text{At } (9, 1), y' = -\frac{1}{3}$$

$$\text{Tangent line: } y - 1 = -\frac{1}{3}(x - 9)$$

$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$



43. $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

At (4, 3):

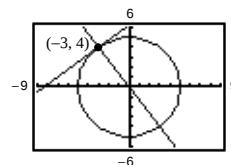
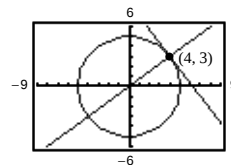
$$\text{Tangent line: } y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$$

$$\text{Normal line: } y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0.$$

At (-3, 4):

$$\text{Tangent line: } y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$$

$$\text{Normal line: } y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0.$$



45. $x^2 + y^2 = r^2$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$

 Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

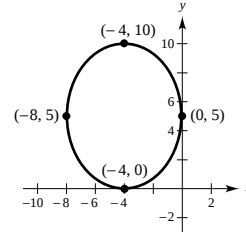
$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

 Horizontal tangents: $(-4, 0), (-4, 10)$.

 Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

 Vertical tangents: $(0, 5), (-8, 5)$.


49. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$$2x^2 + 4x = 6 \quad \text{and} \quad (x + 3)(x - 1) = 0$$

 The curves intersect at $(1, \pm 2)$.

Ellipse:

$$4x + 2yy' = 0$$

$$y' = -\frac{2x}{y}$$

 At $(1, 2)$, the slopes are:

$$y' = -1$$

 At $(1, -2)$, the slopes are:

$$y' = 1$$

Parabola:

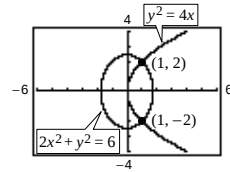
$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y' = 1$$

$$y' = -1$$

Tangents are perpendicular.



51. $y = -x$ and $x = \sin y$

 Point of intersection: $(0, 0)$

$$y = -x:$$

$$y' = -1$$

$$x = \sin y:$$

$$1 = y' \cos y$$

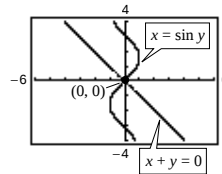
$$y' = \sec y$$

 At $(0, 0)$, the slopes are:

$$y' = -1$$

$$y' = 1$$

Tangents are perpendicular.

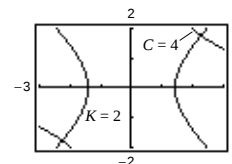
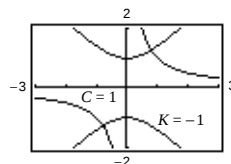


53. $xy = C$ $x^2 - y^2 = K$

$$xy' + y = 0 \quad 2x - 2yy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = \frac{x}{y}$$

 At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.


55. $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$$4yy' = 12x^3$$

$$y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$$

(b) $4y \frac{dy}{dt} - 12x^3 \frac{dx}{dt} = 0$

$$y \frac{dy}{dt} = 3x^3 \frac{dx}{dt}$$

57. $\cos \pi y - 3 \sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b) $-\pi \sin(\pi y) \frac{dy}{dt} - 3\pi \cos(\pi x) \frac{dx}{dt} = 0$

$$-\sin(\pi y) \frac{dy}{dt} = 3 \cos(\pi x) \frac{dx}{dt}$$

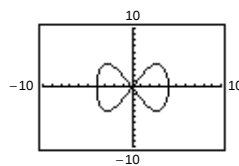
59. A function is in explicit form if y is written as a function of x : $y = f(x)$. For example, $y = x^3$. An implicit equation is not in the form $y = f(x)$. For example, $x^2 + y^2 = 5$.

61. (a) $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm \sqrt{4x^2 - \frac{1}{4}x^4}$$



(b) $y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$$36 = 16x^2 - x^4$$

$$x^4 - 16x^2 + 36 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$.

Hence, there are four values of x :

$$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \frac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} + 7)$, and the line is

$$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23].$$

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} - 7)$, and the line is

$$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}].$$

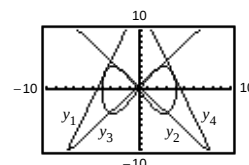
For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} - 7)$, and the line is

$$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})].$$

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} + 7)$, and the line is

$$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)].$$

—CONTINUED—



61. —CONTINUED—

(c) Equating y_3 and y_4 ,

$$\begin{aligned}
 -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 &= -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 \\
 (\sqrt{7} - 7)(x + 1 - \sqrt{7}) &= (\sqrt{7} + 7)(x - 1 - \sqrt{7}) \\
 \sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} &= \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7} \\
 16\sqrt{7} &= 14x \\
 x &= \frac{8\sqrt{7}}{7}
 \end{aligned}$$

If $x = \frac{8\sqrt{7}}{7}$, then $y = 5$ and the lines intersect at $\left(\frac{8\sqrt{7}}{7}, 5\right)$.

63. Let $f(x) = x^n = x^{p/q}$, where p and q are nonzero integers and $q > 0$. First consider the case where $p = 1$. The derivative of $f(x) = x^{1/q}$ is given by

$$\frac{d}{dx}[x^{1/q}] = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$$

where $t = x + \Delta x$. Observe that

$$\begin{aligned}
 \frac{f(t) - f(x)}{t - x} &= \frac{t^{1/q} - x^{1/q}}{t - x} = \frac{t^{1/q} - x^{1/q}}{(t^{1/q})^q - (x^{1/q})^q} \\
 &= \frac{t^{1/q} - x^{1/q}}{(t^{1/q} - x^{1/q})(t^{1-(1/q)} + t^{1-(2/q)}x^{1/q} + \cdots + t^{1/q}x^{1-(2/q)} + x^{1-(1/q)})} \\
 &= \frac{1}{t^{1-(1/q)} + t^{1-(2/q)}x^{1/q} + \cdots + t^{1/q}x^{1-(2/q)} + x^{1-(1/q)}}.
 \end{aligned}$$

As $t \rightarrow x$, the denominator approaches $qx^{1-(1/q)}$. That is,

$$\frac{d}{dx}[x^{1/q}] = \frac{1}{qx^{1-(1/q)}} = \frac{1}{q}x^{(1/q)-1}.$$

Now consider $f(x) = x^{p/q} = (x^p)^{1/q}$. From the Chain Rule,

$$f'(x) = \frac{1}{q}(x^p)^{(1/q)-1} \frac{d}{dx}[x^p] = \frac{1}{q}(x^p)^{(1/q)-1} p x^{p-1} = \frac{p}{q} x^{[(p/q)-p] + (p-1)} = \frac{p}{q} x^{(p/q)-1} = n x^{n-1} \left(n = \frac{p}{q}\right).$$

Section 2.6 Related Rates

1. $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt}$$

(a) When $x = 4$ and $dx/dt = 3$,

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}.$$

(b) When $x = 25$ and $dy/dt = 2$,

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20.$$

3. $xy = 4$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right) \frac{dy}{dt}$$

(a) When $x = 8$, $y = 1/2$, and $dx/dt = 10$,

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}.$$

(b) When $x = 1$, $y = 4$, and $dy/dt = -6$,

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}.$$

5. $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

9. (a) $\frac{dx}{dt}$ negative $\Rightarrow \frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\Rightarrow \frac{dx}{dt}$ negative

13. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x) \frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}} \frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

15. $A = \pi r^2$

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2/\text{min.}$$

(b) When $r = 24$,

$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2/\text{min.}$$

7. $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x \frac{dx}{dt}$$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = (\sqrt{2})^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

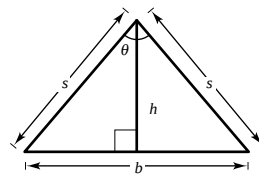
11. Yes, y changes at a constant rate: $\frac{dy}{dt} = a \cdot \frac{dx}{dt}$.

No, the rate $\frac{dy}{dt}$ is a multiple of $\frac{dx}{dt}$.

17. (a) $\sin \frac{\theta}{2} = \frac{(1/2)b}{s} \Rightarrow b = 2s \sin \frac{\theta}{2}$

$$\cos \frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s \cos \frac{\theta}{2}$$

$$\begin{aligned} A &= \frac{1}{2}bh = \frac{1}{2}\left(2s \sin \frac{\theta}{2}\right)\left(s \cos \frac{\theta}{2}\right) \\ &= \frac{s^2}{2}\left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right) = \frac{s^2}{2} \sin \theta \end{aligned}$$



(b) $\frac{dA}{dt} = \frac{s^2}{2} \cos \theta \frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2} \text{ rad/min.}$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{s^2}{8}$$

(c) If $d\theta/dt$ is constant, dA/dt is proportional to $\cos \theta$.

19. $V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \left(\frac{dV}{dt} \right) = \frac{1}{4\pi r^2} (800)$$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2} (800) = \frac{2}{9\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2} (800) = \frac{1}{18\pi}$ cm/min.

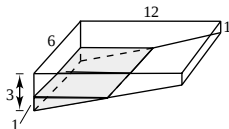
23. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{9}{4}h^2 \right) h$ [since $2r = 3h$]
 $= \frac{3\pi}{4}h^3$

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.

25.



(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$

(see similar triangle diagram)

% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Since for $0 \leq h \leq 2$, $b = 6h$, you have

$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h \frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

21. $s = 6x^2$

$$\frac{dx}{dt} = 3$$

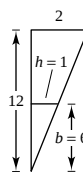
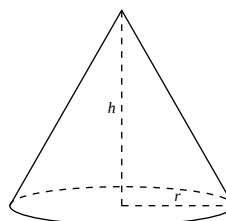
$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec.}$$

(b) When $x = 10$,

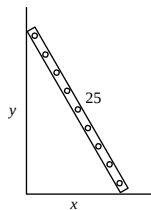
$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec.}$$



27. $x^2 + y^2 = 25^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$



(a) When $x = 7$, $y = \sqrt{576} = 24$, $\frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$ ft/sec.

When $x = 15$, $y = \sqrt{400} = 20$, $\frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$ ft/sec.

When $x = 24$, $y = 7$, $\frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$ ft/sec.

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right)$$

From part (a) we have $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$,

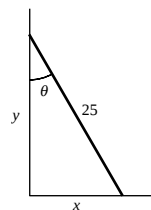
and $\frac{dy}{dt} = -\frac{7}{12}$.

$$\begin{aligned} \text{Thus, } \frac{dA}{dt} &= \frac{1}{2} \left[7 \left(-\frac{7}{12} \right) + 24(2) \right] \\ &= \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec}. \end{aligned}$$

(c) $\tan \theta = \frac{x}{y}$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left[\frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt} \right]$$



Using $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -\frac{7}{12}$ and $\cos \theta = \frac{24}{25}$, we have $\frac{d\theta}{dt} = \left(\frac{24}{25} \right)^2 \left[\frac{1}{24}(2) - \frac{7}{(24)^2} \left(-\frac{7}{12} \right) \right] = \frac{1}{12}$ rad/sec.

29. When $y = 6$, $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and

$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108 + 36} = 12.$$

$$x^2 + (12 - y)^2 = s^2$$

$$2x \frac{dx}{dt} + 2(12 - y)(-1) \frac{dy}{dt} = 2s \frac{ds}{dt}$$

$$x \frac{dx}{dt} + (y - 12) \frac{dy}{dt} = s \frac{ds}{dt}$$

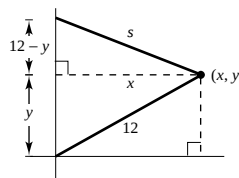
Also, $x^2 + y^2 = 12^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}.$$

Thus, $x \frac{dx}{dt} + (y - 12) \left(\frac{-x}{y} \frac{dx}{dt} \right) = s \frac{ds}{dt}$

$$\frac{dx}{dt} \left[x - x + \frac{12x}{y} \right] = s \frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})} (-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical).}$$



31. (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

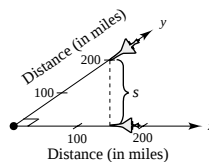
$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$

When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

(b) $t = \frac{250}{750} = \frac{1}{3} \text{ hr} = 20 \text{ min}$



33. $s^2 = 90^2 + x^2$

$$x = 30$$

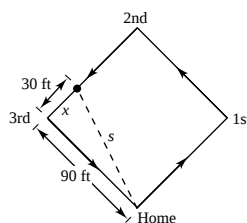
$$\frac{dx}{dt} = -28$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When $x = 30$,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$



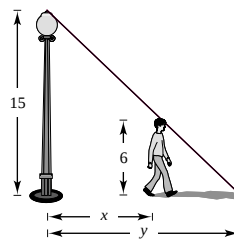
35. (a) $\frac{15}{6} = \frac{y}{y-x} \Rightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$

(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3} \text{ ft/sec}$



37. $x(t) = \frac{1}{2} \sin \frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2} \left(\frac{\pi}{6}\right) \cos \frac{\pi t}{6} = \frac{\pi}{12} \cos \frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus,

$$\begin{aligned} \frac{dy}{dt} &= -\frac{1/4}{\sqrt{15}/4} \cdot \frac{\pi}{12} \cos\left(\frac{\pi}{6}\right) \\ &= \frac{-\pi}{\sqrt{15}} \left(\frac{1}{12}\right) \frac{\sqrt{3}}{2} = \frac{-\pi}{24} \frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120} \end{aligned}$$

$$\text{Speed} = \left| \frac{-\sqrt{5}\pi}{120} \right| = \frac{\sqrt{5}\pi}{120} \text{ m/sec}$$

41. $pV^{1.3} = k$

$$1.3 pV^{0.3} \frac{dV}{dt} + V^{1.3} \frac{dp}{dt} = 0$$

$$V^{0.3} \left(1.3p \frac{dV}{dt} + V \frac{dp}{dt} \right) = 0$$

$$1.3p \frac{dV}{dt} = -V \frac{dp}{dt}$$

43. $\tan \theta = \frac{y}{30}$

$$\frac{dy}{dt} = 3 \text{ m/sec.}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{30} \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{30} \cos^2 \theta \cdot \frac{dy}{dt}$$

When $y = 30, \theta = \pi/4$ and $\cos \theta = \sqrt{2}/2$. Thus,

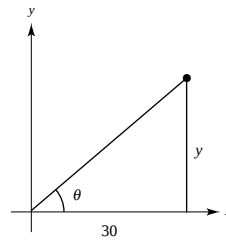
$$\frac{d\theta}{dt} = \frac{1}{30} \left(\frac{1}{2}\right) (3) = \frac{1}{20} \text{ rad/sec.}$$

39. Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

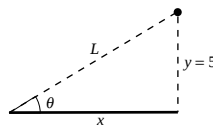
Therefore,

$$k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}.$$



45. $\tan \theta = \frac{y}{x}, y = 5$

$$\frac{dx}{dt} = -600 \text{ mi/hr}$$



$$(\sec^2 \theta) \frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left(-\frac{5}{x^2} \right) \frac{dx}{dt} = \frac{x^2}{L^2} \left(-\frac{5}{x^2} \right) \frac{dx}{dt}$$

$$= \left(-\frac{5^2}{L^2} \right) \left(\frac{1}{5} \right) \frac{dx}{dt} = (-\sin^2 \theta) \left(\frac{1}{5} \right) (-600) = 120 \sin^2 \theta$$

(a) When $\theta = 30^\circ$, $\frac{d\theta}{dt} = \frac{120}{4} = 30 \text{ rad/hr} = \frac{1}{2} \text{ rad/min}$.

(b) When $\theta = 60^\circ$, $\frac{d\theta}{dt} = 120 \left(\frac{3}{4} \right) = 90 \text{ rad/hr} = \frac{3}{2} \text{ rad/min}$.

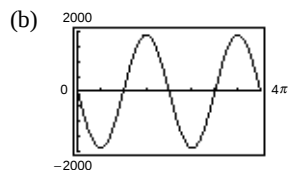
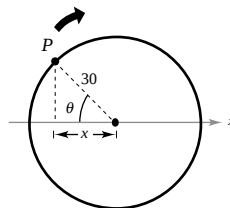
(c) When $\theta = 75^\circ$, $\frac{d\theta}{dt} = 120 \sin^2 75^\circ \approx 111.96 \text{ rad/hr} \approx 1.87 \text{ rad/min}$.

47. $\frac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

(a) $\cos \theta = \frac{x}{30}$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{30} \frac{dx}{dt}$$

$$\frac{dx}{dt} = -30 \sin \theta \frac{d\theta}{dt} = -30 \sin \theta (20\pi) = -600\pi \sin \theta$$



(c) $|dx/dt| = |-600\pi \sin \theta|$ is greatest when $\sin \theta = 1 \Rightarrow \theta = (\pi/2) + n\pi$ (or $90^\circ + n \cdot 180^\circ$)

$|dx/dt|$ is least when $\theta = n\pi$ (or $n \cdot 180^\circ$).

(d) For $\theta = 30^\circ$, $\frac{dx}{dt} = -600\pi \sin(30^\circ) = -600\pi \frac{1}{2} = -300\pi \text{ cm/sec}$.

For $\theta = 60^\circ$, $\frac{dx}{dt} = -600\pi \sin(60^\circ) = -600\pi \frac{\sqrt{3}}{2} = -300\sqrt{3} \pi \text{ cm/sec}$

49. $\tan \theta = \frac{x}{50} \Rightarrow x = 50 \tan \theta$

$$\frac{dx}{dt} = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$2 = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25} \cos^2 \theta, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \frac{d^2y}{dt^2}$

First derivative: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Second derivative: $x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[-x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When $x = 7$, $y = 24$, $\frac{dy}{dt} = -\frac{7}{12}$, and $\frac{dx}{dt} = 2$ (see Exercise 27). Since $\frac{dx}{dt}$ is constant, $\frac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[-4 - \frac{49}{144} \right] = \frac{1}{24} \left[-\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\frac{dm}{dt} = \frac{dm}{ds} \frac{ds}{dt} = (-1.762s + 29.10) \frac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\frac{ds}{dt} = 1.2$.

Thus, $\frac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

3. $f(x) = \sqrt{x} + 1$

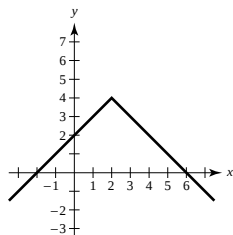
5. f is differentiable for all $x \neq -1$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.

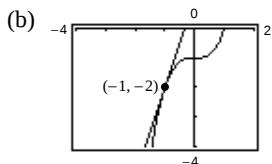


11. (a) Using the limit definition, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1$$

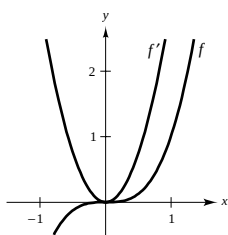


9. Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

At $x = -1$, $g'(-1) = -\frac{4}{3} - \frac{1}{6} = -\frac{3}{2}$

$$\begin{aligned} 13. \quad g'(2) &= \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^2(x - 1) - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x^2 + x + 2) = 8 \end{aligned}$$

15.



17. $y = 25$

$$y' = 0$$

19. $f(x) = x^8$

$$f'(x) = 8x^7$$

21. $h(t) = 3t^4$

$$h'(t) = 12t^3$$

23. $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$$

27. $g(t) = \frac{2}{3}t^{-2}$

$$g'(t) = \frac{-4}{3}t^{-3} = -\frac{4}{3t^3}$$

29. $f(\theta) = 2\theta - 3\sin \theta$

$$f'(\theta) = 2 - 3\cos \theta$$

31. $f(\theta) = 3\cos \theta - \frac{\sin \theta}{4}$

$$f'(\theta) = -3\sin \theta - \frac{\cos \theta}{4}$$

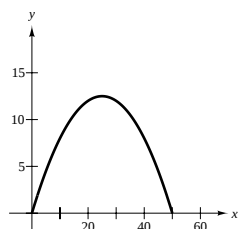
33. $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$$a(t) = v'(t) = 2$$

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

45. $f(x) = 2x - x^{-2}$

$$\begin{aligned} f'(x) &= 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right) \\ &= \frac{2(x^3 + 1)}{x^3} \end{aligned}$$

49. $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

53. $y = 3x^2 \sec x$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

35. $s(t) = -16t^2 + s_0$

$$s(9.2) = -16(9.2)^2 + s_0 = 0$$

$$s_0 = 1354.24$$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e) $y'(25) = 0$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(d) $x(t) = 0$ for $t = 1, 2$.

$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

43. $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47. $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

51. $y = \frac{x^2}{\cos x}$

$$y' = \frac{\cos x (2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

55. $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

57. $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

61. $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) = 6 \sec^2 \theta \tan \theta$$

65. $f(x) = (1 - x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{1 - x^3}}$$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$$

$$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$$

$$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$$

73. $y = \frac{1}{2} \csc 2x$

$$y' = \frac{1}{2}(-\csc 2x \cot 2x)(2)$$

$$= -\csc 2x \cot 2x$$

77. $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$$y' = \sin^{1/2} x \cos x - \sin^{5/2} x \cos x$$

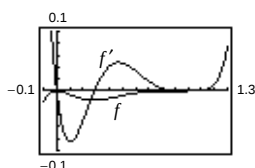
$$= (\cos x) \sqrt{\sin x}(1 - \sin^2 x)$$

$$= (\cos^3 x) \sqrt{\sin x}$$

81. $f(t) = t^2(t - 1)^5$

$$f'(t) = t(t - 1)^4(7t - 2)$$

The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



59. $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

63. $y = 2 \sin x + 3 \cos x$

$$y' = 2 \cos x - 3 \sin x$$

$$y'' = -2 \sin x - 3 \cos x$$

$$y'' + y = -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) = 0$$

67. $h(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$h'(x) = 2\left(\frac{x-3}{x^2+1}\right)\left(\frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}\right)$$

$$= \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$$

71. $y = 3 \cos(3x + 1)$

$$y' = -9 \sin(3x + 1)$$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$$y' = \frac{1}{2} - \frac{1}{4} \cos 2x(2)$$

$$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

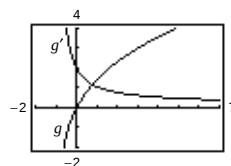
79. $y = \frac{\sin \pi x}{x+2}$

$$y' = \frac{(x+2)\pi \cos \pi x - \sin \pi x}{(x+2)^2}$$

83. $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

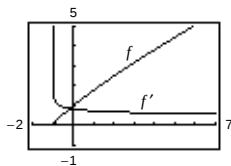
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



85. $f(t) = (t + 1)^{1/2}(t + 1)^{1/3} = (t + 1)^{5/6}$

$$f'(t) = \frac{5}{6(t + 1)^{1/6}}$$

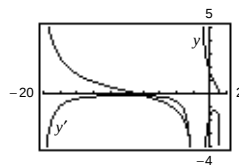
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan \sqrt{1 - x}$

$$y' = -\frac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

91. $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$\begin{aligned} f'' &= -2 \csc x (-\csc x \cdot \cot x) \\ &= 2 \csc^2 x \cot x \end{aligned}$$

93. $f(t) = \frac{t}{(1 - t)^2}$

$$f'(t) = \frac{t + 1}{(1 - t)^3}$$

$$f''(t) = \frac{2(t + 2)}{(1 - t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

97. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

101. $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103. $x \sin y = y \cos x$

$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$

$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

105. $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

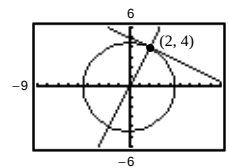
At (2, 4): $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$

$$2x - y = 0$$



107. $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

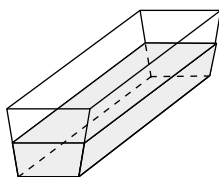
(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$



Width of water at depth h :

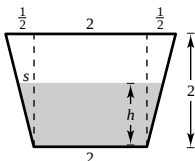
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.



111. $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

$$4.9t^2 = 25$$

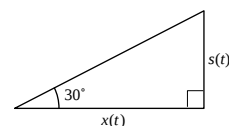
$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}}$$

$$\approx -38.34 \text{ m/sec}$$



CHAPTER 2

Differentiation

Section 2.1	The Derivative and the Tangent Line Problem	. . 330
Section 2.2	Basic Differentiation Rules and Rates of Change	338
Section 2.3	The Product and Quotient Rules and Higher-Order Derivatives	 344
Section 2.4	The Chain Rule	 350
Section 2.5	Implicit Differentiation	 356
Section 2.6	Related Rates	 361
Review Exercises		 367
Problem Solving		 373

CHAPTER 2

Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Even-Numbered Exercises

2. (a) $m = \frac{1}{4}$

(b) $m = 1$

4. (a) $\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$

$$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$$

Thus, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

6. $g(x) = \frac{3}{2}x + 1$ is a line. Slope = $\frac{3}{2}$

8. Slope at $(2, 1) = \lim_{\Delta x \rightarrow 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x}$

$$= \lim_{\Delta x \rightarrow 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (-4 - \Delta x) = -4$$

10. Slope at $(-2, 7) = \lim_{\Delta t \rightarrow 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} (-4 + \Delta t) = -4$$

12. $g(x) = -5$

$$g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-5 - (-5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

14. $f(x) = 3x + 2$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 3 = 3$$

16. $f(x) = 9 - \frac{1}{2}x$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

18. $f(x) = 1 - x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-2x - \Delta x) = -2x \end{aligned}$$

20. $f(x) = x^3 + x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x \end{aligned}$$

22. $f(x) = \frac{1}{x^2}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2x^2} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2x^2} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x - \Delta x}{(x + \Delta x)^2x^2} \\ &= \frac{-2x}{x^4} \\ &= -\frac{2}{x^3} \end{aligned}$$

24. $f(x) = \frac{4}{\sqrt{x}}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{4}{\sqrt{x + \Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x\sqrt{x}\sqrt{x + \Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{4x - 4(x + \Delta x)}{\Delta x\sqrt{x}\sqrt{x + \Delta x}(\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-4}{\sqrt{x}\sqrt{x + \Delta x}(\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \frac{-4}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{-2}{x\sqrt{x}} \end{aligned}$$

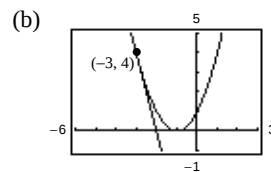
26. (a) $f(x) = x^2 + 2x + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x + 2) = 2x + 2
 \end{aligned}$$

At $(-3, 4)$, the slope of the tangent line is $m = 2(-3) + 2 = -4$.

The equation of the tangent line is

$$\begin{aligned}
 y - 4 &= -4(x + 3) \\
 y &= -4x - 8.
 \end{aligned}$$



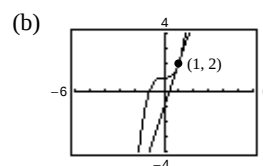
28. (a) $f(x) = x^3 + 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2
 \end{aligned}$$

At $(1, 2)$, the slope of the tangent line is $m = 3(1)^2 = 3$.

The equation of the tangent line is

$$\begin{aligned}
 y - 2 &= 3(x - 1) \\
 y &= 3x - 1.
 \end{aligned}$$



30. (a) $f(x) = \sqrt{x - 1}$

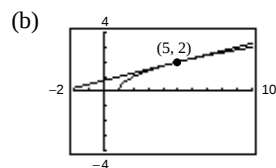
$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x(\sqrt{x + \Delta x - 1} + \sqrt{x - 1})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}}
 \end{aligned}$$

At $(5, 2)$, the slope of the tangent line is

$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$\begin{aligned}
 y - 2 &= \frac{1}{4}(x - 5) \\
 y &= \frac{1}{4}x + \frac{3}{4}
 \end{aligned}$$



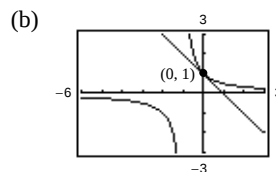
32. (a) $f(x) = \frac{1}{x+1}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + 1} - \frac{1}{x + 1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + 1) - (x + \Delta x + 1)}{\Delta x(x + \Delta x + 1)(x + 1)} \\ &= \lim_{\Delta x \rightarrow 0} -\frac{1}{(x + \Delta x + 1)(x + 1)} \\ &= -\frac{1}{(x + 1)^2} \end{aligned}$$

At $(0, 1)$, the slope of the tangent line is

$$m = \frac{-1}{(0 + 1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.



34. Using the limit definition of derivative, $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x^2 = 1 \Rightarrow x = \pm 1.$$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$$y - 3 = 3(x - 1) \text{ and } y - 1 = 3(x + 1)$$

$$y = 3x \qquad y = 3x + 4$$

36. Using the limit definition of derivative, $f'(x) = \frac{-1}{2(x-1)^{3/2}}$.

Since the slope of the given line is $-\frac{1}{2}$, we have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$1 = (x-1)^{3/2}$$

$$1 = x - 1 \Rightarrow x = 2$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2$$

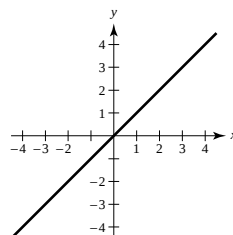
38. $h(-1) = 4$ because the tangent line passes through $(-1, 4)$

$$h'(-1) = \frac{6 - 4}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$

40. $f(x) = x^2 \Rightarrow f'(x) = 2x$ (d)

42. f' does not exist at $x = 0$. Matches (c)

44.



Answers will vary.

Sample answer: $y = x$

46. (a) Yes. $\lim_{\Delta x \rightarrow 0} \frac{f(x + 2\Delta x) - f(x)}{2\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

(b) No. The numerator does not approach zero.

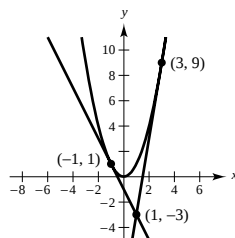
(c) Yes.
$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x) - f(x - \Delta x) + f(x)}{2\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{2\Delta x} + \frac{f(x - \Delta x) - f(x)}{2(-\Delta x)} \right] \\ &= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x) \end{aligned}$$

(d) Yes. $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

48. Let (x_0, y_0) be a point of tangency on the graph of f . By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0 :

$$\begin{aligned} \frac{-3 - y_0}{1 - x_0} &= 2x_0 \\ -3 - y_0 &= (1 - x_0)2x_0 \\ -3 - x_0^2 &= 2x_0 - 2x_0^2 \\ x_0^2 - 2x_0 - 3 &= 0 \end{aligned}$$

$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$



Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2 . The equations of the tangent lines are

$$\begin{aligned} y + 3 &= 6(x - 1) & y + 3 &= -2(x - 1) \\ y &= 6x - 9 & y &= -2x - 1 \end{aligned}$$

50. (a) $f(x) = x^2$

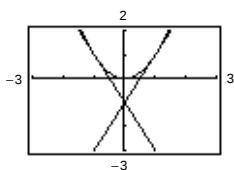
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At $x = -1$, $f'(-1) = -2$ and the tangent line is

$$y - 1 = -2(x + 1) \quad \text{or} \quad y = -2x - 1.$$

At $x = 0$, $f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $f'(1) = 2$ and the tangent line is $y = 2x - 1$.



For this function, the slopes of the tangent lines are always distinct for different values of x .

(b)
$$\begin{aligned} g'(x) &= \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2 \end{aligned}$$

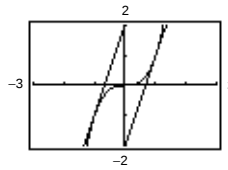
At $x = -1$, $g'(-1) = 3$ and the tangent line is

$$y + 1 = 3(x + 1) \quad \text{or} \quad y = 3x + 2.$$

At $x = 0$, $g'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $g'(1) = 3$ and the tangent line is

$$y - 1 = 3(x - 1) \quad \text{or} \quad y = 3x - 2.$$

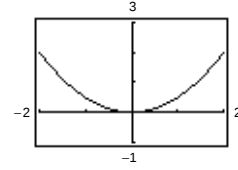


For this function, the slopes of the tangent lines are sometimes the same.

52. $f(x) = \frac{1}{2}x^2$

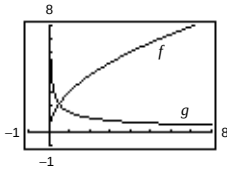
By the limit definition of the derivative we have $f'(x) = x$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2



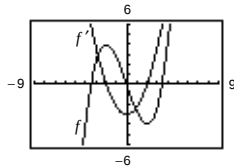
54.
$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= (3\sqrt{x + 0.01} - 3\sqrt{x})100$$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

58. $f(x) = \frac{x^3}{4} - 3x$ and $f'(x) = \frac{3}{4}x^2 - 3$



60. $f(x) = x + \frac{1}{x}$

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2) = \frac{(2 + \Delta x) + \frac{1}{2 + \Delta x} - \frac{5}{2}}{\Delta x}(x - 2) + \frac{5}{2}$$

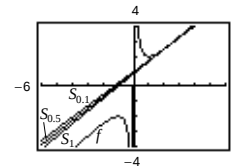
$$= \frac{2(2 + \Delta x)^2 + 2 - 5(2 + \Delta x)}{2(2 + \Delta x)\Delta x}(x - 2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2 + \Delta x)}(x - 2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \frac{5}{6}(x - 2) + \frac{5}{2} = \frac{5}{6}x + \frac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \frac{4}{5}(x - 2) + \frac{5}{2} = \frac{4}{5}x + \frac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \frac{16}{21}(x - 2) + \frac{5}{2} = \frac{16}{21}x + \frac{41}{42}$

(b) As $\Delta x \rightarrow 0$, the line approaches the tangent line to f at $(2, \frac{5}{2})$.



62. $g(x) = x(x - 1) = x^2 - x$, $c = 1$

$$g'(1) = \lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - x - 0}{x - 1} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} x = 1$$

64. $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 3)}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 3) = 5$$

66. $f(x) = \frac{1}{x}, c = 3$

$$f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x \rightarrow 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x \rightarrow 3} \left(-\frac{1}{3x} \right) = -\frac{1}{9}$$

68. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x \rightarrow -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x \rightarrow -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

70. $f(x) = |x - 4|, c = 4$

$$f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

72. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)

74. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

76. $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)

78. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

80. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

82. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x \rightarrow 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \text{ (Vertical tangent)}$$

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

84. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

86. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

The derivative from the left is

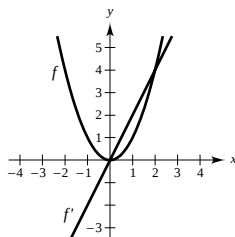
$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(\frac{1}{2}x + 1) - 2}{x - 2} = \lim_{x \rightarrow 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

The derivative from the right is

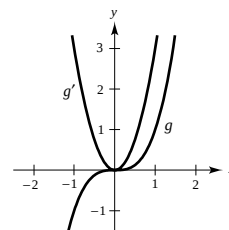
$$\begin{aligned} \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2} \\ &= \lim_{x \rightarrow 2^+} \frac{2x - 4}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2(x - 2)}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}. \end{aligned}$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. ($f'(2) = \frac{1}{2}$)

88. (a) $f(x) = x^2$ and $f'(x) = 2x$



- (b) $g(x) = x^3$ and $g'(x) = 3x^2$



- (c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

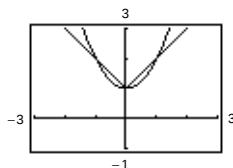
$$\begin{aligned} \text{(d) If } f(x) = x^4, \text{ then } f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3 \end{aligned}$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of n , $n \geq 2$.

90. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

92. True—see Theorem 2.1

- 94.



As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

2. (a) $y = x^{-1/2}$
 $y' = -\frac{1}{2}x^{-3/2}$
 $y'(1) = -\frac{1}{2}$
- (b) $y = x^{-1}$
 $y' = -x^{-2}$
 $y'(1) = -1$
- (c) $y = x^{-3/2}$
 $y' = -\frac{3}{2}x^{-5/2}$
 $y'(1) = -\frac{3}{2}$
- (d) $y = x^{-2}$
 $y' = -2x^{-3}$
 $y'(1) = -2$
4. $f(x) = -2$
 $f'(x) = 0$
6. $y = x^8$
 $y' = 8x^7$
8. $y = \frac{1}{x^8} = x^{-8}$
 $y' = 8x^{-9} = \frac{-8}{x^9}$
10. $y = \sqrt[4]{x} = x^{1/4}$
 $y' = \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}}$
12. $g(x) = 3x - 1$
 $g'(x) = 3$
14. $y = t^2 + 2t - 3$
 $y' = 2t + 2$
16. $y = 8 - x^3$
 $y' = -3x^2$
18. $f(x) = 2x^3 - x^2 + 3x$
 $f'(x) = 6x^2 - 2x + 3$
20. $g(t) = \pi \cos t$
 $g'(t) = -\pi \sin t$
22. $y = 5 + \sin x$
 $y' = \cos x$
24. $y = \frac{5}{(2x)^3} + 2 \cos x = \frac{5}{8}x^{-3} + 2 \cos x$
 $y' = \frac{5}{8}(-3)x^{-4} - 2 \sin x = \frac{-15}{8x^4} - 2 \sin x$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
26. $y = \frac{2}{3x^2}$	$y = \frac{2}{3}x^{-2}$	$y' = -\frac{4}{3}x^{-3}$	$y' = -\frac{4}{3x^3}$
28. $y = \frac{\pi}{(3x)^2}$	$y = \frac{\pi}{9}x^{-2}$	$y' = -\frac{2\pi}{9}x^{-3}$	$y' = -\frac{2\pi}{9x^3}$
30. $y = \frac{4}{x^{-3}}$	$y = 4x^3$	$y' = 12x^2$	$y' = 12x^2$
32. $f(t) = 3 - \frac{3}{5t}, \left(\frac{3}{5}, 2\right)$ $f'(t) = \frac{3}{5t^2}$ $f'\left(\frac{3}{5}\right) = \frac{5}{3}$	34. $y = 3x^3 - 6, (2, 18)$ $y' = 9x^2$ $y'(2) = 36$	36. $f(x) = 3(5 - x)^2, (5, 0)$ $= 3x^2 - 30x + 75$ $f'(x) = 6x - 30$ $f'(5) = 0$	
38. $g(t) = 2 + 3 \cos t, (\pi, -1)$ $g'(t) = -3 \sin t$ $g'(\pi) = 0$	40. $f(x) = x^2 - 3x - 3x^{-2}$ $f'(x) = 2x - 3 + 6x^{-3}$ $= 2x - 3 + \frac{6}{x^3}$	42. $f(x) = x + x^{-2}$ $f'(x) = 1 - 2x^{-3}$ $= 1 - \frac{2}{x^3}$	
44. $h(x) = \frac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$ $h'(x) = 2 - \frac{1}{x^2} = \frac{2x^2 - 1}{x^2}$	46. $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$ $y' = 36x - 45x^2$		
48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$ $f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5} = \frac{1}{3x^{2/3}} + \frac{1}{5x^{4/5}}$	50. $f(t) = t^{2/3} - t^{1/3} + 4$ $f'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{3}t^{-2/3} = \frac{2}{3t^{1/3}} - \frac{1}{3t^{2/3}}$		

$$52. f(x) = \frac{2}{\sqrt[3]{x}} + 3 \cos x = 2x^{-1/3} + 3 \cos x$$

$$f'(x) = \frac{-2}{3}x^{-4/3} - 3 \sin x = \frac{-2}{3x^{4/3}} - 3 \sin x$$

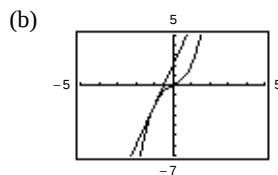
$$54. (a) y = x^3 + x$$

$$y' = 3x^2 + 1$$

$$\text{At } (-1, -2): y' = 3(-1)^2 + 1 = 4.$$

$$\text{Tangent line: } y + 2 = 4(x + 1)$$

$$4x - y + 2 = 0$$



$$56. (a) y = (x^2 + 2x)(x + 1)$$

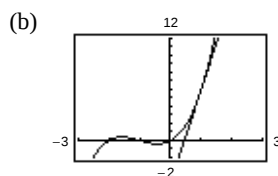
$$= x^3 + 3x^2 + 2x$$

$$y' = 3x^2 + 6x + 2$$

$$\text{At } (1, 6): y' = 3(1)^2 + 6(1) + 2 = 11.$$

$$\text{Tangent line: } y - 6 = 11(x - 1)$$

$$0 = 11x - y - 5$$



$$58. y = x^3 + x$$

$$y' = 3x^2 + 1 > 0 \text{ for all } x.$$

Therefore, there are no horizontal tangents.

$$60. y = x^2 + 1$$

$$y' = 2x = 0 \Rightarrow x = 0$$

$$\text{At } x = 0, y = 1.$$

Horizontal tangent: $(0, 1)$

$$62. y = \sqrt{3}x + 2 \cos x, 0 \leq x < 2\pi$$

$$y' = \sqrt{3} - 2 \sin x = 0$$

$$\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$$\text{At } x = \frac{\pi}{3}, y = \frac{\sqrt{3}\pi + 3}{3}.$$

$$\text{At } x = \frac{2\pi}{3}, y = \frac{2\sqrt{3}\pi - 3}{3}.$$

$$\text{Horizontal tangents: } \left(\frac{\pi}{3}, \frac{\sqrt{3}\pi + 3}{3}\right), \left(\frac{2\pi}{3}, \frac{2\sqrt{3}\pi - 3}{3}\right)$$

$$64. k - x^2 = -4x + 7 \quad \text{Equate functions}$$

$$-2x = -4 \quad \text{Equate derivatives}$$

$$\text{Hence, } x = 2 \text{ and } k - 4 = -8 + 7 \Rightarrow k = 3$$

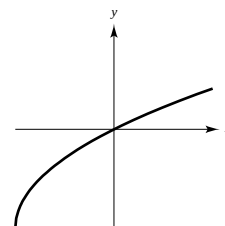
$$66. k\sqrt{x} = x + 4 \quad \text{Equate functions}$$

$$\frac{k}{2\sqrt{x}} = 1 \quad \text{Equate derivatives}$$

$$\text{Hence, } k = 2\sqrt{x} \text{ and}$$

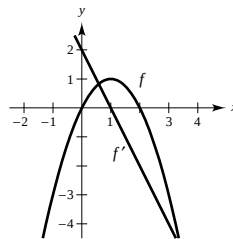
$$(2\sqrt{x})\sqrt{x} = x + 4 \Rightarrow 2x = x + 4 \Rightarrow x = 4 \Rightarrow k = 4$$

68. The graph of a function f such that $f' > 0$ for all x and the rate of change the function is decreasing (i.e. $f'' < 0$) would, in general, look like the graph at the right.



70. $g(x) = -5f(x) \Rightarrow g'(x) = -5f'(x)$

72.



If f is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

74. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Since

$$y = x \Rightarrow y' = 1 \Rightarrow m_1 = 1 \text{ and } y = \frac{1}{x} \Rightarrow y' = \frac{-1}{x^2} \Rightarrow m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

76. $f(x) = \frac{2}{x}, (5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

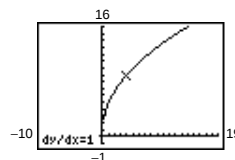
The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f . The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

78. $f'(4) = 1$

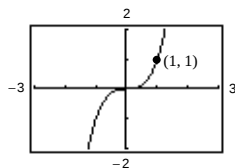


80. (a) Nearby point: (1.0073138, 1.0221024)

$$\text{Secant line: } y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$$

$$y = 3.022(x - 1) + 1$$

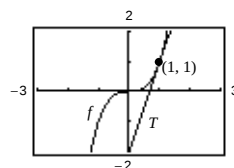
(Answers will vary.)



(b) $f'(x) = 3x^2$

$$T(x) = 3(x - 1) + 1 = 3x - 2$$

- (c) The accuracy worsens as you move away from (1, 1).



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	-8	-1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	-8	-5	-2	-0.5	0.7	1	1.3	2.5	4	7	10

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less “linear” than $y = x^{3/2}$.

82. True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

84. True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = (1/\pi)(1) = 1/\pi$.

86. False. If $f(x) = \frac{1}{x^n} = x^{-n}$, then $f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}$

88. $f(t) = t^2 - 3, [2, 2.1]$

$$f'(t) = 2t$$

Instantaneous rate of change:

$$(2, 1) \Rightarrow f'(2) = 2(2) = 4$$

$$(2.1, 1.41) \Rightarrow f'(2.1) = 4.2$$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

90. $f(x) = \sin x, \left[0, \frac{\pi}{6}\right]$

$$f'(x) = \cos x$$

Instantaneous rate of change:

$$(0, 0) \Rightarrow f'(0) = 1$$

$$\left(\frac{\pi}{6}, \frac{1}{2}\right) \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

92. $s(t) = -16t^2 - 22t + 220$

$$v(t) = -32t - 22$$

$$v(3) = -118 \text{ ft/sec}$$

$$s(t) = -16t^2 - 22t + 220$$

$$= 112 \text{ (height after falling 108 ft)}$$

$$-16t^2 - 22t + 108 = 0$$

$$-2(t - 2)(8t + 27) = 0$$

$$t = 2$$

$$v(2) = -32(2) - 22$$

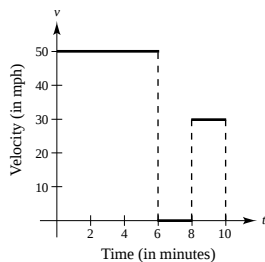
$$= -86 \text{ ft/sec}$$

94. $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + s_0 = 0 \text{ when } t = 6.8.$$

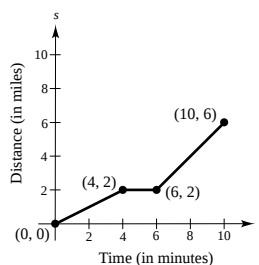
$$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6 \text{ m}$$

96.



(The velocity has been converted to miles per hour)

98. This graph corresponds with Exercise 75.



100. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

$$\begin{aligned} \text{Average velocity: } \frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} &= \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c]}{2\Delta t} \\ &= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t} \\ &= \frac{-2at_0\Delta t}{2\Delta t} \\ &= -at_0 \\ &= s'(t_0) \quad \text{Instantaneous velocity at } t = t_0 \end{aligned}$$

102. $V = s^3, \frac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\frac{dV}{ds} = 48 \text{ cm}^2$.

104. $C = (\text{gallons of fuel used})(\text{cost per gallon})$

$$= \left(\frac{15,000}{x}\right)(1.25) = \frac{18,750}{x}$$

$$\frac{dC}{dx} = -\frac{18,750}{x^2}$$

x	10	15	20	25	30	35	40
C	1875	1250	537.5	750	625	535.71	468.75
$\frac{dC}{dx}$	-187.5	-83.333	-46.875	-30	-20.833	-15.306	-11.719

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

106. $\frac{dT}{dt} = K(T - T_a)$

108. $y = \frac{1}{x}, x > 0$

$$y' = -\frac{1}{x^2}$$

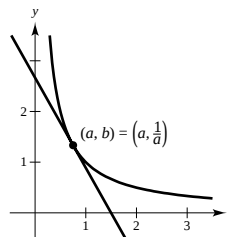
At (a, b) , the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The x -intercept is $(2a, 0)$.

The y -intercept is $(0, \frac{2}{a})$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$.



110. $y = x^2$

$$y' = 2x$$

(a) Tangent lines through $(0, a)$:

$$y - a = 2x(x - 0)$$

$$x^2 - a = 2x^2$$

$$-a = x^2$$

$$\pm \sqrt{-a} = x$$

The points of tangency are $(\pm \sqrt{-a}, -a)$. At $(\sqrt{-a}, -a)$ the slope is $y'(\sqrt{-a}) = 2\sqrt{-a}$. At $(-\sqrt{-a}, -a)$ the slope is $y'(-\sqrt{-a}) = -2\sqrt{-a}$.

$$\text{Tangent lines: } y + a = 2\sqrt{-a}(x - \sqrt{-a}) \quad \text{and} \quad y + a = -2\sqrt{-a}(x + \sqrt{-a})$$

$$y = 2\sqrt{-a}x + a$$

$$y = -2\sqrt{-a}x + a$$

Restriction: a must be negative.

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$

$$x^2 = 2x^2 - 2ax$$

$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

$$\text{Tangent lines: } y - 0 = 0(x - 0) \quad \text{and} \quad y - 4a^2 = 4a(x - 2a)$$

$$y = 0$$

$$y = 4ax - 4a^2$$

Restriction: None, a can be any real number.

112. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$$f_2(x) = \sin|x| \text{ is differentiable for all } x \neq 0.$$

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

2. $f(x) = (6x + 5)(x^3 - 2)$

$$\begin{aligned} f'(x) &= (6x + 5)(3x^2) + (x^3 - 2)(6) \\ &= 18x^3 + 15x^2 + 6x^3 - 12 \\ &= 24x^3 + 15x^2 - 12 \end{aligned}$$

6. $g(x) = \sqrt{x} \sin x$

$$g'(x) = \sqrt{x} \cos x + \sin x \left(\frac{1}{2\sqrt{x}} \right) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$$

10. $h(s) = \frac{s}{\sqrt{s} - 1}$

$$\begin{aligned} h'(s) &= \frac{(\sqrt{s} - 1)(1) - s\left(\frac{1}{2}s^{-1/2}\right)}{(\sqrt{s} - 1)^2} \\ &= \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{(\sqrt{s} - 1)^2} = \frac{\sqrt{s} - 2}{2(\sqrt{s} - 1)^2} \end{aligned}$$

14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$\begin{aligned} f'(x) &= (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2) \\ &= 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1) \\ &= (x - 1)^2(5x^2 + 2x + 2) \\ f'(1) &= 0 \end{aligned}$$

18. $f(x) = \frac{\sin x}{x}$

$$\begin{aligned} f'(x) &= \frac{(x)(\cos x) - (\sin x)(1)}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2} \\ f'\left(\frac{\pi}{6}\right) &= \frac{(\pi/6)(\sqrt{3}/2) - (1/2)}{\pi^2/36} \\ &= \frac{3\sqrt{3}\pi - 18}{\pi^2} \\ &= \frac{3(\sqrt{3}\pi - 6)}{\pi^2} \end{aligned}$$

4. $g(s) = \sqrt{s}(4 - s^2) = s^{1/2}(4 - s^2)$

$$\begin{aligned} g'(s) &= s^{1/2}(-2s) + (4 - s^2)\frac{1}{2}s^{-1/2} = -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}} \\ &= \frac{4 - 5s^2}{2s^{1/2}} \end{aligned}$$

8. $g(t) = \frac{t^2 + 2}{2t - 7}$

$$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$$

12. $f(t) = \frac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t \sin t + 3 \cos t}{t^4}$$

16. $f(x) = \frac{x + 1}{x - 1}$

$$\begin{aligned} f'(x) &= \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} \\ &= \frac{x - 1 - x - 1}{(x - 1)^2} \\ &= -\frac{2}{(x - 1)^2} \\ f'(2) &= -\frac{2}{(2 - 1)^2} = -2 \end{aligned}$$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
20. $y = \frac{5x^2 - 3}{4}$	$y = \frac{5}{4}x^2 - \frac{3}{4}$	$y' = \frac{10}{4}x$	$y' = \frac{5x}{2}$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
22. $y = \frac{4}{5x^2}$	$y = \frac{4}{5}x^{-2}$	$y' = -\frac{8}{5}x^{-3}$	$y' = -\frac{8}{5x^3}$
24. $y = \frac{3x^2 - 5}{7}$	$y = \frac{3}{7}x^2 - \frac{5}{7}$	$y' = \frac{6x}{7}$	$y' = \frac{6}{7}x$
26. $f(x) = \frac{x^3 + 3x + 2}{x^2 - 1}$ $f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2}$ $= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$		28. $f(x) = x^4 \left[1 - \frac{2}{x+1} \right] = x^4 \left[\frac{x-1}{x+1} \right]$ $f'(x) = x^4 \left[\frac{(x+1) - (x-1)}{(x+1)^2} \right] + \left[\frac{x-1}{x+1} \right] (4x^3)$ $= 2x^3 \left[\frac{2x^2 + x - 2}{(x+1)^2} \right]$	
30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3) = x^{1/3}(x^{1/2} + 3)$ $f'(x) = x^{1/3} \left(\frac{1}{2}x^{-1/2} \right) + (x^{1/2} + 3) \left(\frac{1}{3}x^{-2/3} \right)$ $= \frac{5}{6}x^{-1/6} + x^{-2/3}$ $= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$		32. $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$ $h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$	
Alternate solution:			
$f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$ $= x^{5/6} + 3x^{1/3}$ $f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$ $= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$			
34. $g(x) = x^2 \left(\frac{2}{x} - \frac{1}{x+1} \right) = 2x - \frac{x^2}{x+1}$ $g'(x) = 2 - \frac{(x+1)2x - x^2(1)}{(x+1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x+1)^2} = \frac{x^2 + 2x + 2}{(x+1)^2}$			
36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$ $f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$ $= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$ $= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$ $= 6x^5 + 4x^3 - 3x^2 - 1$			
38. $f(x) = \frac{c^2 - x^2}{c^2 + x^2}$ $f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$ $= \frac{-4xc^2}{(c^2 + x^2)^2}$		40. $f(\theta) = (\theta + 1) \cos \theta$ $f'(\theta) = (\theta + 1)(-\sin \theta) + (\cos \theta)(1)$ $= \cos \theta - (\theta + 1) \sin \theta$	

$$42. f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

$$46. h(s) = \frac{1}{s} - 10 \csc s$$

$$h'(s) = -\frac{1}{s^2} + 10 \csc s \cot s$$

$$50. y = x \sin x + \cos x$$

$$y' = x \cos x + \sin x - \sin x = x \cos x$$

$$54. h(\theta) = 5\theta \sec \theta + \theta \tan \theta$$

$$h'(\theta) = 5\theta \sec \theta \tan \theta + 5 \sec \theta + \theta \sec^2 \theta + \tan \theta$$

$$58. f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$$

$$f'(\theta) = \frac{1}{\cos \theta - 1} = \frac{\cos \theta - 1}{(1 - \cos \theta)^2}$$

(form of answer may vary)

$$62. f(x) = \sin x(\sin x + \cos x)$$

$$\begin{aligned} f'(x) &= \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x \\ &= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x \\ &= \sin 2x + \cos 2x \end{aligned}$$

$$f'\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1$$

$$64. (a) f(x) = (x-1)(x^2-2), (0, 2)$$

$$f'(x) = (x-1)(2x) + (x^2-2)(1) = 3x^2 - 2x - 2$$

$$f'(0) = -2 = \text{slope at } (0, 2).$$

$$\text{Tangent line: } y - 2 = -2x \Rightarrow y = -2x + 2$$

$$66. (a) f(x) = \frac{x-1}{x+1}, \left(2, \frac{1}{3}\right)$$

$$f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$f'(2) = \frac{2}{9} = \text{slope at } \left(2, \frac{1}{3}\right).$$

$$\text{Tangent line: } y - \frac{1}{3} = \frac{2}{9}(x-2) \Rightarrow y = \frac{2}{9}x - \frac{1}{9}$$

$$44. y = x + \cot x$$

$$y' = 1 - \csc^2 x = -\cot^2 x$$

$$48. y = \frac{\sec x}{x}$$

$$y' = \frac{x \sec x \tan x - \sec x}{x^2}$$

$$= \frac{\sec x(x \tan x - 1)}{x^2}$$

$$52. f(x) = \sin x \cos x$$

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos 2x$$

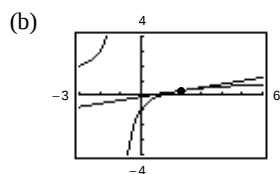
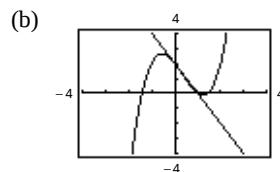
$$56. f(x) = \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$$

$$f'(x) = 2 \frac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2} \quad (\text{form of answer may vary})$$

$$60. f(x) = \tan x \cot x = 1$$

$$f'(x) = 0$$

$$f'(1) = 0$$



68. (a) $f(x) = \sec x, \left(\frac{\pi}{3}, 2\right)$

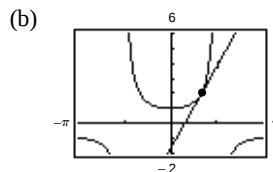
$$f'(x) = \sec x \tan x$$

$$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3} = \text{slope at } \left(\frac{\pi}{3}, 2\right).$$

Tangent line:

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$



70. $f(x) = \frac{x^2}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$$

$$f'(x) = 0 \text{ when } x = 0.$$

Horizontal tangent is at (0, 0).

72. $f'(x) = \frac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$

$$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$$

f and g differ by a constant.

74. $f(x) = \frac{\cos x}{x^n} = x^{-n} \cos x$

$$f'(x) = -x^{-n} \sin x - nx^{-n-1} \cos x$$

$$= -x^{-n-1}(x \sin x + n \cos x)$$

$$= -\frac{x \sin x + n \cos x}{x^{n+1}}$$

$$\text{When } n = 1: f'(x) = -\frac{x \sin x + \cos x}{x^2}.$$

$$\text{When } n = 2: f'(x) = -\frac{x \sin x + 2 \cos x}{x^3}.$$

$$\text{When } n = 3: f'(x) = -\frac{x \sin x + 3 \cos x}{x^4}.$$

$$\text{When } n = 4: f'(x) = -\frac{x \sin x + 4 \cos x}{x^5}.$$

$$\text{For general } n, f'(x) = -\frac{x \sin x + n \cos x}{x^{n+1}}.$$

76. $V = \pi r^2 h = \pi(t + 2)\left(\frac{1}{2}\sqrt{t}\right)$

$$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$$

$$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi \text{ cubic inches/sec}$$

78. $P = \frac{k}{V}$

$$\frac{dP}{dV} = -\frac{k}{V^2}$$

80. $f(x) = \sec x$

$$g(x) = \csc x, [0, 2\pi)$$

$$f'(x) = g'(x)$$

$$\sec x \tan x = -\csc x \cot x \Rightarrow \frac{\sec x \tan x}{\csc x \cot x} = -1 \Rightarrow \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = -1 \Rightarrow$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \Rightarrow \tan^3 x = -1 \Rightarrow \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

82. (a) $n(t) = -9.6643t^2 + 90.7414t + 77.5029$

$$v(t) = -276.4643t^2 + 2987.6929t + 1809.9714$$

(b) $A = \frac{v(t)}{n(t)} \approx \frac{-276.46t^2 + 2987.69t + 1809.97}{-9.66t^2 + 90.74t + 77.50}$

A represents the average retail value (in millions of dollars) per 1000 motor homes.

(c) $A'(t) \approx \frac{40.46(x^2 - 2.09x + 17.83)}{(x^2 - 9.39x - 8.02)^2}$

86. $f(x) = \frac{x^2 + 2x - 1}{x} = x + 2 - \frac{1}{x}$

$$f'(x) = 1 + \frac{1}{x^2}$$

$$f''(x) = -\frac{2}{x^3}$$

90. $f''(x) = 2 - 2x^{-1}$

$$f'''(x) = 2x^{-2} = \frac{2}{x^2}$$

92. $f^{(4)}(x) = 2x + 1$

$$f^{(5)}(x) = 2$$

$$f^{(6)}(x) = 0$$

84. $f(x) = x + \frac{32}{x^2}$

$$f'(x) = 1 - \frac{64}{x^3}$$

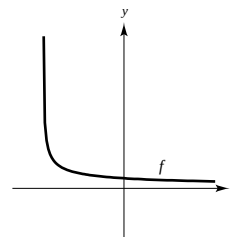
$$f''(x) = \frac{192}{x^4}$$

88. $f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

$$\begin{aligned} f''(x) &= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\ &= \sec x(\sec^2 x + \tan^2 x) \end{aligned}$$

94. The graph of a differentiable function f such that $f' > 0$ and $f'' < 0$ for all real numbers x would in general look like the graph below.



96. $f(x) = 4 - h(x)$

$$f'(x) = -h'(x)$$

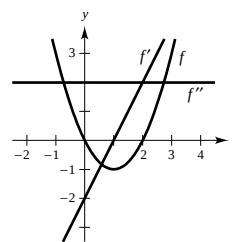
$$f'(2) = -h'(2) = -4$$

98. $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

$$\begin{aligned} f'(2) &= g(2)h'(2) + h(2)g'(2) \\ &= (3)(4) + (-1)(-2) \\ &= 14 \end{aligned}$$

100.



It appears that f is quadratic; so f' would be linear and f'' would be constant.

102. $s(t) = -8.25t^2 + 66t$

$$v(t) = -16.50t + 66$$

$$a(t) = -16.50$$

$t(\text{sec})$	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft/sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft/sec ²)	-16.5	-16.5	-16.5	-16.5	-16.5

Average velocity on:

$$[0, 1] \text{ is } \frac{57.75 - 0}{1 - 0} = 57.75.$$

$$[1, 2] \text{ is } \frac{99 - 57.75}{2 - 1} = 41.25.$$

$$[2, 3] \text{ is } \frac{123.75 - 99}{3 - 2} = 24.75.$$

$$[3, 4] \text{ is } \frac{132 - 123.75}{4 - 3} = 8.25.$$

104. (a) $f(x) = x^n$

$$f^n(x) = n(n-1)(n-2) \cdots (2)(1) = n!$$

(b) $f(x) = \frac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2) \cdots (2)(1)}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}$$

Note: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ (read “ n factorial.”)

106. $[xf(x)]' = xf'(x) + f(x)$

$$[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$$

$$[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$$

$$\text{In general, } [xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x).$$

108. $f(x) = \sin x \quad f\left(\frac{\pi}{2}\right) = 1$

$$f'(x) = \cos x \quad f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x \quad f''\left(\frac{\pi}{2}\right) = -1$$

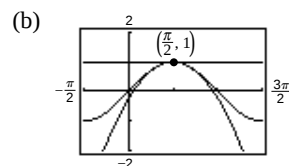
(a) $P_1(x) = f'(a)(x-a) + f(a) = 0\left(x - \frac{\pi}{2}\right) + 1 = 1$

$$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) = \frac{1}{2}(-1)\left(x - \frac{\pi}{2}\right)^2 + 1$$

$$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

(c) P_2 is a better approximation than P_1 .

(d) The accuracy worsens as you move farther away from $x = a = \frac{\pi}{2}$.



110. True. y is a fourth-degree polynomial.

$$\frac{d^n y}{dx^n} = 0 \text{ when } n > 4.$$

112. True

114. True. If $v(t) = c$ then

$$a(t) = v'(t) = 0.$$

116. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$

$$= fg'' - f''g \quad \text{True}$$

(b) $(fg)'' = (fg' + f'g)'$

$$= fg'' + f'g' + f'g' + f''g$$

$$= fg'' + 2f'g' + f''g$$

$$\neq fg'' + f''g \quad \text{False}$$

Section 2.4 The Chain Rule

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

$$2. y = \frac{1}{\sqrt{x+1}}$$

$$u = x + 1$$

$$y = u^{-1/2}$$

$$4. y = 3 \tan(\pi x^2)$$

$$u = \pi x^2$$

$$y = 3 \tan u$$

$$6. y = \cos \frac{3x}{2}$$

$$u = \frac{3x}{2}$$

$$y = \cos u$$

$$8. y = (2x^3 + 1)^2$$

$$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$$

$$10. y = 3(4 - x^2)^5$$

$$y' = 15(4 - x^2)(-2x) = -30x(4 - x^2)$$

$$12. f(t) = (9t + 2)^{2/3}$$

$$f'(t) = \frac{2}{3}(9t + 2)^{-1/3}(9) = \frac{6}{\sqrt[3]{9t + 2}}$$

$$14. g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$$

$$g'(x) = \frac{1}{2}(5 - 3x)^{-1/2}(-3) = \frac{-3}{2\sqrt{5 - 3x}}$$

$$16. g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$$

$$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

$$18. f(x) = -3(2 - 9x)^{1/4}$$

$$f'(x) = -\frac{3}{4}(2 - 9x)^{-3/4}(-9) = \frac{27}{4(2 - 9x)^{3/4}}$$

$$20. s(t) = (t^2 + 3t - 1)^{-1}$$

$$\begin{aligned} s'(t) &= -1(t^2 + 3t - 1)^{-2}(2t + 3) \\ &= \frac{-(2t + 3)}{(t^2 + 3t - 1)^2} \end{aligned}$$

$$22. y = -5(t + 3)^{-3}$$

$$y' = 15(t + 3)^{-4} = \frac{15}{(t + 3)^4}$$

$$24. g(t) = (t^2 - 2)^{-1/2}$$

$$g'(t) = -\frac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\frac{t}{(t^2 - 2)^{3/2}}$$

$$26. f(x) = x(3x - 9)^3$$

$$\begin{aligned} f'(x) &= x[3(3x - 9)^2(3)] + (3x - 9)^3(1) \\ &= (3x - 9)^2[9x + 3x - 9] \\ &= 27(x - 3)^2(4x - 3) \end{aligned}$$

$$28. y = \frac{1}{2}x^2\sqrt{16 - x^2}$$

$$\begin{aligned} y' &= \frac{1}{2}x^2\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2} \\ &= \frac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2} \\ &= \frac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}} \end{aligned}$$

$$30. y = \frac{x}{\sqrt{x^4 + 4}}$$

$$\begin{aligned} y' &= \frac{(x^4 + 4)^{1/2}(1) - x\frac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4} \\ &= \frac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \frac{4 - x^4}{(x^4 + 4)^{3/2}} \end{aligned}$$

$$32. h(t) = \left(\frac{t^2}{t^3 + 2}\right)^2$$

$$\begin{aligned} h'(t) &= 2\left(\frac{t^2}{t^3 + 2}\right)\left(\frac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2}\right) \\ &= \frac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \frac{2t^3(4 - t^3)}{(t^3 + 2)^3} \end{aligned}$$

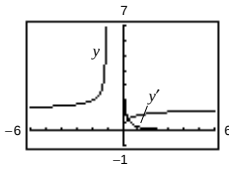
$$34. g(x) = \left(\frac{3x^2 - 2}{2x + 3} \right)^3$$

$$\begin{aligned} g'(x) &= 3 \left(\frac{3x^2 - 2}{2x + 3} \right)^2 \left(\frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2} \right) \\ &= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} = \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4} \end{aligned}$$

$$36. y = \sqrt{\frac{2x}{x+1}}$$

$$y' = \frac{1}{\sqrt{2x}(x+1)^{3/2}}$$

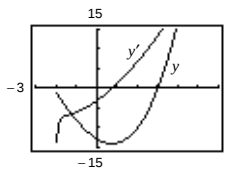
y' has no zeros.



$$40. y = (t^2 - 9)\sqrt{t+2}$$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t+2}}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



$$44. y = x^2 \tan \frac{1}{x}$$

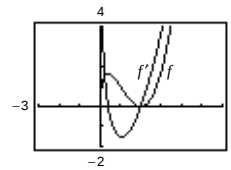
$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

$$38. f(x) = \sqrt{x}(2-x)^2$$

$$f'(x) = \frac{(x-2)(5x-2)}{2\sqrt{x}}$$

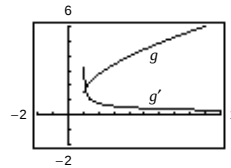
The zeros of f' correspond to the points on the graph of f where the tangent lines are horizontal.



$$42. g(x) = \sqrt{x-1} + \sqrt{x+1}$$

$$g'(x) = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$$

g' has no zeros.



$$46. (a) y = \sin 3x$$

$$y' = 3 \cos 3x$$

$$y'(0) = 3$$

3 cycles in $[0, 2\pi]$

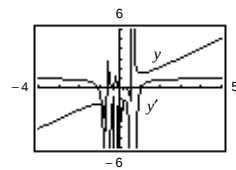
$$(b) y = \sin\left(\frac{x}{2}\right)$$

$$y' = \left(\frac{1}{2}\right) \cos\left(\frac{x}{2}\right)$$

$$y'(0) = \frac{1}{2}$$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a .



48. $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

52. $y = \cos(1 - 2x)^2 = \cos((1 - 2x)^2)$

$$y' = -\sin(1 - 2x)^2(2(1 - 2x)(-2)) = 4(1 - 2x) \sin(1 - 2x)^2$$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)$

$$\begin{aligned} g'(\theta) &= \sec\left(\frac{1}{2}\theta\right) \sec^2\left(\frac{1}{2}\theta\right) \frac{1}{2} + \tan\left(\frac{1}{2}\theta\right) \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right) \frac{1}{2} \\ &= \frac{1}{2} \sec\left(\frac{1}{2}\theta\right) \left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right) \right] \end{aligned}$$

56. $g(v) = \frac{\cos v}{\csc v} = \cos v \cdot \sin v$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v) = \cos^2 v - \sin^2 v = \cos 2v$$

58. $y = 2 \tan^3 x$

$$y' = 6 \tan^2 x \cdot \sec^2 x$$

50. $h(x) = \sec(x^2)$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

60. $g(t) = 5 \cos^2 \pi t = 5(\cos \pi t)^2$

$$\begin{aligned} g'(t) &= 10 \cos \pi t (-\sin \pi t)(\pi) \\ &= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t \end{aligned}$$

62. $h(t) = 2 \cot^2(\pi t + 2)$

$$\begin{aligned} h'(t) &= 4 \cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi)) \\ &= -4\pi \cot(\pi t + 2) \csc^2(\pi t + 2) \end{aligned}$$

64. $y = 3x - 5 \cos(\pi x)^2$

$$= 3x - 5 \cos(\pi^2 x^2)$$

$$\begin{aligned} \frac{dy}{dx} &= 3 + 5 \sin(\pi^2 x^2)(2\pi^2 x) \\ &= 3 + 10\pi^2 x \sin(\pi x)^2 \end{aligned}$$

66. $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$\begin{aligned} y' &= \cos x^{1/3} \left(\frac{1}{3} x^{-2/3} \right) + \frac{1}{3} (\sin x)^{-2/3} \cos x \\ &= \frac{1}{3} \left[\frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}} \right] \end{aligned}$$

68. $y = (3x^3 + 4x)^{1/5}, (2, 2)$

$$\begin{aligned} y' &= \frac{1}{5} (3x^3 + 4x)^{-4/5} (9x^2 + 4) \\ &= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}} \end{aligned}$$

$$y'(2) = \frac{1}{2}$$

70. $f(x) = \frac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \left(4, \frac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

72. $f(x) = \frac{x+1}{2x-3}, (2, 3)$

$$f'(x) = \frac{(2x-3)(1) - (x+1)(2)}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$$

$$f'(2) = -5$$

74. $y = \frac{1}{x} + \sqrt{\cos x}, \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$$y'(\pi/2) \text{ is undefined.}$$

76. (a) $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$, $(2, 2)$

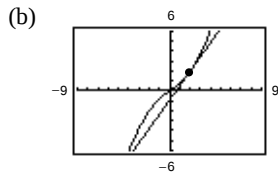
$$f'(x) = \frac{1}{3}x \left[\frac{1}{2}(x^2 + 5)^{-1/2}(2x) \right] + \frac{1}{3}(x^2 + 5)^{1/2}$$

$$= \frac{x^2}{3\sqrt{x^2 + 5}} + \frac{1}{3}\sqrt{x^2 + 5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \Rightarrow 13x - 9y - 8 = 0$$



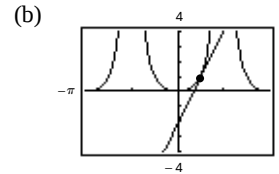
78. (a) $f(x) = \tan^2 x$, $\left(\frac{\pi}{4}, 1\right)$

$$f'(x) = 2 \tan x \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$$



80. $f(x) = (x - 2)^{-1}$

$$f'(x) = -(x - 2)^{-2} = \frac{-1}{(x - 2)^2}$$

$$f''(x) = 2(x - 2)^{-3} = \frac{2}{(x - 2)^3}$$

82. $f(x) = \sec^2 \pi x$

$$f'(x) = 2 \sec \pi x (\pi \sec \pi x \tan \pi x)$$

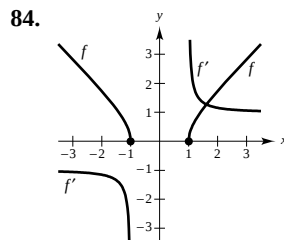
$$= 2\pi \sec^2 \pi x \tan \pi x$$

$$f''(x) = 2\pi \sec^2 \pi x (\sec^2 \pi x)(\pi) + 2\pi \tan \pi x (2\pi \sec^2 \pi x \tan \pi x)$$

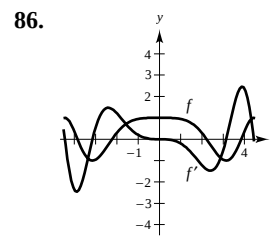
$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$= 2\pi^2 \sec^2 \pi x (\sec^2 \pi x + 2 \tan^2 \pi x)$$

$$= 2\pi^2 \sec^2 \pi x (3 \sec^2 \pi x - 2)$$



f is decreasing on $(-\infty, -1)$ so f' must be negative there. f is increasing on $(1, \infty)$ so f' must be positive there.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

88. $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \Rightarrow g'(x) = 2x f'(x^2)$$

90. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Rightarrow g'(x) = 0$

$$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$$

(b) $\tan^2 x + 1 = \sec^2 x$

$$g(x) + 1 = f(x)$$

Taking derivatives of both sides,

$$g'(x) = f'(x).$$

Equivalently, $f'(x) = 2 \sec x \cdot \sec x \cdot \tan x$ and

$g'(x) = 2 \tan x \cdot \sec^2 x$, which are the same.

94. $y = A \cos \omega t$

(a) Amplitude: $A = \frac{3.5}{2} = 1.75$

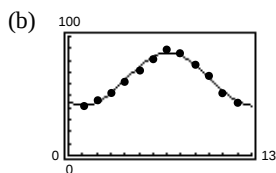
$$y = 1.75 \cos \omega t$$

Period: $10 \Rightarrow \omega = \frac{2\pi}{10} = \frac{\pi}{5}$

$$y = 1.75 \cos \frac{\pi t}{5}$$

96. (a) Using a graphing utility, or by trial and error, you obtain a model of the form

$$T(t) = 64.18 - 22.15 \sin\left(\frac{\pi t}{6} + 1\right)$$



92. $y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$

$$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$$

$$= -4 \sin 12t - 3 \cos 12t$$

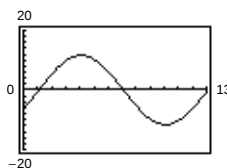
When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

(b) $v = y' = 1.75 \left[-\frac{\pi}{5} \sin \frac{\pi t}{5} \right]$

$$= -0.35 \pi \sin \frac{\pi t}{5}$$

(c) $T'(t) = -22.15 \cos\left(\frac{\pi t}{6} + 1\right) \left(\frac{\pi}{6}\right)$

$$= -11.60 \cos\left(\frac{\pi t}{6} + 1\right)$$



(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

98. (a) $g(x) = f(x) - 2 \Rightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Rightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Rightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$$

$$r'(-1) = -3f'(3) = (-3)(-4) = 12$$

(d) $s(x) = f(x + 2) \Rightarrow s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$$s'(-2) = f'(0) = -\frac{1}{3}, \text{ etc.}$$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

100. $f(x + p) = f(x)$ for all x .

(a) Yes, $f'(x + p) = f'(x)$, which shows that f' is periodic as well.

(b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$. Since f' is periodic, so is g' .

102. If $f(-x) = -f(x)$, then

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[-f(x)]$$

$$f'(-x)(-1) = -f'(x)$$

$$f'(-x) = f'(x).$$

Thus, $f'(x)$ is even.

104. $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}[\sqrt{u^2}] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u' \frac{u}{|u|}, u \neq 0$$

106. $f(x) = |x^2 - 4|$

$$f'(x) = 2x \left(\frac{x^2 - 4}{|x^2 - 4|} \right), x \neq \pm 2$$

110. (a) $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$

$$\begin{aligned} f''(x) &= 2[2(\sec 2x)(\tan 2x)] \tan 2x + 2(\sec 2x)(\sec^2 2x)(2) \\ &= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x] \end{aligned}$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

$$f'\left(\frac{\pi}{6}\right) = 2 \sec\left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

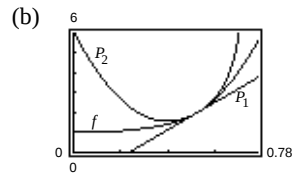
$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(c) P_2 is a better approximation than P_1 .

108. $f(x) = |\sin x|$

$$f'(x) = \cos x \left(\frac{\sin x}{|\sin x|} \right), x \neq k\pi$$



(d) The accuracy worsens as you move away from $x = \pi/6$.

112. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2 \cos 2x)$.

114. False. First apply the Product Rule.

Section 2.5 Implicit Differentiations

2. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

6. $x^2y + y^2x = -2$

$$x^2y' + 2xy + y^2 + 2yxy' = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

10. $2 \sin x \cos y = 1$

$$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

$$= \cot x \cot y$$

14. $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

18. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$$(x - 2)^2 + (y + 3)^2 = 4 \text{ (Circle)}$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(c) Explicitly:

$$\frac{dy}{dx} = \pm \frac{1}{2} [4 - (x - 2)^2]^{-1/2} (-2)(x - 2)$$

$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2})^2}$$

$$= \frac{-(x - 2)}{\pm \sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$

$$= \frac{-(x - 2)}{y + 3}$$

4. $x^3 + y^3 = 8$

$$3x^2 + 3y^2y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

8. $(xy)^{1/2} - x + 2y = 0$

$$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$$

$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$$

$$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}y' = 0$$

$$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$

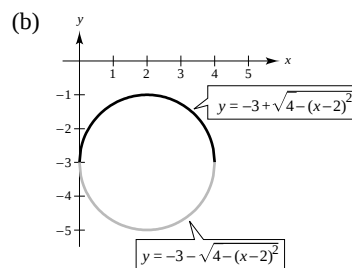
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$

$$y' = \frac{\cos \pi x}{\sin \pi y}$$

16. $x = \sec \frac{1}{y}$

$$1 = -\frac{y'}{y^2} \sec \frac{1}{y} \tan \frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y) \tan(1/y)} = -y^2 \cos\left(\frac{1}{y}\right) \cot\left(\frac{1}{y}\right)$$



(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

$$(2y + 6)y' = -2(x - 2)$$

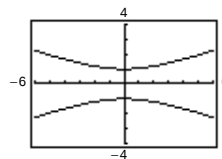
$$y' = \frac{-(x - 2)}{y + 3}$$

20. (a) $9y^2 = x^2 + 9$

$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm\sqrt{x^2 + 9}}{3}$$

(b)



(c) Explicitly: $\frac{dy}{dx} = \frac{\pm\frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3} = \frac{\pm x}{3\sqrt{x^2 + 9}} = \frac{\pm x}{3(\pm 3y)} = \frac{x}{9y}$

(d) Implicitly: $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

22. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At (1, 1): $y' = \frac{2}{3}$.

24.

$$(x + y)^3 = x^3 + y^3$$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At (-1, 1): $y' = -1$.

26. $x^3 + y^3 = 4xy + 1$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

At (2, 1), $y' = \frac{4 - 12}{3 - 8} = \frac{8}{5}$

28.

$$x \cos y = 1$$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x} \cot y = \frac{\cot y}{x}$$

At $\left(2, \frac{\pi}{3}\right)$: $y' = \frac{1}{2\sqrt{3}}$.

30. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At (2, 2): $y' = 2$.

32.

$$x^3 + y^3 - 6xy = 0$$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $\left(\frac{4}{3}, \frac{8}{3}\right)$: $y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$.

34. $\cos y = x$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{-1}{\sin y}, 0 < y < \pi$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}}, -1 < x < 1$$

36. $x^2 y^2 - 2x = 3$

$$2x^2 y y' + 2xy^2 - 2 = 0$$

$$x^2 y y' + xy^2 - 1 = 0$$

$$y' = \frac{1 - xy^2}{x^2 y}$$

$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$

$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$

$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2 y^2} + x^2 yy'' + y^2 = 0$$

$$4xy^2 - 4x^2 y^4 + 1 - 2xy^2 + x^2 y^4 + x^4 y^3 y'' + x^2 y^4 = 0$$

$$x^4 y^3 y'' = 2x^2 y^4 - 2xy^2 - 1$$

$$y'' = \frac{2x^2 y^4 - 2xy^2 - 1}{x^4 y^3}$$

38. $1 - xy = x - y$

$$y - xy = x - 1$$

$$y = \frac{x - 1}{1 - x} = -1$$

$$y' = 0$$

$$y'' = 0$$

40. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y'' = -2y^{-2} y' = \left[\frac{-2}{y^2} \right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

42. $y^2 = \frac{x - 1}{x^2 + 1}$

$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$

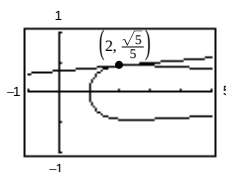
$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \frac{1 + 4 - 4}{[(2\sqrt{5})/5](4 + 1)^2} = \frac{1}{10\sqrt{5}}$.

Tangent line: $y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$

$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$



44. $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

At $(0, 3)$:

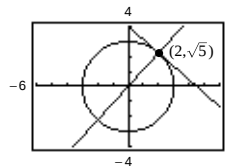
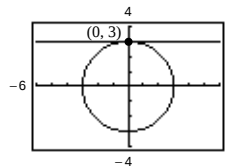
Tangent line: $y = 3$

Normal line: $x = 0$.

At $(2, \sqrt{5})$:

$$\text{Tangent line: } y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2) \Rightarrow 2x + \sqrt{5}y - 9 = 0$$

$$\text{Normal line: } y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2) \Rightarrow \sqrt{5}x - 2y = 0.$$



46. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

$$\text{Equations: } (x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$$

$$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$$

48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

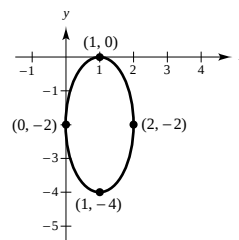
Horizontal tangents: $(1, 0), (1, -4)$.

Vertical tangents occur when $y = -2$:

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$.



50. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$$2x^2 + 3x^3 = 5 \quad \text{and} \quad 3x^3 + 2x^2 - 5 = 0$$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

$$\begin{array}{ll} y^2 = x^3: & 2x^2 + 3y^2 = 5: \\ 2yy' = 3x^2 & 4x + 6yy' = 0 \\ y' = \frac{3x^2}{2y} & y' = -\frac{2x}{3y} \end{array}$$

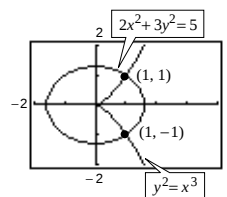
At $(1, 1)$, the slopes are:

$$y' = \frac{3}{2} \qquad y' = -\frac{2}{3}$$

At $(1, -1)$, the slopes are:

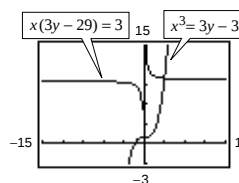
$$y' = -\frac{3}{2} \qquad y' = \frac{2}{3}$$

Tangents are perpendicular.



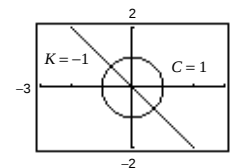
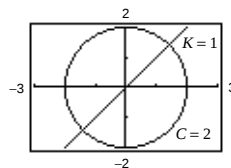
52. Rewriting each equation and differentiating,

$$\begin{array}{ll} x^3 = 3(y - 1) & x(3y - 29) = 3 \\ y = \frac{x^3}{3} + 1 & y = \frac{1}{3}\left(\frac{3}{x} + 29\right) \\ y' = x^2 & y' = -\frac{1}{x^2} \end{array}$$



For each value of x , the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

54. $x^2 + y^2 = C^2 \quad y = Kx$
 $2x + 2yy' = 0 \quad y' = K$
 $y' = -\frac{x}{y}$



At the point of intersection (x, y) the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

56. $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$
 $(-6xy + 3y^2)y' = 3y^2 - 2x$
 $y' = \frac{3y^2 - 2x}{3y^2 - 6xy}$

(b) $2x \frac{dx}{dt} - 3y^2 \frac{dx}{dt} - 6xy \frac{dy}{dt} + 3y^2 \frac{dy}{dt} = 0$
 $(2x - 3y^2) \frac{dx}{dt} = (6xy - 3y^2) \frac{dy}{dt}$

58. (a) $4 \sin x \cos y = 1$

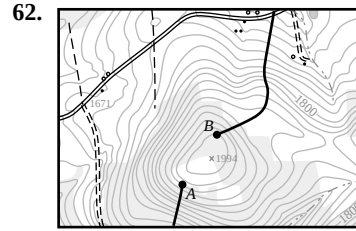
$$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

(b) $4 \sin x(-\sin y) \frac{dy}{dt} + 4 \cos x \frac{dx}{dt} \cos y = 0$

$$\cos x \cos y \frac{dx}{dt} = \sin x \sin y \frac{dy}{dt}$$

60. Given an implicit equation, first differentiate both sides with respect to x . Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y' .



Use starting point B.

64. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0) :

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$$x\text{-intercept: } (x_0 + \sqrt{x_0}\sqrt{y_0}, 0)$$

$$y\text{-intercept: } (0, y_0 + \sqrt{x_0}\sqrt{y_0})$$

Sum of intercepts:

$$(x_0 + \sqrt{x_0}\sqrt{y_0}) + (y_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c.$$

Section 2.6 Related Rates

2. $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6} \frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = [4(3) - 6](2) = 12$

(b) When $x = 1$ and $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}$

4. $x^2 + y^2 = 25$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right) \frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

$$6. \quad y = \frac{1}{1+x^2}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \left[\frac{-2x}{(1+x^2)^2} \right] \frac{dx}{dt}$$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25} \text{ cm/sec.}$$

$$10. \quad (a) \quad \frac{dx}{dt} \text{ negative} \Rightarrow \frac{dy}{dt} \text{ negative}$$

$$(b) \quad \frac{dy}{dt} \text{ positive} \Rightarrow \frac{dx}{dt} \text{ positive}$$

$$14. \quad D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2 \sin x \cos x) \frac{dx}{dt} = \frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}} \frac{dx}{dt} = \frac{2 + 2 \sin x \cos x}{\sqrt{x^2 + \sin^2 x}}$$

$$16. \quad A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

If dr/dt is constant, dA/dt is not constant.

$$\frac{dA}{dt} \text{ depends on } r \text{ and } \frac{dr}{dt}.$$

$$20. \quad V = x^3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3/\text{sec.}$$

$$8. \quad y = \sin x$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \cos x \frac{dx}{dt}$$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{6} \right) (2) = \sqrt{3} \text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{4} \right) (2) = \sqrt{2} \text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{3} \right) (2) = 1 \text{ cm/sec.}$$

12. Answers will vary. See page 145.

$$18. \quad V = \frac{4}{3}\pi r^3$$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$(a) \quad \text{When } r = 6, \frac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{min.}$$

$$\text{When } r = 24, \frac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3/\text{min.}$$

(b) If dr/dt is constant, dV/dt is proportional to r^2 .

(b) When $x = 10$,

$$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3/\text{sec.}$$

$$22. \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min}.$$

(b) When $r = 24$,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min}.$$

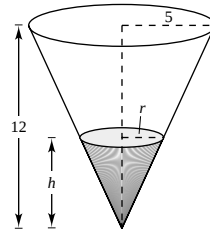
$$24. \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144} h^3 = \frac{25\pi}{3(144)} h^3$$

(By similar triangles, $\frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h$.)

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{25\pi}{144} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2} \right) \frac{dV}{dt}$$

$$\text{When } h = 8, \frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi} \text{ ft/min}.$$

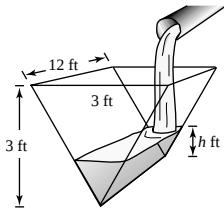


$$26. \quad V = \frac{1}{2}bh(12) = 6bh = 6h^2 \text{ (since } b = h)$$

$$(a) \quad \frac{dV}{dt} = 12h \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt}$$

$$\text{When } h = 1 \text{ and } \frac{dV}{dt} = 2, \frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6} \text{ ft/min}$$

$$(b) \quad \text{If } \frac{dh}{dt} = \frac{3}{8} \text{ and } h = 2, \text{ then } \frac{dV}{dt} = 12(2)\left(\frac{3}{8}\right) = 9 \text{ ft}^3/\text{min}.$$



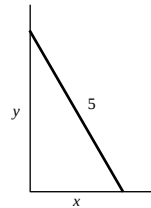
$$28. \quad x^2 + y^2 = 25$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x} \text{ since } \frac{dy}{dt} = 0.15.$$

When $x = 2.5$,

$$y = \sqrt{18.75}, \frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5} 0.15 \approx -0.26 \text{ m/sec}$$



30. Let L be the length of the rope.

$$(a) \quad L^2 = 144 + x^2$$

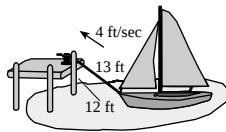
$$2L \frac{dL}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x} \text{ since } \frac{dL}{dt} = -4 \text{ ft/sec}.$$

When $L = 13$,

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec}.$$



Speed of the boat increases as it approaches the dock.

$$(b) \quad \text{If } \frac{dx}{dt} = -4, \text{ and } L = 13,$$

$$\frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt}$$

$$= \frac{5}{13}(-4)$$

$$= -\frac{20}{13} \text{ ft/sec}$$

As $L \rightarrow 0$, $\frac{dL}{dt}$ increases.

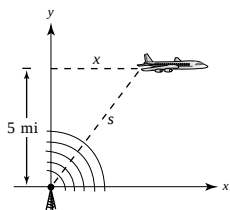
32. $x^2 + y^2 = s^2$

$$2x \frac{dx}{dt} + 0 = 2s \frac{ds}{dt} \quad \left(\text{since } \frac{dy}{dt} = 0 \right)$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(-240) = \frac{-480}{\sqrt{3}} = -160\sqrt{3} \approx -277.13 \text{ mph.}$$



36. (a) $\frac{20}{6} = \frac{y}{y-x}$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$

$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7} \frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$

(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$

38. $x(t) = \frac{3}{5} \sin \pi t$, $x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}$, $y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

Lowest point: $\left(0, \frac{4}{5}\right)$

34. $s^2 = 90^2 + x^2$

$$x = 60$$

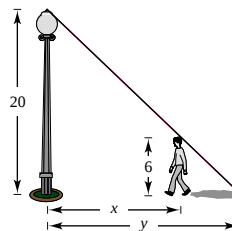
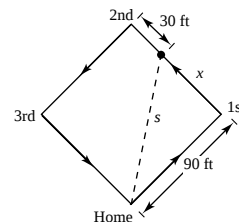
$$\frac{dx}{dt} = 28$$

$$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When $x = 60$,

$$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$$

$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$



(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5} \sin \pi t \Rightarrow \sin \pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5} \pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus, $\frac{dy}{dt} = \frac{-3/10}{\sqrt{15}/4} \cdot \frac{3}{5} \pi \cos\left(\frac{\pi}{6}\right)$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}$$

Speed = $\left| \frac{-9\sqrt{5}\pi}{125} \right| \approx 0.5058 \text{ m/sec}$

$$40. \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$,

$$R = 30$$

$$\frac{dR}{dt} = (30)^2 \left[\frac{1}{(50)^2} (1) + \frac{1}{(75)^2} (1.5) \right]$$

$$= 0.6 \text{ ohms/sec.}$$

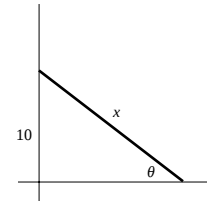
$$44. \quad \sin \theta = \frac{10}{x}$$

$$\frac{dx}{dt} = (-1) \text{ ft/sec}$$

$$\cos \theta \left(\frac{d\theta}{dt} \right) = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2} \frac{dx}{dt} (\sec \theta)$$

$$= \frac{-10}{25^2} (-1) \frac{25}{\sqrt{25^2 - 10^2}} = \frac{10}{25} \frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}} = \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$



$$46. \quad \tan \theta = \frac{x}{50}$$

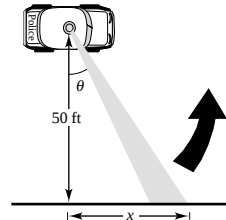
$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2 \theta \left(\frac{d\theta}{dt} \right) = \frac{1}{50} \left(\frac{dx}{dt} \right)$$

$$\frac{dx}{dt} = 50 \sec^2 \theta \left(\frac{d\theta}{dt} \right)$$

(a) When $\theta = 30^\circ$, $\frac{dx}{dt} = \frac{200\pi}{3} \text{ ft/sec.}$

(c) When $\theta = 70^\circ$, $\frac{dx}{dt} \approx 427.43\pi \text{ ft/sec.}$

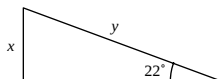


(b) When $\theta = 60^\circ$, $\frac{dx}{dt} = 200\pi \text{ ft/sec.}$

$$48. \quad \sin 22^\circ = \frac{x}{y}$$

$$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 22^\circ)(240) \approx 89.9056 \text{ mi/hr}$$



$$42. \quad rg \tan \theta = v^2$$

$$32r \tan \theta = v^2, r \text{ is a constant.}$$

$$32r \sec^2 \theta \frac{d\theta}{dt} = 2v \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v} \sec^2 \theta \frac{d\theta}{dt}$$

$$\text{Likewise, } \frac{d\theta}{dt} = \frac{v}{16r} \cos^2 \theta \frac{dv}{dt}.$$

50. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

52. $L^2 = 144 + x^2$; acceleration of the boat $= \frac{d^2x}{dt^2}$.

First derivative: $2L \frac{dL}{dt} = 2x \frac{dx}{dt}$

$$L \frac{dL}{dt} = x \frac{dx}{dt}$$

Second derivative: $L \frac{d^2L}{dt^2} + \frac{dL}{dt} \cdot \frac{dL}{dt} = x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right) \left[L \frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2 \right]$$

When $L = 13$, $x = 5$, $\frac{dx}{dt} = -10.4$, and $\frac{dL}{dt} = -4$ (see Exercise 30). Since $\frac{dL}{dt}$ is constant, $\frac{d^2L}{dt^2} = 0$.

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{1}{5} [13(0) + (-4)^2 - (-10.4)^2] \\ &= \frac{1}{5} [16 - 108.16] = \frac{1}{5} [-92.16] = -18.432 \text{ ft/sec}^2 \end{aligned}$$

54. $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles, $\frac{20}{x} = \frac{y}{x - 12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

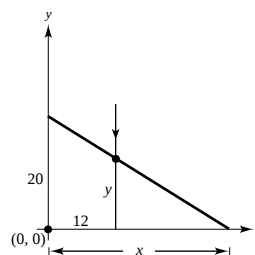
$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20 \frac{dx}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y} \frac{dy}{dt}$$

At $t = 1$, $\frac{dx}{dt} = \frac{240/4.9}{20 - 15.1} (-9.8) \approx -97.96 \text{ m/sec}.$



51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \frac{d^2y}{dt^2}$

First derivative: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Second derivative: $x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[-x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When $x = 7$, $y = 24$, $\frac{dy}{dt} = -\frac{7}{12}$, and $\frac{dx}{dt} = 2$ (see Exercise 27). Since $\frac{dx}{dt}$ is constant, $\frac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[-4 - \frac{49}{144} \right] = \frac{1}{24} \left[-\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\frac{dm}{dt} = \frac{dm}{ds} \frac{ds}{dt} = (-1.762s + 29.10) \frac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\frac{ds}{dt} = 1.2$.

Thus, $\frac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

3. $f(x) = \sqrt{x} + 1$

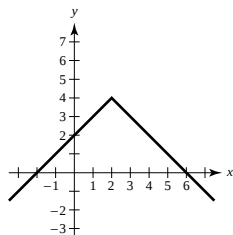
5. f is differentiable for all $x \neq -1$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.

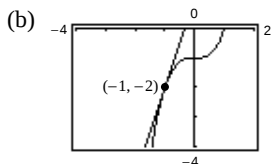


11. (a) Using the limit definition, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1$$

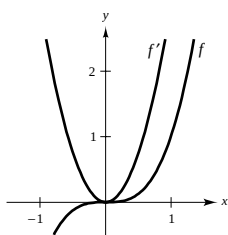


9. Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

At $x = -1$, $g'(-1) = -\frac{4}{3} - \frac{1}{6} = -\frac{3}{2}$

$$\begin{aligned} 13. \quad g'(2) &= \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^2(x - 1) - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x^2 + x + 2) = 8 \end{aligned}$$

15.



17. $y = 25$

$$y' = 0$$

19. $f(x) = x^8$

$$f'(x) = 8x^7$$

21. $h(t) = 3t^4$

$$h'(t) = 12t^3$$

23. $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$$

27. $g(t) = \frac{2}{3}t^{-2}$

$$g'(t) = \frac{-4}{3}t^{-3} = -\frac{4}{3t^3}$$

29. $f(\theta) = 2\theta - 3\sin \theta$

$$f'(\theta) = 2 - 3\cos \theta$$

31. $f(\theta) = 3\cos \theta - \frac{\sin \theta}{4}$

$$f'(\theta) = -3\sin \theta - \frac{\cos \theta}{4}$$

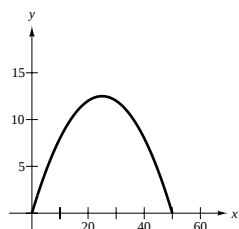
33. $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$$a(t) = v'(t) = 2$$

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

45. $f(x) = 2x - x^{-2}$

$$\begin{aligned} f'(x) &= 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right) \\ &= \frac{2(x^3 + 1)}{x^3} \end{aligned}$$

49. $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

53. $y = 3x^2 \sec x$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

35. $s(t) = -16t^2 + s_0$

$$s(9.2) = -16(9.2)^2 + s_0 = 0$$

$$s_0 = 1354.24$$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e) $y'(25) = 0$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(d) $x(t) = 0$ for $t = 1, 2$.

$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

43. $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47. $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

51. $y = \frac{x^2}{\cos x}$

$$y' = \frac{\cos x (2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

55. $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

57. $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

61. $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) = 6 \sec^2 \theta \tan \theta$$

65. $f(x) = (1 - x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{1 - x^3}}$$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$$

$$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$$

$$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$$

73. $y = \frac{1}{2} \csc 2x$

$$y' = \frac{1}{2}(-\csc 2x \cot 2x)(2)$$

$$= -\csc 2x \cot 2x$$

77. $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$$y' = \sin^{1/2} x \cos x - \sin^{5/2} x \cos x$$

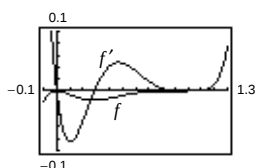
$$= (\cos x) \sqrt{\sin x}(1 - \sin^2 x)$$

$$= (\cos^3 x) \sqrt{\sin x}$$

81. $f(t) = t^2(t - 1)^5$

$$f'(t) = t(t - 1)^4(7t - 2)$$

The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



59. $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

63. $y = 2 \sin x + 3 \cos x$

$$y' = 2 \cos x - 3 \sin x$$

$$y'' = -2 \sin x - 3 \cos x$$

$$y'' + y = -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) = 0$$

67. $h(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$h'(x) = 2\left(\frac{x-3}{x^2+1}\right)\left(\frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}\right)$$

$$= \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$$

71. $y = 3 \cos(3x + 1)$

$$y' = -9 \sin(3x + 1)$$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$$y' = \frac{1}{2} - \frac{1}{4} \cos 2x(2)$$

$$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

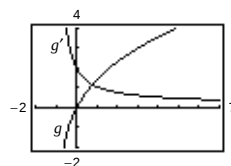
79. $y = \frac{\sin \pi x}{x+2}$

$$y' = \frac{(x+2)\pi \cos \pi x - \sin \pi x}{(x+2)^2}$$

83. $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

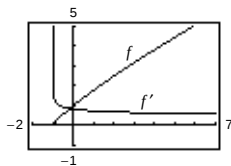
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



85. $f(t) = (t + 1)^{1/2}(t + 1)^{1/3} = (t + 1)^{5/6}$

$$f'(t) = \frac{5}{6(t + 1)^{1/6}}$$

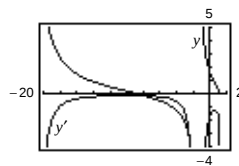
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan \sqrt{1 - x}$

$$y' = -\frac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

91. $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$\begin{aligned} f'' &= -2 \csc x (-\csc x \cdot \cot x) \\ &= 2 \csc^2 x \cot x \end{aligned}$$

93. $f(t) = \frac{t}{(1 - t)^2}$

$$f'(t) = \frac{t + 1}{(1 - t)^3}$$

$$f''(t) = \frac{2(t + 2)}{(1 - t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

97. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

101. $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103. $x \sin y = y \cos x$
 $(x \cos y)y' + \sin y = -y \sin x + y' \cos x$
 $y'(x \cos y - \cos x) = -y \sin x - \sin y$
 $y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$

105. $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

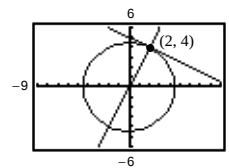
At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$

$$2x - y = 0$$



107. $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

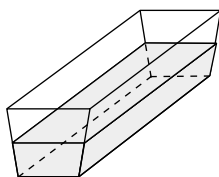
(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$



Width of water at depth h :

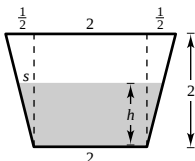
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.



111. $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

$$4.9t^2 = 25$$

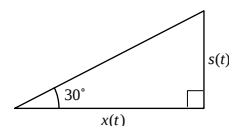
$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}}$$

$$\approx -38.34 \text{ m/sec}$$



Problem Solving for Chapter 2

1. (a)
- $x^2 + (y - r)^2 = r^2$
- Circle

$$x^2 = y \quad \text{Parabola}$$

Substituting,

$$(y - r)^2 = r^2 - y$$

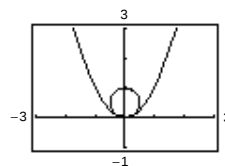
$$y^2 - 2ry + r^2 = r^2 - y$$

$$y^2 - 2ry + y = 0$$

$$y(y - 2r + 1) = 0$$

Since you want only one solution, let $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$

$$\text{Graph } y = x^2 \text{ and } x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$



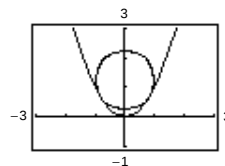
- (b) Let
- (x, y)
- be a point of tangency:
- $x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \frac{x}{b - y}$
- (circle).

 $y = x^2 \Rightarrow y' = 2x$ (parabola). Equating,

$$2x = \frac{x}{b - y}$$

$$2(b - y) = 1$$

$$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$$

Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply

$$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \frac{1}{2}\right)\right]^2 = 1 \Rightarrow y - \frac{1}{2} = 1 \Rightarrow y = \frac{3}{4} \text{ and } b = \frac{5}{4}.$$

$$\text{Center: } \left(0, \frac{5}{4}\right)$$

$$\text{Graph } y = x^2 \text{ and } x^2 + \left(y - \frac{5}{4}\right)^2 = 1$$

3. (a) $f(x) = \cos x$ $P_1(x) = a_0 + a_1x$
 $f(0) = 1$ $P_1(0) = a_0 \Rightarrow a_0 = 1$
 $f'(0) = 0$ $P'_1(0) = a_1 \Rightarrow a_1 = 0$
 $P_1(x) = 1$

- (b) $f(x) = \cos x$ $P_2(x) = a_0 + a_1x + a_2x^2$
 $f(0) = 1$ $P_2(0) = a_0 \Rightarrow a_0 = 1$
 $f'(0) = 0$ $P'_2(0) = a_1 \Rightarrow a_1 = 0$
 $f''(0) = -1$ $P''_2(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$
 $P_2(x) = 1 - \frac{1}{2}x^2$

(c)

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$	0.5403	0.9950	≈ 1	1	≈ 1	0.9950	0.5403
$P_2(x)$	0.5	0.9950	≈ 1	1	≈ 1	0.9950	0.5

 $P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is near 0.

- (d) $f(x) = \sin x$ $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
 $f(0) = 0$ $P_3(0) = a_0 \Rightarrow a_0 = 0$
 $f'(0) = 1$ $P'_3(0) = a_1 \Rightarrow a_1 = 1$
 $f''(0) = 0$ $P''_3(0) = 2a_2 \Rightarrow a_2 = 0$
 $f'''(0) = -1$ $P'''_3(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$
 $P_3(x) = x - \frac{1}{6}x^3$

5. Let $p(x) = Ax^3 + Bx^2 + Cx + D$

$$p'(x) = 3Ax^2 + 2Bx + C$$

At (1, 1): $A + B + C + D = 1$ Equation 1

$$3A + 2B + C = 14$$
 Equation 2

At (-1, -3): $-A + B - C + D = -3$ Equation 3

$$3A - 2B + C = -2$$
 Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

Hence, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$

Subtracting $2A + 2C = 4$ and $6A + 2C = 12$, you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$

Thus, $p(x) = 2x^3 + 4x^2 - 5$.

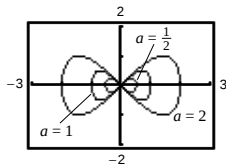
7. (a) $x^4 = a^2x^2 - a^2y^2$

$$a^2y^2 = a^2x^2 - x^4$$

$$y = \frac{\pm \sqrt{a^2x^2 - x^4}}{a}$$

Graph: $y_1 = \frac{\sqrt{a^2x^2 - x^4}}{a}$ and $y_2 = -\frac{\sqrt{a^2x^2 - x^4}}{a}$

(b)



$(\pm a, 0)$ are the x -intercepts, along with $(0, 0)$.

(c) Differentiating implicitly,

$$4x^3 = 2a^2x - 2a^2yy'$$

$$y' = \frac{2a^2x - 4x^3}{2a^2y} = \frac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{\pm a}{\sqrt{2}}$$

$$\left(\frac{a^2}{2}\right)^2 = a^2\left(\frac{a^2}{2}\right) - a^2y^2$$

$$\frac{a^4}{4} = \frac{a^4}{2} - a^2y^2$$

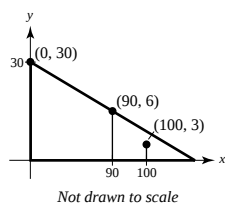
$$a^2y^2 = \frac{a^4}{4}$$

$$y^2 = \frac{a^2}{4}$$

$$y = \pm \frac{a}{2}$$

Four points: $\left(\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(\frac{a}{\sqrt{2}}, -\frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{a}{2}\right)$

9. (a)

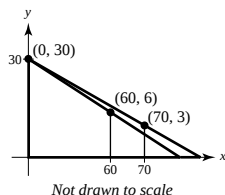


Line determined by (0, 30) and (90, 6):

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$, $y = -\frac{4}{15}(100) + 30 = \frac{10}{3} > 3 \Rightarrow$ Shadow determined by man.

(b)



Line determined by (0, 30) and (60, 6):

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$, $y = -\frac{2}{5}(70) + 30 = 2 < 3 \Rightarrow$ Shadow determined by child.

(c) Need (0, 30), (d, 6), (d + 10, 3) collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Rightarrow \frac{24}{-d} = \frac{3}{-10} \Rightarrow d = 80 \text{ feet}$$

(d) Let y be the length of the street light to the tip of the shadow. We know that $\frac{dx}{dt} = -5$.For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4} \frac{dx}{dt} = \frac{-25}{4}.$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9} \frac{dx}{dt} = \frac{-50}{9}.$$

Therefore,

$$\frac{dy}{dt} = \begin{cases} -\frac{25}{4} & x > 80 \\ -\frac{50}{9} & 0 < x < 80 \end{cases}$$

 $\frac{dy}{dt}$ is not continuous at $x = 80$.

$$\begin{aligned} 11. L'(x) &= \lim_{\Delta x \rightarrow 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x)}{\Delta x} \end{aligned}$$

$$\text{Also, } L'(0) = \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x) - L(0)}{\Delta x}$$

But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0$.Thus, $L'(x) = L'(0)$, for all x .The graph of L is a line through the origin of slope $L'(0)$.

13. (a)

z (degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$	0.0174524	0.0174533	0.0174533

(b) $\lim_{z \rightarrow 0} \frac{\sin z}{z} \approx 0.0174533$

In fact, $\lim_{z \rightarrow 0} \frac{\sin z}{z} = \frac{\pi}{180}$

(c) $\frac{d}{dz}(\sin z) = \lim_{\Delta z \rightarrow 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z}$

$$= \lim_{\Delta z \rightarrow 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left[\sin z \left(\frac{\cos \Delta z - 1}{\Delta z} \right) \right] + \lim_{\Delta z \rightarrow 0} \left[\cos z \left(\frac{\sin \Delta z}{\Delta z} \right) \right]$$

$$= \sin z(0) + \cos z \left(\frac{\pi}{180} \right) = \frac{\pi}{180} \cos z$$

(d) $S(90) = \sin\left(\frac{\pi}{180} 90\right) = \sin \frac{\pi}{2} = 1$; $C(180) = \cos\left(\frac{\pi}{180} 180\right) = -1$

$$\frac{d}{dz}S(z) = \frac{d}{dz}\sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180}C(z)$$

(e) The formulas for the derivatives are more complicated in degrees.

15. $j(t) = a'(t)$

(a) $j(t)$ is the rate of change of the acceleration.

(b) From Exercise 102 in Section 2.3,

$$s(t) = -8.25t^2 + 66t$$

$$v(t) = -16.5t + 66$$

$$a(t) = -16.5$$

$$a'(t) = j(t) = 0$$

Review Exercises for Chapter 2

$$2. f(x) = \frac{x+1}{x-1}$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x+1}{x+\Delta x-1} - \frac{x+1}{x-1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x+1)(x-1) - (x+\Delta x-1)(x+1)}{\Delta x(x+\Delta x-1)(x-1)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + x\Delta x + x - x - \Delta x - 1) - (x^2 + x\Delta x - x + x + \Delta x - 1)}{\Delta x(x+\Delta x-1)(x-1)} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x(x+\Delta x-1)(x-1)} = \lim_{\Delta x \rightarrow 0} \frac{-2}{(x+\Delta x-1)(x-1)} = \frac{-2}{(x-1)^2} \end{aligned}$$

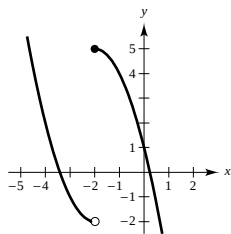
$$4. f(x) = \frac{2}{x}$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{2}{x+\Delta x} - \frac{2}{x}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x - (2x + 2\Delta x)}{\Delta x(x+\Delta x)x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{\Delta x(x+\Delta x)x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2}{(x+\Delta x)x} = \frac{-2}{x^2} \end{aligned}$$

$$8. f(x) = \begin{cases} x^2 + 4x + 2, & \text{if } x < -2 \\ 1 - 4x - x^2, & \text{if } x \geq -2 \end{cases}$$

(a) Nonremovable discontinuity at $x = -2$.

(b) Not differentiable at $x = -2$ because the function is discontinuous there.



$$12. (a) \text{ Using the limit definition, } f'(x) = \frac{-2}{(x+1)^2}.$$

At $x = 0$, $f'(0) = -2$. The tangent line is

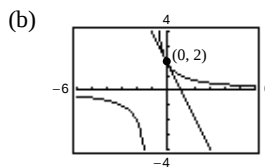
$$y - 2 = -2(x - 0)$$

$$y = -2x + 2$$

6. f is differentiable for all $x \neq -3$.

10. Using the limit definition, you obtain $h'(x) = \frac{3}{8} - 4x$.

$$\text{At } x = -2, h'(-2) = \frac{3}{8} - 4(-2) = \frac{67}{8}.$$



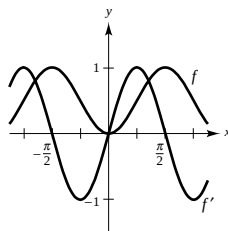
$$14. f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{\frac{1}{x+1} - \frac{1}{3}}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{3 - x - 1}{(x - 2)(x + 1)3}$$

$$= \lim_{x \rightarrow 2} \frac{-1}{(x + 1)3} = \frac{-1}{9}$$

16.



$$18. y = -12$$

$$y' = 0$$

$$20. g(x) = x^{12}$$

$$g'(x) = 12x^{11}$$

$$22. f(t) = -8t^5$$

$$f'(t) = -40t^4$$

$$24. g(s) = 4s^4 - 5s^2$$

$$g'(s) = 16s^3 - 10s$$

$$26. f(x) = x^{1/2} - x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{x+1}{2x^{3/2}}$$

$$28. h(x) = \frac{2}{9}x^{-2}$$

$$h'(x) = \frac{-4}{9}x^{-3} = \frac{-4}{9x^3}$$

$$30. g(\alpha) = 4 \cos \alpha + 6$$

$$g'(\alpha) = -4 \sin \alpha$$

$$32. g(\alpha) = \frac{5 \sin \alpha}{3} - 2\alpha$$

$$g'(\alpha) = \frac{5 \cos \alpha}{3} - 2$$

$$34. s = -16t^2 + s_0$$

First ball:

$$-16t^2 + 100 = 0$$

$$t = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5 \text{ seconds to hit ground}$$

Second ball:

$$-16t^2 + 75 = 0$$

$$t^2 = \sqrt{\frac{75}{16}} = \frac{5\sqrt{3}}{4} \approx 2.165 \text{ seconds to hit ground}$$

Since the second ball was released one second after the first ball, the first ball will hit the ground first. The second ball will hit the ground $3.165 - 2.5 = 0.665$ second later.

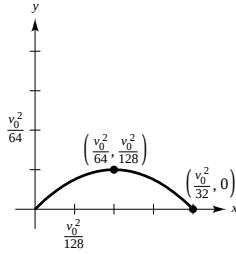
$$36. s(t) = -16t^2 + 14,400 = 0$$

$$16t^2 = 14,400$$

$$t = 30 \text{ sec}$$

Since $600 \text{ mph} = \frac{1}{6} \text{ mi/sec}$, in 30 seconds the bomb will move horizontally $(\frac{1}{6})(30) = 5$ miles.

38.



$$(a) \ y = x - \frac{32}{v_0^2}x^2 = x\left(1 - \frac{32}{v_0^2}x\right)$$

$$= 0 \text{ if } x = 0 \text{ or } x = \frac{v_0^2}{32}.$$

Projectile strikes the ground when $x = v_0^2/32$.

Projectile reaches its maximum height at $x = v_0^2/64$.
(one-half the distance)

$$(c) \ y = x - \frac{32}{v_0^2}x^2 = x\left(1 - \frac{32}{v_0^2}x\right) = 0$$

when $x = 0$ and $x = v_0^2/32$. Therefore, the range is $x = v_0^2/32$. When the initial velocity is doubled the range is

$$x = \frac{(2v_0)^2}{32} = \frac{4v_0^2}{32}$$

or four times the initial range. From part (a), the maximum height occurs when $x = v_0^2/64$. The maximum height is

$$y\left(\frac{v_0^2}{64}\right) = \frac{v_0^2}{64} - \frac{32}{v_0^2}\left(\frac{v_0^2}{64}\right)^2 = \frac{v_0^2}{64} - \frac{v_0^2}{128} = \frac{v_0^2}{128}.$$

If the initial velocity is doubled, the maximum height is

$$y\left[\frac{(2v_0)^2}{64}\right] = \frac{(2v_0)^2}{128} = 4\left(\frac{v_0^2}{128}\right)$$

or four times the original maximum height.

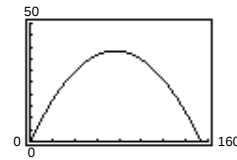
$$(b) \ y' = 1 - \frac{64}{v_0^2}x$$

$$\text{When } x = \frac{v_0^2}{64}, y' = 1 - \frac{64}{v_0^2}\left(\frac{v_0^2}{64}\right) = 0.$$

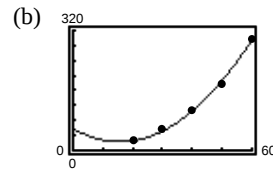
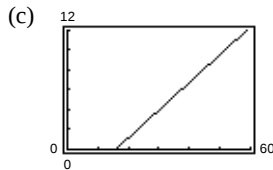
$$(d) \ v_0 = 70 \text{ ft/sec}$$

$$\text{Range: } x = \frac{v_0^2}{32} = \frac{(70)^2}{32} = 153.125 \text{ ft}$$

$$\text{Maximum height: } y = \frac{v_0^2}{128} = \frac{(70)^2}{128} \approx 38.28 \text{ ft}$$



$$40. (a) \ y = 0.14x^2 - 4.43x + 58.4$$



$$(d) \ \text{If } x = 65, y \approx 362 \text{ feet.}$$

(e) As the speed increases, the stopping distance increases at an increasing rate.

$$42. g(x) = (x^3 - 3x)(x + 2)$$

$$\begin{aligned} g'(x) &= (x^3 - 3x)(1) + (x + 2)(3x^2 - 3) \\ &= x^3 - 3x + 3x^3 + 6x^2 - 3x - 6 \\ &= 4x^3 + 6x^2 - 6x - 6 \end{aligned}$$

$$46. f(x) = \frac{x+1}{x-1}$$

$$\begin{aligned} f'(x) &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} \\ &= \frac{-2}{(x-1)^2} \end{aligned}$$

$$50. f(x) = 9(3x^2 - 2x)^{-1}$$

$$f'(x) = -9(3x^2 - 2x)^{-2}(6x - 2) = \frac{18(1 - 3x)}{(3x^2 - 2x)^2}$$

$$54. y = 2x - x^2 \tan x$$

$$y' = 2 - x^2 \sec^2 x - 2x \tan x$$

$$58. v(t) = 36 - t^2, 0 \leq t \leq 6$$

$$a(t) = v'(t) = -2t$$

$$v(4) = 36 - 16 = 20 \text{ m/sec}$$

$$a(4) = -8 \text{ m/sec}$$

$$62. h(t) = 4 \sin t - 5 \cos t$$

$$h'(t) = 4 \cos t + 5 \sin t$$

$$h''(t) = -4 \sin t + 5 \cos t$$

$$66. f(x) = (x^2 - 1)^{1/3}$$

$$\begin{aligned} f'(x) &= \frac{1}{3}(x^2 - 1)^{-2/3}(2x) \\ &= \frac{2x}{3(x^2 - 1)^{2/3}} \end{aligned}$$

$$44. f(t) = t^3 \cos t$$

$$\begin{aligned} f'(t) &= t^3(-\sin t) + \cos t(3t^2) \\ &= -t^3 \sin t + 3t^2 \cos t \end{aligned}$$

$$48. f(x) = \frac{6x - 5}{x^2 + 1}$$

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1)(6) - (6x - 5)(2x)}{(x^2 + 1)^2} \\ &= \frac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2} \end{aligned}$$

$$52. y = \frac{\sin x}{x^2}$$

$$y' = \frac{(x^2) \cos x - (\sin x)(2x)}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}$$

$$56. y = \frac{1 + \sin x}{1 - \sin x}$$

$$\begin{aligned} y' &= \frac{(1 - \sin x) \cos x - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2} \\ &= \frac{2 \cos x}{(1 - \sin x)^2} \end{aligned}$$

$$60. f(x) = 12x^{1/4}$$

$$f'(x) = 3x^{-3/4}$$

$$f''(x) = \frac{-9}{4}x^{-7/4} = \frac{-9}{4x^{7/4}}$$

$$64. y = \frac{(10 - \cos x)}{x}$$

$$xy + \cos x = 10$$

$$xy' + y - \sin x = 0$$

$$xy' = \sin x - y$$

$$xy' + y = (\sin x - y) + y = \sin x$$

$$68. f(x) = \left(x^2 + \frac{1}{x}\right)^5$$

$$f'(x) = 5\left(x^2 + \frac{1}{x}\right)^4\left(2x - \frac{1}{x^2}\right)$$

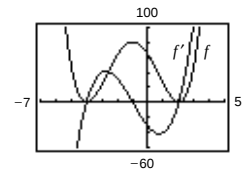
$$\begin{aligned}
 70. \quad h(\theta) &= \frac{\theta}{(1-\theta)^3} \\
 h'(\theta) &= \frac{(1-\theta)^3 - \theta[3(1-\theta)^2(-1)]}{(1-\theta)^6} \\
 &= \frac{(1-\theta)^2(1-\theta+3\theta)}{(1-\theta)^6} = \frac{2\theta+1}{(1-\theta)^4}
 \end{aligned}$$

$$\begin{aligned}
 74. \quad y &= \csc 3x + \cot 3x \\
 y' &= -3 \csc 3x \cot 3x - 3 \csc^2 3x \\
 &= -3 \csc 3x (\cot 3x + \csc 3x)
 \end{aligned}$$

$$\begin{aligned}
 78. \quad f(x) &= \frac{3x}{\sqrt{x^2+1}} \\
 f'(x) &= \frac{3(x^2+1)^{1/2} - 3x \frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1} \\
 &= \frac{3(x^2+1) - 3x^2}{(x^2+1)^{3/2}} \\
 &= \frac{3}{(x^2+1)^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 82. \quad f(x) &= [(x-2)(x+4)]^2 = (x^2+2x-8)^2 \\
 f'(x) &= 4(x^3+3x^2-6x-8) \\
 &= 4(x-2)(x+1)(x+4)
 \end{aligned}$$

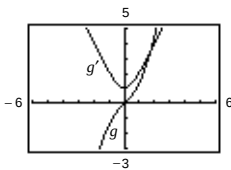
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



$$84. \quad g(x) = x(x^2+1)^{1/2}$$

$$g'(x) = \frac{2x^2+1}{\sqrt{x^2+1}}$$

g' does not equal zero for any value of x . The graph of g has no horizontal tangent lines.



$$\begin{aligned}
 72. \quad y &= 1 - \cos 2x + 2 \cos^2 x \\
 y' &= 2 \sin 2x - 4 \cos x \sin x \\
 &= 2[2 \sin x \cos x] - 4 \sin x \cos x \\
 &= 0
 \end{aligned}$$

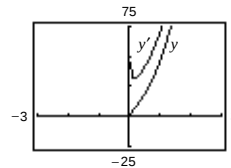
$$\begin{aligned}
 76. \quad y &= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} \\
 y' &= \sec^6 x (\sec x \tan x) - \sec^4 x (\sec x \tan x) \\
 &= \sec^5 x \tan x (\sec^2 x - 1) \\
 &= \sec^5 x \tan^3 x
 \end{aligned}$$

$$\begin{aligned}
 80. \quad y &= \frac{\cos(x-1)}{x-1} \\
 y' &= \frac{-(x-1) \sin(x-1) - \cos(x-1)(1)}{(x-1)^2} \\
 &= -\frac{1}{(x-1)^2} [(x-1) \sin(x-1) + \cos(x-1)]
 \end{aligned}$$

$$86. \quad y = \sqrt{3x}(x+2)^3$$

$$y' = \frac{3(x+2)^2(7x+2)}{2\sqrt{3x}}$$

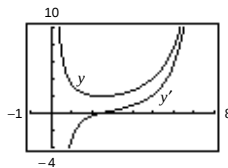
y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



88. $y = 2 \csc^3(\sqrt{x})$

$$y' = -\frac{3}{\sqrt{x}} \csc^3 \sqrt{x} \cot \sqrt{x}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



90. $y = x^{-1} + \tan x$

$$y' = -x^{-2} + \sec^2 x$$

$$y'' = 2x^{-3} + 2 \sec x (\sec x \tan x)$$

$$= \frac{2}{x^3} + 2 \sec^2 x \tan x$$

92. $y = \sin^2 x$

$$y' = 2 \sin x \cos x = \sin 2x$$

$$y'' = 2 \cos 2x$$

94. $g(x) = \frac{6x - 5}{x^2 + 1}$

$$g'(x) = \frac{2(-3x^2 + 5x + 3)}{(x^2 + 1)^2}$$

$$g''(x) = \frac{2(6x^3 - 15x^2 - 18x + 5)}{(x^2 + 1)^3}$$

96. $h(x) = x\sqrt{x^2 - 1}$

$$h'(x) = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$h''(x) = \frac{x(2x^2 - 3)}{(x^2 - 1)^{3/2}}$$

98. $v = \sqrt{2gh} = \sqrt{2(32)h} = 8\sqrt{h}$

$$\frac{dv}{dh} = \frac{4}{\sqrt{h}}$$

(a) When $h = 9$, $\frac{dv}{dh} = \frac{4}{3}$ ft/sec.

(b) When $h = 4$, $\frac{dv}{dh} = 2$ ft/sec.

100. $x^2 + 9y^2 - 4x + 3y = 0$

$$2x + 18yy' - 4 + 3y' = 0$$

$$3(6y + 1)y' = 4 - 2x$$

$$y' = \frac{4 - 2x}{3(6y + 1)}$$

102.

$$y^2 = x^3 - x^2y + xy - y^2$$

$$0 = x^3 - x^2y + xy - 2y^2$$

$$0 = 3x^2 - x^2y' - 2xy + xy' + y - 4yy'$$

$$(x^2 - x + 4y)y' = 3x^2 - 2xy + y$$

$$y' = \frac{3x^2 - 2xy + y}{x^2 - x + 4y}$$

104. $\cos(x + y) = x$

$$-(1 + y') \sin(x + y) = 1$$

$$-y' \sin(x + y) = 1 + \sin(x + y)$$

$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)}$$

$$= -\csc(x + 1) - 1$$

106. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

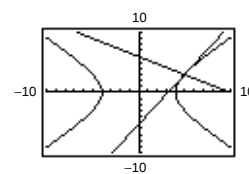
At (5, 3): $y' = \frac{5}{3}$

Tangent line: $y - 3 = \frac{5}{3}(x - 5)$

$$5x - 3y - 16 = 0$$

Normal line: $y - 3 = -\frac{3}{5}(x - 5)$

$$3x + 5y - 30 = 0$$



108. Surface area = $A = 6x^2$, x length of edge.

$$\frac{dx}{dt} = 5$$

$$\frac{da}{dt} = 12x \frac{dx}{dt} = 12(4.5)(5) = 270 \text{ cm}^2/\text{sec}$$

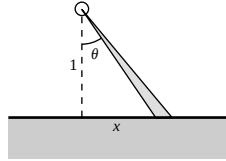
110. $\tan \theta = x$

$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$

$$\sec^2 \theta \left(\frac{d\theta}{dt} \right) = \frac{dx}{dt}$$

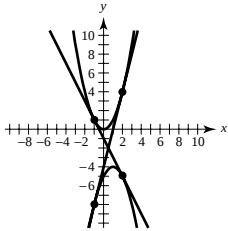
$$\frac{dx}{dt} = (\tan^2 \theta + 1)(6\pi) = 6\pi(x^2 + 1)$$

When $x = \frac{1}{2}$, $\frac{dx}{dt} = 6\pi \left(\frac{1}{4} + 1 \right) = \frac{15\pi}{2} \text{ km/min} = 450\pi \text{ km/hr.}$



Problem Solving for Chapter 2

2.



Let (a, a^2) and $(b, -b^2 + 2b - 5)$ be the points of tangency.

For $y = x^2$, $y' = 2x$ and for $y = -x^2 + 2x - 5$, $y' = -2x + 2$.

Thus, $2a = -2b + 2 \Rightarrow a + b = 1$, or $a = 1 - b$. Furthermore, the slope of the common tangent line is

$$\frac{a^2 - (-b^2 + 2b - 5)}{a - b} = \frac{(1 - b)^2 + b^2 - 2b + 5}{(1 - b) - b} = -2b + 2$$

$$\Rightarrow \frac{1 - 2b + b^2 + b^2 - 2b + 5}{1 - 2b} = -2b + 2$$

$$\Rightarrow 2b^2 - 4b + 6 = 4b^2 - 6b + 2$$

$$\Rightarrow 2b^2 - 2b - 4 = 0$$

$$\Rightarrow b^2 - b - 2 = 0$$

$$\Rightarrow (b - 2)(b + 1) = 0$$

$$b = 2, -1$$

For $b = 2$, $a = 1 - b = -1$ and the points of tangency are $(-1, 1)$, $(2, -5)$. The tangent line has slope -2 :

$$y - 1 = -2(x - 1) \Rightarrow y = -2x + 1$$

For $b = -1$, $a = 1 - b = 2$ and the points of tangency are $(2, 4)$ and $(-1, -8)$. The tangent line has slope 4 :

$$y - 4 = 4(x - 2) \Rightarrow y = 4x - 4.$$

4. (a) $y = x^2$, $y' = 2x$. Slope = 4 at $(2, 4)$.

Tangent line: $y - 4 = 4(x - 2)$

$$y = 4x - 4$$

(b) Slope of normal line: $-\frac{1}{4}$.

Normal line: $y - 4 = -\frac{1}{4}(x - 2)$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$$y = -\frac{1}{4}x + \frac{9}{2} = x^2 \Rightarrow 4x^2 + x - 18 = 0 \Rightarrow (4x + 9)(x - 2) = 0$$

$$x = 2, -\frac{9}{4}. \text{ Second intersection point: } \left(-\frac{9}{4}, \frac{81}{16}\right)$$

(c) Tangent line: $y = 0$

Normal line: $x = 0$

—CONTINUED—

4. —CONTINUED—

(d) Let (a, a^2) , $a \neq 0$, be a point on the parabola $y = x^2$. Tangent line at (a, a^2) is $y = 2a(x - a) + a^2$. Normal line at (a, a^2) is $y = -\frac{1}{2a}(x - a) + a^2$. To find points of intersection, solve

$$x^2 = -\frac{1}{2a}(x - a) + a^2$$

$$x^2 + \frac{1}{2a}x = a^2 + \frac{1}{2}$$

$$x^2 + \frac{1}{2a}x + \frac{1}{16a^2} = a^2 + \frac{1}{2} + \frac{1}{16a^2}$$

$$\left(x + \frac{1}{4a}\right)^2 = \left(a + \frac{1}{4a}\right)^2$$

$$x + \frac{1}{4a} = \pm \left(a + \frac{1}{4a}\right)$$

$$x + \frac{1}{4a} = a + \frac{1}{4a} \Rightarrow x = a \text{ (Point of tangency)}$$

$$x + \frac{1}{4a} = -\left(a + \frac{1}{4a}\right) \Rightarrow x = -a - \frac{1}{2a} = -\frac{2a^2 + 1}{2a}$$

The normal line intersects a second time at $x = -\frac{2a^2 + 1}{2a}$.

6. $f(x) = a + b \cos cx$

$$f'(x) = -bc \sin cx$$

At $(0, 1)$: $a + b = 1$ Equation 1

At $\left(\frac{\pi}{4}, \frac{3}{2}\right)$: $a + b \cos\left(\frac{c\pi}{4}\right) = \frac{3}{2}$ Equation 2

$$-bc \sin\left(\frac{c\pi}{4}\right) = 1 \quad \text{Equation 3}$$

From Equation 1, $a = 1 - b$. Equation 2 becomes $(1 - b) + b \cos\left(\frac{c\pi}{4}\right) = \frac{3}{2} \Rightarrow -b + b \cos\frac{c\pi}{4} = \frac{1}{2}$

From Equation 3, $b = \frac{-1}{c \sin\left(\frac{c\pi}{4}\right)}$. Thus $\frac{1}{c \sin\left(\frac{c\pi}{4}\right)} + \frac{-1}{c \sin\left(\frac{c\pi}{4}\right)} \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2}$

$$1 - \cos\left(\frac{c\pi}{4}\right) = \frac{1}{2} c \sin\left(\frac{c\pi}{4}\right)$$

Graphing the equation $g(c) = \frac{1}{2} c \sin\left(\frac{c\pi}{4}\right) + \cos\left(\frac{c\pi}{4}\right) - 1$, you see that many values of c will work.

One answer: $c = 2$, $b = -\frac{1}{2}$, $a = \frac{3}{2} \Rightarrow f(x) = \frac{3}{2} - \frac{1}{2} \cos 2x$

8. (a) $b^2y^2 = x^3(a - x)$; $a, b > 0$

$$y^2 = \frac{x^3(a - x)}{b^2}$$

$$\text{Graph } y_1 = \frac{\sqrt{x^3(a - x)}}{b} \text{ and } y_2 = -\frac{\sqrt{x^3(a - x)}}{b}$$

- (c) Differentiating implicitly.

$$2b^2yy' = 3x^2(a - x) - x^3 = 3ax^2 - 4x^3$$

$$y' = \frac{(3ax^2 - 4x^3)}{2b^2y} = 0$$

$$\Rightarrow 3ax^2 = 4x^3$$

$$3a = 4x$$

$$x = \frac{3a}{4}$$

$$b^2y^2 = \left(\frac{3a}{4}\right)^3 \left(a - \frac{3a}{4}\right) = \frac{27a^3}{64} \left(\frac{1}{4}a\right)$$

$$y^2 = \frac{27a^4}{256b^2} \Rightarrow y = \pm \frac{3\sqrt{3}a^2}{16b}$$

$$\text{Two points: } \left(\frac{3a}{4}, \frac{3\sqrt{3}a^2}{16b}\right), \left(\frac{3a}{4}, -\frac{3\sqrt{3}a^2}{16b}\right)$$

10. (a) $y = x^{1/3} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^{-2/3} \frac{dx}{dt}$

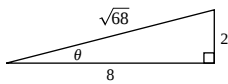
$$1 = \frac{1}{3}(8)^{-2/3} \frac{dx}{dt}$$

$$\frac{dx}{dt} = 12 \text{ cm/sec}$$

- (b) $D = \sqrt{x^2 + y^2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(x^2 + y^2) \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt}\right)$

$$\begin{aligned} &= \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{\sqrt{x^2 + y^2}} \\ &= \frac{8(12) + 2(1)}{\sqrt{64 + 4}} = \frac{98}{\sqrt{68}} = \frac{49}{\sqrt{17}} \text{ cm/sec.} \end{aligned}$$

- (c) $\tan \theta = \frac{y}{x} \Rightarrow \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2}$



$$\text{From the triangle, } \sec \theta = \frac{\sqrt{68}}{8}. \text{ Hence } \frac{d\theta}{dt} = \frac{8(1) - 2(12)}{64 \left(\frac{68}{64}\right)} = \frac{-16}{68} = \frac{-4}{17} \text{ rad/sec}$$

- (b) a determines the x -intercept on the right: $(a, 0)$.

b affects the height.

$$\begin{aligned} 12. E'(x) &= \lim_{\Delta x \rightarrow 0} \frac{E(x + \Delta x) - E(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{E(x)E(\Delta x) - E(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} E(x) \left(\frac{E(\Delta x) - 1}{\Delta x} \right) \\ &= E(x) \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - 1}{\Delta x} \end{aligned}$$

$$\text{But, } E'(0) = \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - E(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{E(\Delta x) - 1}{\Delta x} = 1.$$

Thus, $E'(x) = E(x)E'(0) = E(x)$ exists for all x .

For example: $E(x) = e^x$.

$$14. (a) v(t) = -\frac{27}{5}t + 27 \text{ ft/sec}$$

$$a(t) = -\frac{27}{5} \text{ ft/sec}^2$$

$$(b) v(t) = -\frac{27}{5}t + 27 = 0 \Rightarrow \frac{27}{5}t = 27 \Rightarrow t = 5 \text{ seconds}$$

$$S(5) = -\frac{27}{10}(5)^2 + 27(5) + 6 = 73.5 \text{ feet}$$

(c) The acceleration due to gravity on Earth is greater in magnitude than that on the moon.

CHAPTER 3

Applications of Differentiation

Section 3.1	Extrema on an Interval	103
Section 3.2	Rolle's Theorem and the Mean Value Theorem	107
Section 3.3	Increasing and Decreasing Functions and the First Derivative Test	113
Section 3.4	Concavity and the Second Derivative Test	121
Section 3.5	Limits at Infinity	129
Section 3.6	A Summary of Curve Sketching	136
Section 3.7	Optimization Problems	145
Section 3.8	Newton's Method	155
Section 3.9	Differentials	160
Review Exercises		163
Problem Solving		172

CHAPTER 3

Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Odd-Numbered Exercises

$$1. f(x) = \frac{x^2}{x^2 + 4}$$

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

$$f'(0) = 0$$

$$5. f(x) = (x + 2)^{2/3}$$

$$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$$

$f'(-2)$ is undefined.

$$9. \text{Critical numbers: } x = 1, 2, 3$$

$x = 1, 3$: absolute maximum

$x = 2$: absolute minimum

$$13. g(t) = t\sqrt{4-t}, t < 3$$

$$g'(t) = t\left[\frac{1}{2}(4-t)^{-1/2}(-1)\right] + (4-t)^{1/2}$$

$$= \frac{1}{2}(4-t)^{-1/2}[-t + 2(4-t)]$$

$$= \frac{8-3t}{2\sqrt{4-t}}$$

Critical number is $t = \frac{8}{3}$.

$$17. f(x) = 2(3-x), [-1, 2]$$

$$f'(x) = -2 \Rightarrow \text{No critical numbers}$$

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

$$3. f(x) = x + \frac{27}{2x^2} = x + \frac{27}{x}x^{-2}$$

$$f'(x) = 1 - 27x^{-3} = 1 - \frac{27}{x^3}$$

$$f'(3) = 1 - \frac{27}{3^3} = 1 - 1 = 0$$

$$7. \text{Critical numbers: } x = 2$$

$x = 2$: absolute maximum

$$11. f(x) = x^2(x-3) = x^3 - 3x^2$$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Critical numbers: $x = 0, x = 2$

$$15. h(x) = \sin^2 x + \cos x, 0 < x < 2\pi$$

$$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$$

On $(0, 2\pi)$, critical numbers: $x = \frac{\pi}{3}, x = \pi, x = \frac{5\pi}{3}$

$$19. f(x) = -x^2 + 3x, [0, 3]$$

$$f'(x) = -2x + 3$$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

21. $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$$f'(x) = 3x^2 - 3x = 3x(x - 1)$$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

25. $g(t) = \frac{t^2}{t^2 + 3}, [-1, 1]$

$$g'(t) = \frac{6t}{(t^2 + 3)^2}$$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

29. $f(x) = \cos \pi x, \left[0, \frac{1}{6}\right]$

$$f'(x) = -\pi \sin \pi x$$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\frac{1}{6}, \frac{\sqrt{3}}{2}\right)$ Minimum

33. (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

23. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

27. $h(s) = \frac{1}{s - 2}, [0, 1]$

$$h'(s) = \frac{-1}{(s - 2)^2}$$

Left endpoint: $\left(0, -\frac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

31. $y = \frac{4}{x} + \tan \frac{\pi x}{8}, [1, 2]$

$$y' = \frac{-4}{x^2} + \frac{\pi}{8} \sec^2 \frac{\pi x}{8} = 0$$

$$\frac{\pi}{8} \sec^2 \frac{\pi x}{8} = \frac{4}{x^2}$$

On the interval $[1, 2]$, this equation has no solutions. Thus, there are no critical numbers.

Left endpoint: $(1, \sqrt{2} + 3) \approx (1, 4.4142)$ Maximum

Right endpoint: $(2, 3)$ Minimum

35. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

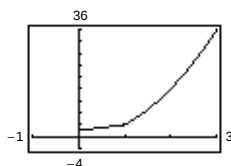
(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

$$37. f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ 4x^2, & 1 < x \leq 3 \end{cases}$$

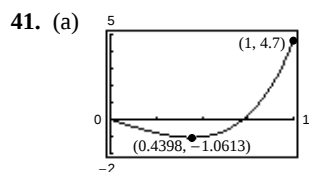
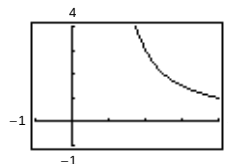
Left endpoint: (0, 2) Minimum

Right endpoint: (3, 36) Maximum



$$39. f(x) = \frac{3}{x-1}, (1, 4]$$

Right endpoint: (4, 1) Minimum



Maximum: (1, 4.7) (endpoint)

Minimum: (0.4398, -1.0613)

(b) $f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum (endpoint)}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: (0.4398, -1.0613)

$$43. f(x) = (1 + x^3)^{1/2}, [0, 2]$$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 + \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$$\left| f''\left(\sqrt[3]{-10 + \sqrt{108}}\right) \right| \approx 1.47 \text{ is the maximum value.}$$

$$45. f(x) = (x + 1)^{2/3}, [0, 2]$$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

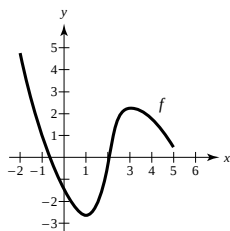
$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$$|f^{(4)}(0)| = \frac{56}{81} \text{ is the maximum value.}$$

47. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\lim_{x \rightarrow \pi/2^-} \tan x = \infty$.

49.



51. (a) Yes

(b) No

53. (a) No

(b) Yes

55. $P = VI - RI^2 = 12I - 0.5I^2, 0 \leq I \leq 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ ampsWhen $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output.

 P is decreasing for $I > 12$.

57.

$$S = 6hs + \frac{3s^2}{2} \left(\frac{\sqrt{3} - \cos \theta}{\sin \theta} \right), \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2} (-\sqrt{3} \csc \theta \cot \theta + \csc^2 \theta)$$

$$= \frac{3s^2}{2} \csc \theta (-\sqrt{3} \cot \theta + \csc \theta) = 0$$

$\csc \theta = \sqrt{3} \cot \theta$

$\sec \theta = \sqrt{3}$

$\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$ radians

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2} (\sqrt{3})$$

$$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2} (\sqrt{3})$$

$$S(\operatorname{arcsec} \sqrt{3}) = 6hs + \frac{3s^2}{2} (\sqrt{2})$$

 S is minimum when $\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$ radians.

59. (a) $y = ax^2 + bx + c$

$y' = 2ax + b$

The coordinates of B are $(500, 30)$, and those of A are $(-500, 45)$.From the slopes at A and B ,

$-1000a + b = -0.09$

$1000a + b = 0.06$.

Solving these two equations, you obtain $a = 3/40000$ and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

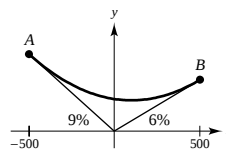
$$30 = \frac{3}{40000} 500^2 + 500 \left(\frac{-3}{200} \right) + c$$

$$45 = \frac{3}{40000} 500^2 - 500 \left(\frac{-3}{200} \right) + c.$$

In both cases, $c = 18.75 = \frac{75}{4}$. Thus,

$$y = \frac{3}{40000} x^2 - \frac{3}{200} x + \frac{75}{4}.$$

—CONTINUED—



59. —CONTINUED—

(b)

x	-500	-400	-300	-200	-100	0	100	200	300	400	500
d	0	.75	3	6.75	12	18.75	12	6.75	3	.75	0

For $-500 \leq x \leq 0$, $d = (ax^2 + bx + c) - (-0.09x)$.For $0 \leq x \leq 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is (100, 18), which is not directly over the point where the two hillsides come together.

61. True. See Exercise 25.

63. True.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since f is not differentiable at $x = 1$.

3. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x -intercepts: $(-1, 0)$, $(2, 0)$

$$f'(x) = 2x - 1 = 0 \text{ at } x = \frac{1}{2}.$$

5. $f(x) = x\sqrt{x+4}$

x -intercepts: $(-4, 0)$, $(0, 0)$

$$f'(x) = x \frac{1}{2}(x+4)^{-1/2} + (x+4)^{1/2}$$

$$= (x+4)^{-1/2} \left(\frac{x}{2} + (x+4) \right)$$

$$f'(x) = \left(\frac{3}{2}x + 4 \right) (x+4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

7. $f(x) = x^2 - 2x$, $[0, 2]$

$$f(0) = f(2) = 0$$

f is continuous on $[0, 2]$. f is differentiable on $(0, 2)$.
Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Rightarrow x = 1$$

c value: 1

9. $f(x) = (x-1)(x-2)(x-3)$, $[1, 3]$

$$f(1) = f(3) = 0$$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$.
Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$$

11. $f(x) = x^{2/3} - 1$, $[-8, 8]$

$$f(-8) = f(8) = 3$$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

13. $f(x) = \frac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$$f(-1) = f(3) = 0$$

f is continuous on $[-1, 3]$. (**Note:** The discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x-3)(1)}{(x+2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x+2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c value: $-2 + \sqrt{5}$

15. $f(x) = \sin x, [0, 2\pi]$

$$f(0) = f(2\pi) = 0$$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = \cos x$$

c values: $\frac{\pi}{2}, \frac{3\pi}{2}$

17. $f(x) = \frac{6x}{\pi} - 4 \sin^2 x, \left[0, \frac{\pi}{6}\right]$

$$f(0) = f\left(\frac{\pi}{6}\right) = 0$$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8 \sin x \cos x = 0$$

$$\frac{6}{\pi} = 8 \sin x \cos x$$

$$\frac{3}{4\pi} = \frac{1}{2} \sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2} \arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

c value: 0.2489

19. $f(x) = \tan x, [0, \pi]$

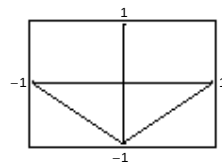
$$f(0) = f(\pi) = 0$$

f is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

21. $f(x) = |x| - 1, [-1, 1]$

$$f(-1) = f(1) = 0$$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



$$23. \quad f(x) = 4x - \tan \pi x, \quad \left[-\frac{1}{4}, \frac{1}{4}\right]$$

$$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$$

f is continuous on $[-1/4, 1/4]$. f is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

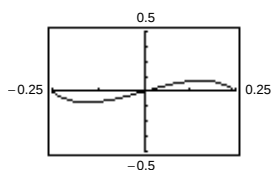
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm \frac{2}{\sqrt{\pi}}$$

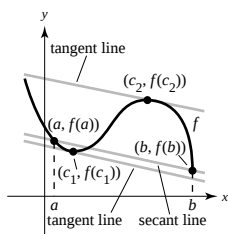
$$x = \pm \frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm \frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

c values: ± 0.1533 radian



27.



31. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\frac{1}{2}$. Therefore,

$$c = -\frac{1}{2}$$

$$25. \quad f(t) = -16t^2 + 48t + 32$$

$$(a) \quad f(1) = f(2) = 64$$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

$$29. \quad f(x) = \frac{1}{x-3}, \quad [0, 6]$$

f has a discontinuity at $x = 3$.

33. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

35. $f(x) = \sqrt{2-x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2-x}} = -\frac{1}{3}$$

$$2\sqrt{2-x} = 3$$

$$\sqrt{2-x} = \frac{3}{2}$$

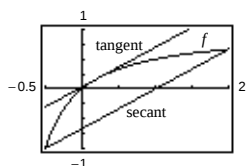
$$2-x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

39. $f(x) = \frac{x}{x+1}$ on $\left[-\frac{1}{2}, 2\right]$.

(a)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-2/3)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

37. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

$$(c) f'(x) = \frac{1}{(x+1)^2} = \frac{2}{3}$$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + (\sqrt{6}/2)$.

$$f(c) = \frac{-1 + (\sqrt{6}/2)}{[-1 + (\sqrt{6}/2)] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

$$\text{Tangent line: } y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}\left(x - \frac{\sqrt{6}}{2} + 1\right)$$

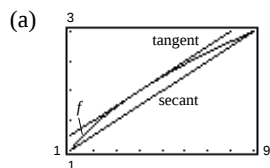
$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$3y - 2x - 5 + 2\sqrt{6} = 0$$

41. $f(x) = \sqrt{x}$, $[1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$



(b) Secant line: $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $f'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

43. $s(t) = -4.9t^2 + 500$

(a) $V_{\text{avg}} = \frac{s(3) - s(0)}{3 - 0} = \frac{455.9 - 500}{3} = -14.7 \text{ m/sec}$

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$.
Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

45. No. Let $f(x) = x^2$ on $[-1, 2]$.

$$f'(x) = 2x$$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but
 $f(-1) \neq f(2)$.

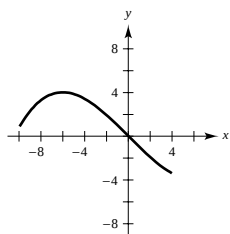
47. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0 , $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. ($0 < 400 < 454.54$)

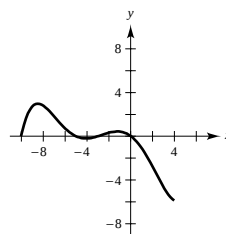
49. (a) f is continuous on $[-10, 4]$ and changes sign, ($f(-8) > 0$, $f(3) < 0$). By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(c)



- (b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

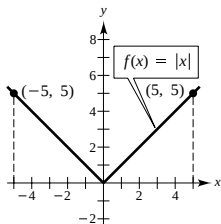
(d)



(e) No, f' did not have to be continuous on $[-10, 4]$.

51. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem.
 $\Rightarrow f$ is not differentiable on $(-5, 5)$.

Example: $f(x) = |x|$



53. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

55. True. A polynomial is continuous and differentiable everywhere.

57. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2 . Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n+1)x^{2n} + a \neq 0$, since $n > 0$, $a > 0$. Therefore, $p(x)$ cannot have two real roots.

59. If $p(x) = Ax^2 + Bx + C$, then

$$\begin{aligned} p'(x) &= 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

61. $f(x) = \frac{1}{2} \cos x$ differentiable on $(-\infty, \infty)$.

$$f'(x) = -\frac{1}{2} \sin x$$

$$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1 \text{ for all real numbers.}$$

Thus, from Exercise 60, f has, at most, one fixed point. ($x \approx 0.4502$)

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$ Decreasing on: $(-\infty, 3)$

5. $f(x) = \frac{1}{x^2} = x^{-2}$

$$f'(x) = \frac{-2}{x^3}$$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$

9. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y' :	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-2\sqrt{2}, 2\sqrt{2})$ Decreasing on $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

11. $f(x) = x^2 - 6x$

$$f'(x) = 2x - 6 = 0$$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(3, \infty)$ Decreasing on: $(-\infty, 3)$ Relative minimum: $(3, -9)$

3. $y = \frac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2), (2, \infty)$ Decreasing on: $(-2, 2)$

7. $g(x) = x^2 - 2x - 8$

$$g'(x) = 2x - 2$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$ Decreasing on: $(-\infty, 1)$

13. $f(x) = -2x^2 + 4x + 3$

$$f'(x) = -4x + 4 = 0$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$ Decreasing on: $(1, \infty)$ Relative maximum: $(1, 5)$

15. $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$ Decreasing on: $(-2, 1)$ Relative maximum: $(-2, 20)$ Relative minimum: $(1, -7)$

17. $f(x) = x^2(3 - x) = 3x^2 - x^3$

$$f'(x) = 6x - 3x^2 = 3x(2 - x)$$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(0, 2)$ Decreasing on: $(-\infty, 0), (2, \infty)$ Relative maximum: $(2, 4)$ Relative minimum: $(0, 0)$

19. $f(x) = \frac{x^5 - 5x}{5}$

$$f'(x) = x^4 - 1$$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$ Decreasing on: $(-1, 1)$ Relative maximum: $(-1, \frac{4}{5})$ Relative minimum: $(1, -\frac{4}{5})$

21. $f(x) = x^{1/3} + 1$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

23. $f(x) = (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3(x - 1)^{1/3}}$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$ Decreasing on: $(-\infty, 1)$ Relative minimum: $(1, 0)$

25. $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x - 5}{|x - 5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$ Decreasing on: $(5, \infty)$ Relative maximum: $(5, 5)$

27. $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Critical numbers: $x = -1, 1$ Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$ Decreasing on: $(-1, 0), (0, 1)$ Relative maximum: $(-1, -2)$ Relative minimum: $(1, 2)$

29. $f(x) = \frac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

31. $f(x) = \frac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

33. $f(x) = \frac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$

Relative maximum: $\left(\frac{\pi}{6}, \frac{\pi + 6\sqrt{3}}{12}\right)$

Decreasing on: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

Relative minimum: $\left(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12}\right)$

35. $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

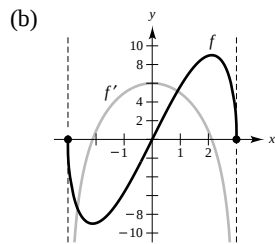
Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

37. $f(x) = 2x\sqrt{9-x^2}, [-3, 3]$

(a) $f'(x) = \frac{2(9-2x^2)}{\sqrt{9-x^2}}$



(c) $\frac{2(9-2x^2)}{\sqrt{9-x^2}} = 0$

Critical numbers: $x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$

(d) Intervals:

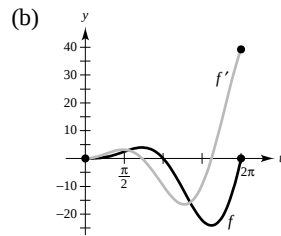
$$\left(-3, -\frac{3\sqrt{2}}{2}\right) \quad \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) \quad \left(\frac{3\sqrt{2}}{2}, 3\right)$$

$$\begin{array}{ccc} f'(x) < 0 & f'(x) > 0 & f'(x) < 0 \\ \text{Decreasing} & \text{Increasing} & \text{Decreasing} \end{array}$$

f is increasing when f' is positive and decreasing when f' is negative.

39. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$
 $= t(t \cos t + 2 \sin t)$



(c) $t(t \cos t + 2 \sin t) = 0$

$$t = 0 \text{ or } t = -2 \tan t$$

$$t \cot t = -2$$

$$t \approx 2.2889, 5.0870 \text{ (graphing utility)}$$

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

$$(0, 2.2889) \quad (2.2889, 5.0870) \quad (5.0870, 2\pi)$$

$$\begin{array}{ccc} f'(t) > 0 & f'(t) < 0 & f'(t) > 0 \\ \text{Increasing} & \text{Decreasing} & \text{Increasing} \end{array}$$

f is increasing when f' is positive and decreasing when f' is negative.

$$41. f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1} = \frac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, x \neq \pm 1$$

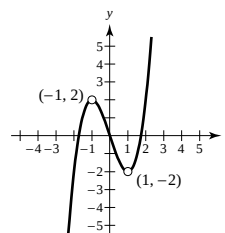
$$f(x) = g(x) = x^3 - 3x \text{ for all } x \neq \pm 1.$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \quad f'(x) \neq 0$$

f symmetric about origin

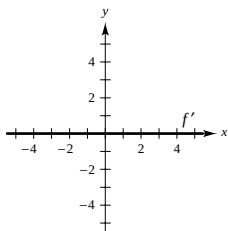
zeros of f : $(0, 0), (\pm\sqrt{3}, 0)$

No relative extrema

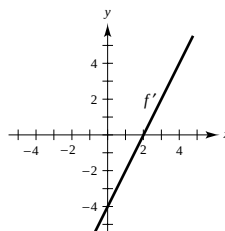


Holes at $(-1, 2)$ and $(1, -2)$

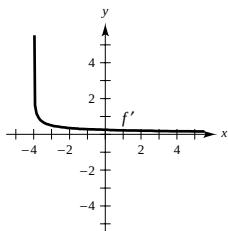
$$43. f(x) = c \text{ is constant} \Rightarrow f'(x) = 0$$



$$45. f \text{ is quadratic} \Rightarrow f' \text{ is a line.}$$



$$47. f \text{ has positive, but decreasing slope}$$



In Exercises 49–53, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

$$49. g(x) = f(x) + 5$$

$$g'(x) = f'(x)$$

$$g'(0) = f'(0) < 0$$

$$51. g(x) = -f(x)$$

$$g'(x) = -f'(x)$$

$$g'(-6) = -f'(-6) < 0$$

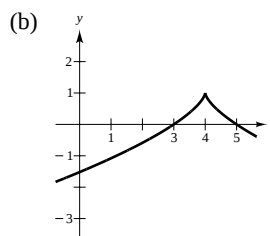
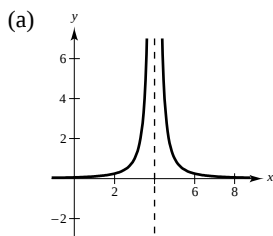
$$53. g(x) = f(x - 10)$$

$$g'(x) = f'(x - 10)$$

$$g'(0) = f'(-10) > 0$$

$$55. f'(x) = \begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$$

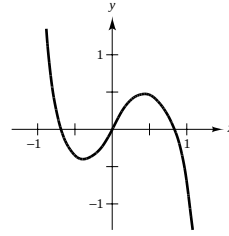
Two possibilities for $f(x)$ are given below.



57. The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

Relative minimum when $x \approx -0.40$.

Relative maximum when $x \approx 0.48$.

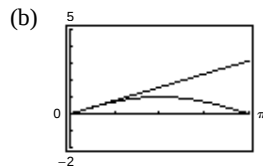


59. $f(x) = x$, $g(x) = \sin x$, $0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.



$x > \sin x$ on $(0, \pi)$

- (c) Let $h(x) = f(x) - g(x) = x - \sin x$

$$h'(x) = 1 - \cos x > 0 \text{ on } (0, \pi).$$

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$$x - \sin x > 0$$

$$x > \sin x$$

$$f(x) > g(x) \text{ on } (0, \pi).$$

61. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$$v' = k(2Rr - 3r^2)$$

$$= kr(2R - 3r) = 0$$

$$r = 0 \text{ or } \frac{2}{3}R$$

Maximum when $r = \frac{2}{3}R$.

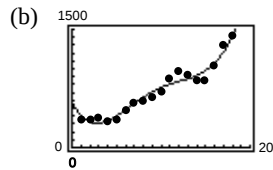
63. $P = \frac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

65. (a) $B = 0.1198t^4 - 4.4879t^3 + 56.9909t^2 - 223.0222t + 579.9541$



- (c) $B' = 0$ for $t \approx 2.78$, or 1983, (311.1 thousand bankruptcies)
Actual minimum: 1984 (344.3 thousand bankruptcies)

67. (a) Use a cubic polynomial
 $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

$(0, 0): \quad 0 = a_0 \quad (f(0) = 0)$

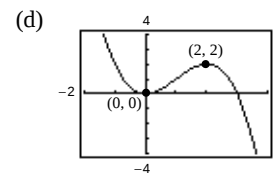
$0 = a_1 \quad (f'(0) = 0)$

$(2, 2): \quad 2 = 8a_3 + 4a_2 \quad (f(2) = 2)$

$0 = 12a_3 + 4a_2 \quad (f'(2) = 0)$

- (c) The solution is $a_0 = a_1 = 0$, $a_2 = \frac{3}{2}$, $a_3 = -\frac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$



69. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(0, 0): 0 = a_0 \quad (f(0) = 0)$

$0 = a_1 \quad (f'(0) = 0)$

$(4, 0): 0 = 256a_4 + 64a_3 + 16a_2 \quad (f(4) = 0)$

$0 = 256a_4 + 48a_3 + 8a_2 \quad (f'(4) = 0)$

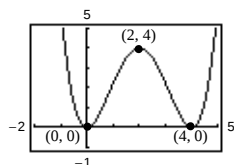
$(2, 4): 4 = 16a_4 + 8a_3 + 4a_2 \quad (f(2) = 4)$

$0 = 32a_4 + 12a_3 + 4a_2 \quad (f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, a_4 = \frac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



71. True

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

73. False

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

75. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

77. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

79. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} - n \\ &= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1. \end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Rightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Rightarrow (1 + x)^n > 1 + nx.$$

Section 3.4 Concavity and the Second Derivative Test

1. $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

3. $f(x) = \frac{24}{x^2 + 12}, y'' = \frac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

5. $f(x) = \frac{x^2 + 1}{x^2 - 1}, y'' = \frac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

7. $f(x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2$

$f''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

9. $y = 2x - \tan x, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2 \sec^2 x \tan x$

Concave upward: $\left(-\frac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \frac{\pi}{2}\right)$

11. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

The concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

13. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm \frac{2}{\sqrt{3}}$.

Test interval:	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(\pm \frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

15. $f(x) = x(x - 4)^3$

$$f'(x) = x[3(x - 4)^2] + (x - 4)^3$$

$$= (x - 4)^2(4x - 4)$$

$$f''(x) = 4(x - 1)[2(x - 4)] + 4(x - 4)^2$$

$$= 4(x - 4)[2(x - 1) + (x - 4)]$$

$$= 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$$

$$f''(x) = 12(x - 4)(x - 2) = 0 \text{ when } x = 2, 4.$$

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16), (4, 0)$

17. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

 $f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection.Concave upward on $(-3, \infty)$

19. $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}$$

Test intervals:	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

21. $f(x) = \sin\left(\frac{x}{2}\right), 0 \leq x \leq 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$$f''(x) = 0 \text{ when } x = 0, 2\pi, 4\pi.$$

Point of inflection: $(2\pi, 0)$

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

23. $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$$f'(x) = \sec\left(x - \frac{\pi}{2}\right) \tan\left(x - \frac{\pi}{2}\right)$$

$$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right) \tan^2\left(x - \frac{\pi}{2}\right) \neq 0 \text{ for any } x \text{ in the domain of } f.$$

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No points of inflection

25. $f(x) = 2 \sin x + \sin 2x, 0 \leq x \leq 2\pi$

$$f'(x) = 2 \cos x + 2 \cos 2x$$

$$f''(x) = -2 \sin x - 4 \sin 2x = -2 \sin x(1 + 4 \cos x)$$

$$f''(x) = 0 \text{ when } x = 0, 1.823, \pi, 4.460.$$

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

27. $f(x) = x^4 - 4x^3 + 2$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

31. $f(x) = x^3 - 3x^2 + 3$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

Critical numbers: $x = 0, x = 2$

$$f''(0) = -6 < 0$$

Therefore, $(0, 3)$ is a relative maximum.

$$f''(2) = 6 > 0$$

Therefore, $(2, -1)$ is a relative minimum.

29. $f(x) = (x - 5)^2$

$$f'(x) = 2(x - 5)$$

$$f''(x) = 2$$

Critical number: $x = 5$

$$f''(5) > 0$$

Therefore, $(5, 0)$ is a relative minimum.

33. $g(x) = x^2(6 - x)^3$

$$g'(x) = x(x - 6)^2(12 - 5x)$$

$$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$$g''(0) = 432 > 0$$

Therefore, $(0, 0)$ is a relative minimum.

$$g''\left(\frac{12}{5}\right) = -155.52 < 0$$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative minimum.

$$g''(6) = 0$$

Test fails by the First Derivative Test, $(6, 0)$ is not an extremum.

35. $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

39. $f(x) = \cos x - x$, $0 \leq x \leq 4\pi$

$$f'(x) = -\sin x - 1 \leq 0$$

Therefore, f is non-increasing and there are no relative extrema.

41. $f(x) = 0.2x^2(x - 3)^3$, $[-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$$\begin{aligned} f''(x) &= (x - 3)(4x^2 - 9.6x + 3.6) \\ &= 0.4(x - 3)(10x^2 - 24x + 9) \end{aligned}$$

(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$$
 is a relative minimum.

Points of inflection:

$$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$$

43. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$, $[0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$$f'(x) = 0 \text{ when } x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}.$$

$$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$$

$$f''(x) = 0 \text{ when } x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$$

(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

37. $f(x) = x + \frac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

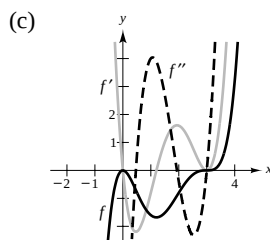
Critical numbers: $x = \pm 2$

$$f''(-2) < 0$$

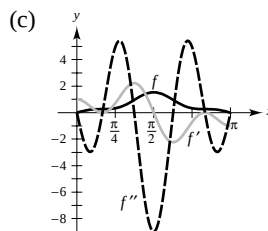
Therefore, $(-2, -4)$ is a relative maximum.

$$f''(2) > 0$$

Therefore, $(2, 4)$ is a relative minimum.

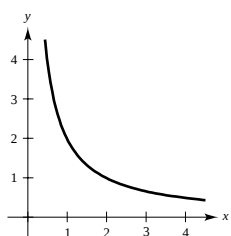


f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.



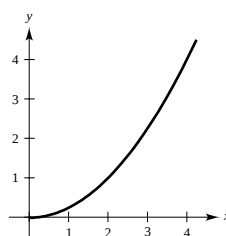
The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

45. (a)


 $f' < 0$ means f decreasing

 f' increasing means
concave upward

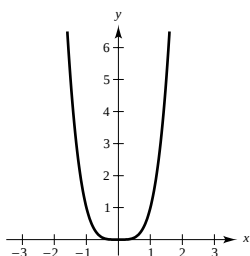
(b)


 $f' > 0$ means f increasing

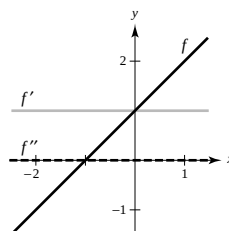
 f' increasing means
concave upward

 47. Let $f(x) = x^4$.

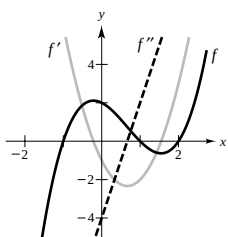
$$f''(x) = 12x^2$$

 $f''(0) = 0$, but $(0, 0)$ is not a point of inflection.


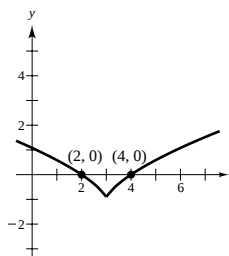
49.



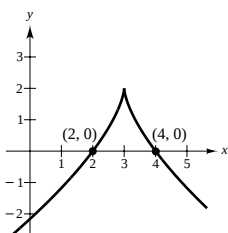
51.



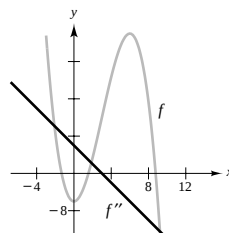
53.



55.



57.


 f'' is linear.

 f' is quadratic.

 f is cubic.

 f concave upwards on $(-\infty, 3)$, downward on $(3, \infty)$.

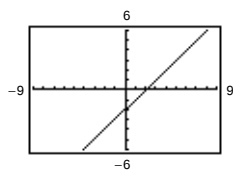
59. (a) $n = 1$:

$$f(x) = x - 2$$

$$f'(x) = 1$$

$$f''(x) = 0$$

No inflection points

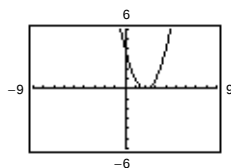
 $n = 2$:

$$f(x) = (x - 2)^2$$

$$f'(x) = 2(x - 2)$$

$$f''(x) = 2$$

No inflection points

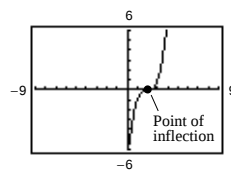
Relative minimum:
(2, 0) $n = 3$:

$$f(x) = (x - 2)^3$$

$$f'(x) = 3(x - 2)^2$$

$$f''(x) = 6(x - 2)$$

Inflection point: (2, 0)

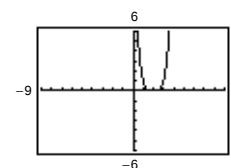
 $n = 4$:

$$f(x) = (x - 2)^4$$

$$f'(x) = 4(x - 2)^3$$

$$f''(x) = 12(x - 2)^2$$

No inflection points:

Relative minimum:
(2, 0)

Conclusion: If $n \geq 3$ and n is odd, then (2, 0) is an inflection point. If $n \geq 2$ and n is even, then (2, 0) is a relative minimum.

(b) Let $f(x) = (x - 2)^n$, $f'(x) = n(x - 2)^{n-1}$, $f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

Thus, $x = 2$ is an inflection point if and only if $n \geq 3$ is odd.

61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: (3, 3)

Relative minimum: (5, 1)

Point of inflection: (4, 2)

$$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$$

$$\left. \begin{array}{l} f(3) = 27a + 9b + 3c + d = 3 \\ f(5) = 125a + 25b + 5c + d = 1 \end{array} \right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$$f'(3) = 27a + 6b + c = 0, f''(4) = 24a + 2b = 0$$

$$49a + 8b + c = -1 \quad 24a + 2b = 0$$

$$27a + 6b + c = 0 \quad 22a + 2b = -1$$

$$22a + 2b = -1 \quad 2a = 1$$

$$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$$

$$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b$

$$f(0) = 0 \Rightarrow d = 0$$

$$f(-4) = 1 \Rightarrow -64a + 16b - 4c = 1$$

$$f'(-4) = 0 \Rightarrow 48a - 8b + c = 0$$

$$f'(0) = 0 \Rightarrow c = 0$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

Two miles from touchdown.

65. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

67. $C = 0.5x^2 + 15x + 5000$

$$\bar{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\bar{C}$ = average cost per unit

$$\frac{d\bar{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\bar{C}$ is minimized when $x = 100$ units.

69. $S = \frac{5000t^2}{8 + t^2}$

$$S'(t) = \frac{80,000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80,000(8 - 3t^2)}{(8 + t^2)^3}$$

$$S''(t) = 0 \text{ for } t = \sqrt{8/3} \approx 1.633.$$

Sales are increasing at the greatest rate at $t = 1.633$ years.

71. $f(x) = 2(\sin x + \cos x)$, $f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$f'(x) = 2(\cos x - \sin x), \quad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$$

$$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$$

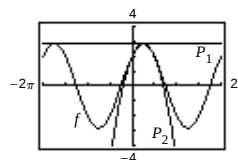
$$P_1'(x) = 0$$

$$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}(-2\sqrt{2})\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2''(x) = -2\sqrt{2}$$

The values of f , P_1 , P_2 , and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.



$$73. \quad f(x) = \sqrt{1-x}, \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2\sqrt{1-x}}, \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1-x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

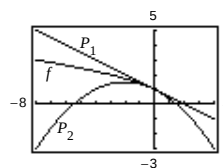
$$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x-0) = 1 - \frac{x}{2}$$

$$P_1'(x) = -\frac{1}{2}$$

$$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x-0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x-0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$$

$$P_2''(x) = -\frac{1}{4}$$



The values of f , P_1 , P_2 , and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

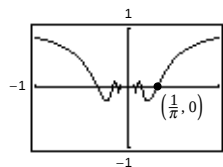
$$75. \quad f(x) = x \sin\left(\frac{1}{x}\right)$$

$$f'(x) = x \left[-\frac{1}{x^2} \cos\left(\frac{1}{x}\right) \right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x} \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x} \left[\frac{1}{x^2} \sin\left(\frac{1}{x}\right) \right] + \frac{1}{x^2} \cos\left(\frac{1}{x}\right) - \frac{1}{x^2} \cos\left(\frac{1}{x}\right) = -\frac{1}{x^3} \sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

$$\text{Point of inflection: } \left(\frac{1}{\pi}, 0\right)$$



When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

77. Assume the zeros of f are all real. Then express the function as $f(x) = a(x - r_1)(x - r_2)(x - r_3)$ where r_1 , r_2 , and r_3 are the distinct zeros of f . From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x - r_1)(x - r_2) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]$$

$$f''(x) = a[(x - r_1) + (x - r_2) + (x - r_1) + (x - r_3) + (x - r_2) + (x - r_3)]$$

$$= a[6x - 2(r_1 + r_2 + r_3)].$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = (\text{Average of } r_1, r_2, \text{ and } r_3).$$

79. True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

81. False.

$$f(x) = 3 \sin x + 2 \cos x$$

$$f'(x) = 3 \cos x - 2 \sin x$$

$$3 \cos x - 2 \sin x = 0$$

$$3 \cos x = 2 \sin x$$

$$\frac{3}{2} = \tan x$$

$$\text{Critical number: } x = \tan^{-1}\left(\frac{3}{2}\right)$$

$$f\left(\tan^{-1}\frac{3}{2}\right) \approx 3.60555 \text{ is the maximum value of } y.$$

 83. False. Concavity is determined by f'' .

Section 3.5 Limits at Infinity

1. $f(x) = \frac{3x^2}{x^2 + 2}$

No vertical asymptotes

 Horizontal asymptote: $y = 3$

Matches (f)

3. $f(x) = \frac{x}{x^2 + 2}$

No vertical asymptotes

 Horizontal asymptote: $y = 0$

Matches (d)

5. $f(x) = \frac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

 Horizontal asymptotes: $y = 0$

Matches (b)

7. $f(x) = \frac{4x + 3}{2x - 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2

$$\lim_{x \rightarrow \infty} f(x) = 2$$

9. $f(x) = \frac{-6x}{\sqrt{4x^2 + 5}}$

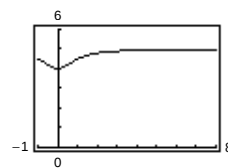
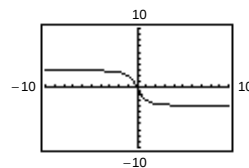
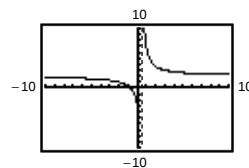
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

$$\lim_{x \rightarrow \infty} f(x) = -3$$

11. $f(x) = 5 - \frac{1}{x^2 + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

$$\lim_{x \rightarrow \infty} f(x) = 5$$



$$13. (a) h(x) = \frac{f(x)}{x^2} = \frac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \frac{10}{x^2}$$

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad (\text{Limit does not exist})$$

$$(b) h(x) = \frac{f(x)}{x^3} = \frac{5x^3 - 3x^2 + 10}{x^3} = 5 - \frac{3}{x} + \frac{10}{x^3}$$

$$\lim_{x \rightarrow \infty} h(x) = 5$$

$$(c) h(x) = \frac{f(x)}{x^4} = \frac{5x^3 - 3x^2 + 10}{x^4} = \frac{5}{x} - \frac{3}{x^2} + \frac{10}{x^4}$$

$$\lim_{x \rightarrow \infty} h(x) = 0$$

$$17. (a) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^2 - 4} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\frac{2}{3}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x - 4} = -\infty \quad (\text{Limit does not exist})$$

$$21. \lim_{x \rightarrow \infty} \frac{x}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1/x}{1 - (1/x^2)} = \frac{0}{1} = 0$$

$$15. (a) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^3 - 1} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^2 - 1} = 1$$

$$(c) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x - 1} = \infty \quad (\text{Limit does not exist})$$

$$19. \lim_{x \rightarrow \infty} \frac{2x - 1}{3x + 2} = \lim_{x \rightarrow \infty} \frac{2 - (1/x)}{3 + (2/x)} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$$

$$23. \lim_{x \rightarrow -\infty} \frac{5x^2}{x + 3} = \lim_{x \rightarrow -\infty} \frac{5x}{1 + (3/x)} = -\infty$$

Limit does not exist.

$$25. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}, \quad (\text{for } x < 0 \text{ we have } x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1 - (1/x)}} = -1$$

$$27. \lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x}}{\left(\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}} \right)} \quad (\text{for } x < 0, x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-2 - (1/x)}{\sqrt{x + (1/x)}} = -2$$

29. Since $(-1/x) \leq (\sin(2x))/x \leq (1/x)$ for all $x \neq 0$, we have by the Squeeze Theorem,

$$\lim_{x \rightarrow \infty} -\frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq 0.$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} = 0.$$

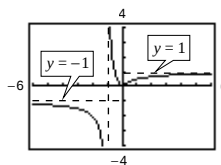
$$31. \lim_{x \rightarrow \infty} \frac{1}{2x + \sin x} = 0$$

33. (a) $f(x) = \frac{|x|}{x+1}$

$$\lim_{x \rightarrow \infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{|x|}{x+1} = -1$$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.



35. $\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$

(Let $x = 1/t$.)

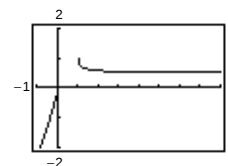
37. $\lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 3}) = \lim_{x \rightarrow -\infty} \left[(x + \sqrt{x^2 + 3}) \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}} \right] = \lim_{x \rightarrow -\infty} \frac{-3}{x - \sqrt{x^2 + 3}} = 0$

39. $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x}) = \lim_{x \rightarrow \infty} \left[(x - \sqrt{x^2 + x}) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right]$
 $= \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$

41.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \sqrt{x(x-1)}) &= \lim_{x \rightarrow \infty} \frac{x - \sqrt{x^2 - x}}{1} \cdot \frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} \\ &= \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 - x}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + \sqrt{1 - (1/x)}} \\ &= \frac{1}{2} \end{aligned}$$

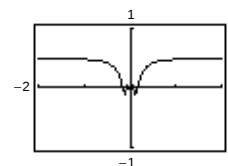


43.

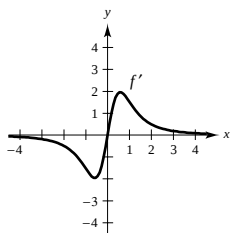
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

Let $x = 1/t$.

$$\lim_{x \rightarrow \infty} x \sin \left(\frac{1}{2x} \right) = \lim_{t \rightarrow 0^+} \frac{\sin(t/2)}{t} = \lim_{t \rightarrow 0^+} \frac{1}{2} \frac{\sin(t/2)}{t/2} = \frac{1}{2}$$



45. (a)



$$(b) \lim_{x \rightarrow \infty} f(x) = 3 \quad \lim_{x \rightarrow \infty} f'(x) = 0$$

(c) Since $\lim_{x \rightarrow \infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x \rightarrow \infty} f'(x) = 0$.

$$49. y = \frac{2+x}{1-x}$$

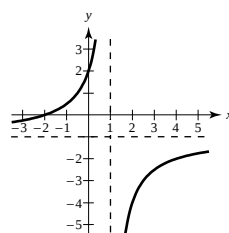
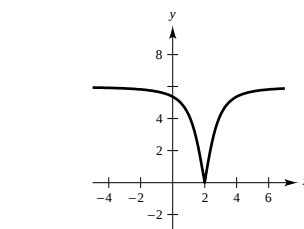
Intercepts: $(-2, 0)$, $(0, 2)$

Symmetry: none

Horizontal asymptote: $y = -1$ since

$$\lim_{x \rightarrow -\infty} \frac{2+x}{1-x} = -1 = \lim_{x \rightarrow \infty} \frac{2+x}{1-x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



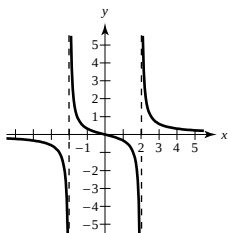
$$51. y = \frac{x}{x^2 - 4}$$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$



$$53. y = \frac{x^2}{x^2 + 9}$$

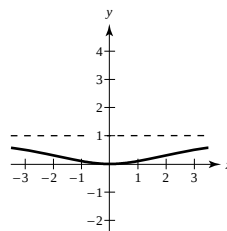
Intercept: $(0, 0)$

Symmetry: y -axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 9} = 1 = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 9}.$$

Relative minimum: $(0, 0)$



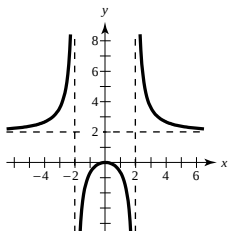
55. $y = \frac{2x^2}{x^2 - 4}$

Intercept: (0, 0)

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = \pm 2$



57. $xy^2 = 4$

Domain: $x > 0$

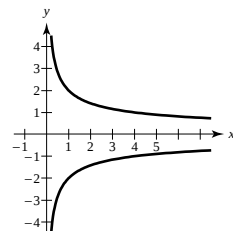
Intercepts: none

Symmetry: x-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \rightarrow \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



59. $y = \frac{2x}{1 - x}$

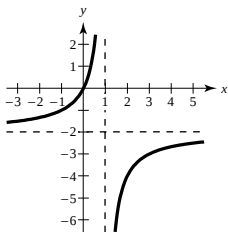
Intercept: (0, 0)

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \rightarrow -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \rightarrow \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



61. $y = 2 - \frac{3}{x^2}$

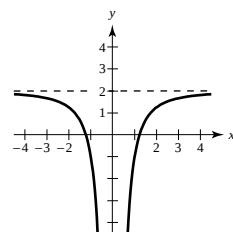
Intercepts: $(\pm\sqrt{3/2}, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x \rightarrow -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \rightarrow \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



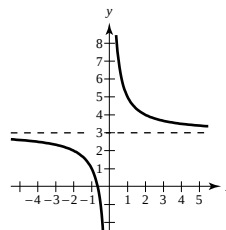
63. $y = 3 + \frac{2}{x}$

Intercept: $y = 0 = 3 + \frac{2}{x} \Rightarrow \frac{2}{x} = -3 \Rightarrow x = -\frac{2}{3} \left(-\frac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



65. $y = \frac{x^3}{\sqrt{x^2 - 4}}$

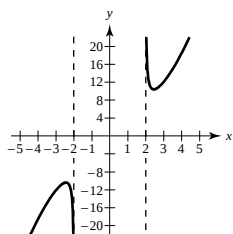
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



67. $f(x) = 5 - \frac{1}{x^2} = \frac{5x^2 - 1}{x^2}$

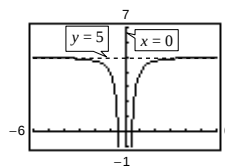
Domain: $(-\infty, 0), (0, \infty)$

$f'(x) = \frac{2}{x^3} \Rightarrow$ No relative extrema

$f''(x) = -\frac{6}{x^4} \Rightarrow$ No points of inflection

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 5$



69. $f(x) = \frac{x}{x^2 - 4}$

$f'(x) = \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2}$

$= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0$ for any x in the domain of f .

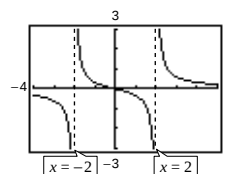
$f''(x) = \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4}$

$= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0$ when $x = 0$.

Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 0$



71. $f(x) = \frac{x - 2}{x^2 - 4x + 3} = \frac{x - 2}{(x - 1)(x - 3)}$

$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$

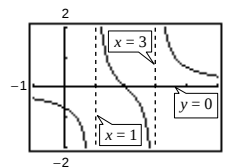
$f''(x) = \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$

$= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = 0$ when $x = 2$.

Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptote: $x = 1, x = 3$

Horizontal asymptote: $y = 0$



73. $f(x) = \frac{3x}{\sqrt{4x^2 + 1}}$

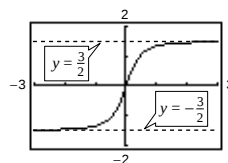
$$f'(x) = \frac{3}{(4x^2 + 1)^{3/2}} \Rightarrow \text{No relative extrema}$$

$$f''(x) = \frac{-36x}{(4x^2 + 1)^{5/2}} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm \frac{3}{2}$

No vertical asymptotes



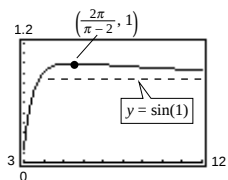
75. $g(x) = \sin\left(\frac{x}{x-2}\right), 3 < x < \infty$

$$g'(x) = \frac{-2 \cos\left(\frac{x}{x-2}\right)}{(x-2)^2}$$

Horizontal asymptote: $y = 1$

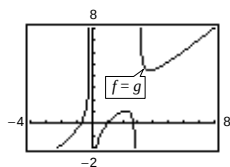
Relative maximum: $\frac{x}{x-2} = \frac{\pi}{2} \Rightarrow x = \frac{2\pi}{\pi-2} \approx 5.5039$

No vertical asymptotes



77. $f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}, g(x) = x + \frac{2}{x(x-3)}$

(a)

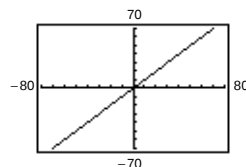


(b) $f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}$

$$= \frac{x^2(x-3)}{x(x-3)} + \frac{2}{x(x-3)}$$

$$= x + \frac{2}{x(x-3)} = g(x)$$

(c)



The graph appears as the slant asymptote $y = x$.

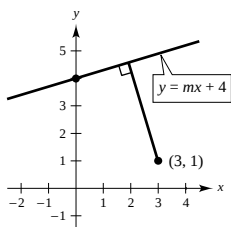
79. $C = 0.5x + 500$

$$\bar{C} = \frac{C}{x}$$

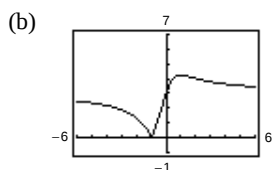
$$\bar{C} = 0.5 + \frac{500}{x}$$

$$\lim_{x \rightarrow \infty} \left(0.5 + \frac{500}{x}\right) = 0.5$$

81. line:
- $mx - y + 4 = 0$



$$(a) d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$

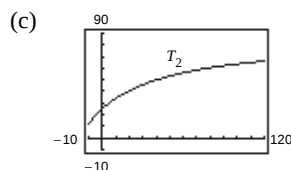
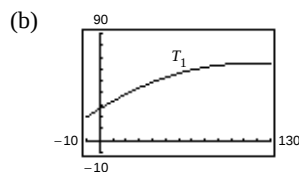


$$(c) \lim_{m \rightarrow \infty} d(m) = 3 = \lim_{m \rightarrow -\infty} d(m)$$

The line approaches the vertical line $x = 0$. Hence, the distance approaches 3.

85. Answers will vary. See page 195.

83. (a)
- $T_1(t) = -0.003t^2 + 0.677t + 26.564$



$$T_2 = \frac{1451 + 86t}{58 + t}$$

$$(d) T_1(0) \approx 26.6$$

$$T_2(0) \approx 25.0$$

$$(e) \lim_{t \rightarrow \infty} T_2 = \frac{86}{1} = 86$$

- (f) The limiting temperature is 86.
 T_1 has no horizontal asymptote.

87. False. Let
- $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$
- . (See Exercise 2.)

Section 3.6 A Summary of Curve Sketching

- 1.
- f
- has constant negative slope. Matches (D)

5. (a)
- $f'(x) = 0$
- for
- $x = -2$
- and
- $x = 2$

f' is negative for $-2 < x < 2$ (decreasing function).

f' is positive for $x > 2$ and $x < -2$
 (increasing function).

- (b) $f''(x) = 0$ at $x = 0$ (Inflection point).

f'' is positive for $x > 0$ (Concave upwards).

f'' is negative for $x < 0$ (Concave downward).

3. The slope is periodic, and zero at
- $x = 0$
- . Matches (A)

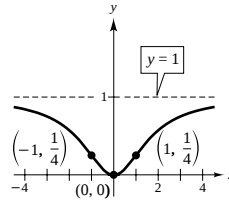
- (c) f' is increasing on $(0, \infty)$. ($f'' > 0$)

- (d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x .

7. $y = \frac{x^2}{x^2 + 3}$

$$y' = \frac{6x}{(x^2 + 3)^2} = 0 \text{ when } x = 0.$$

$$y'' = \frac{18(1 - x^2)}{(x^2 + 3)^3} = 0 \text{ when } x = \pm 1.$$

 Horizontal asymptote: $y = 1$


	y	y'	y''	Conclusion
$-\infty < x < -1$		-	-	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	-	0	Point of inflection
$-1 < x < 0$		-	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < 1$		+	+	Increasing, concave up
$x = 1$	$\frac{1}{4}$	+	0	Point of inflection
$1 < x < \infty$		+	-	Increasing, concave down

9. $y = \frac{1}{x-2} - 3$

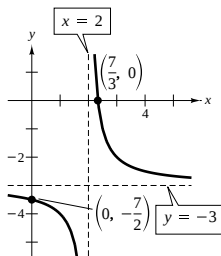
$$y' = -\frac{1}{(x-2)^2} < 0 \text{ when } x \neq 2.$$

$$y'' = \frac{2}{(x-2)^3}$$

No relative extrema, no points of inflection

 Intercepts: $\left(\frac{7}{3}, 0\right), \left(0, -\frac{7}{2}\right)$

 Vertical asymptote: $x = 2$

 Horizontal asymptote: $y = -3$


11. $y = \frac{2x}{x^2 - 1}$

$$y' = \frac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0 \text{ if } x \neq \pm 1.$$

$$y'' = \frac{4x(x^2 + 3)}{(x^2 - 1)^3} = 0 \text{ if } x = 0.$$

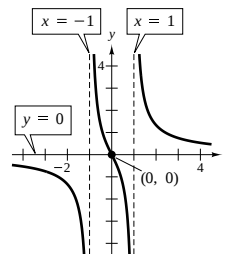
 Inflection point: $(0, 0)$

 Intercept: $(0, 0)$

 Vertical asymptote: $x = \pm 1$

 Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



13. $g(x) = x + \frac{4}{x^2 + 1}$

$$g'(x) = 1 - \frac{8x}{(x^2 + 1)^2} = \frac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0 \text{ when } x \approx 0.1292, 1.6085$$

$$g''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}$$

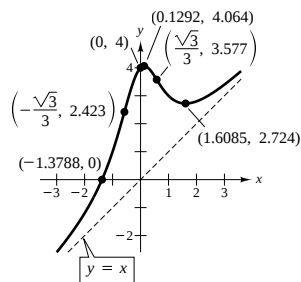
$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\frac{\sqrt{3}}{3}, 2.423\right), \left(\frac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$



15. $f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = 0 \text{ when } x = \pm 1.$$

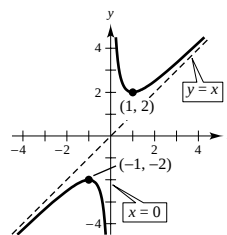
$$f''(x) = \frac{2}{x^3} \neq 0$$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



17. $y = \frac{x^2 - 6x + 12}{x - 4} = x - 2 + \frac{4}{x - 4}$

$$y' = 1 - \frac{4}{(x - 4)^2} = \frac{(x - 2)(x - 6)}{(x - 4)^2} = 0 \text{ when } x = 2, 6.$$

$$y'' = \frac{8}{(x - 4)^3}$$

$y'' < 0$ when $x = 2$.

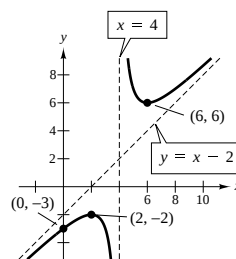
Therefore, $(2, -2)$ is a relative maximum.

$y'' > 0$ when $x = 6$.

Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

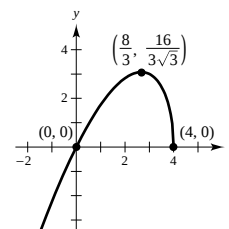


19. $y = x\sqrt{x-4}$,

 Domain: $(-\infty, 4]$

$$y' = \frac{8-3x}{2\sqrt{4-x}} = 0 \text{ when } x = \frac{8}{3} \text{ and undefined when } x = 4.$$

$$y'' = \frac{3x-16}{4(4-x)^{3/2}} = 0 \text{ when } x = \frac{16}{3} \text{ and undefined when } x = 4.$$

Note: $x = \frac{16}{3}$ is not in the domain.


	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		+	−	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	−	Relative maximum
$\frac{8}{3} < x < 4$		−	−	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

21. $h(x) = x\sqrt{9-x^2}$ Domain: $-3 \leq x \leq 3$

$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$$

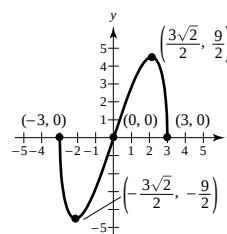
$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0$$

 Relative maximum: $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$

 Relative minimum: $\left(-\frac{3\sqrt{2}}{2}, -\frac{9}{2}\right)$

 Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

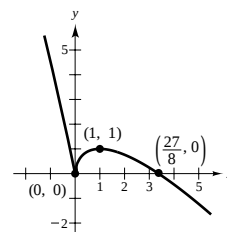
 Point of inflection: $(0, 0)$


23. $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1-x^{1/3})}{x^{1/3}}$$

 $= 0$ when $x = 1$ and undefined when $x = 0$.

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \neq 0.$$



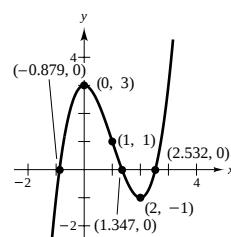
	y	y'	y''	Conclusion
$-\infty < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		+	−	Increasing, concave down
$x = 1$	1	0	−	Relative maximum
$1 < x < \infty$		−	−	Decreasing, concave down

25. $y = x^3 - 3x^2 + 3$

$$y' = 3x^2 - 6x = 3x(x - 2) = 0 \text{ when } x = 0, x = 2$$

$$y'' = 6x - 6 = 6(x - 1) = 0 \text{ when } x = 1$$

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	-	Increasing, concave down
$x = 0$	3	0	-	Relative maximum
$0 < x < 1$		-	-	Decreasing, concave down
$x = 1$	1	-	0	Point of inflection
$1 < x < 2$		-	+	Decreasing, concave up
$x = 2$	-1	0	+	Relative minimum
$2 < x < \infty$		+	+	Increasing, concave up



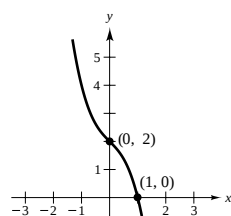
27. $y = 2 - x - x^3$

$$y' = -1 - 3x^2$$

No critical numbers

$$y'' = -6x = 0 \text{ when } x = 0.$$

	y	y'	y''	Conclusion
$-\infty < x < 0$		-	+	Decreasing, concave up
$x = 0$	2	-	0	Point of inflection
$0 < x < \infty$		-	-	Decreasing, concave down

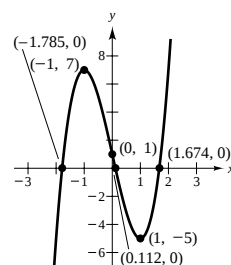


29. $f(x) = 3x^3 - 9x + 1$

$$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0 \text{ when } x = \pm 1$$

$$f''(x) = 18x = 0 \text{ when } x = 0$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	7	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	1	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-5	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

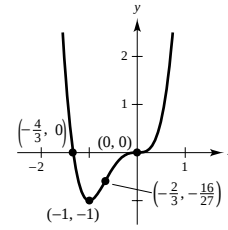


31. $y = 3x^4 + 4x^3$

$$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0 \text{ when } x = 0, x = -1.$$

$$y'' = 36x^2 + 24x = 12x(3x + 2) = 0 \text{ when } x = 0, x = -\frac{2}{3}.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-1	0	+	Relative minimum
$-1 < x < -\frac{2}{3}$		+	+	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	+	0	Point of inflection
$-\frac{2}{3} < x < 0$		+	-	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		+	+	Increasing, concave up

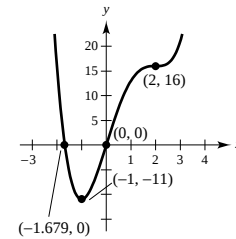


33. $f(x) = x^4 - 4x^3 + 16x$

$$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0 \text{ when } x = -1, x = 2.$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0 \text{ when } x = 0, x = 2.$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-11	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 2$		+	-	Increasing, concave down
$x = 2$	16	0	0	Point of inflection
$2 < x < \infty$		+	+	Increasing, concave up

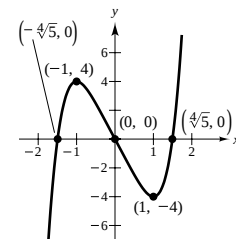


35. $y = x^5 - 5x$

$$y' = 5x^4 - 5 = 5(x^4 - 1) = 0 \text{ when } x = \pm 1.$$

$$y'' = 20x^3 = 0 \text{ when } x = 0.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	4	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	0	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

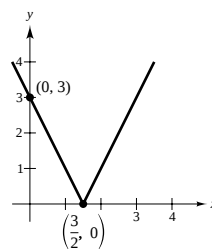


37. $y = |2x - 3|$

$$y' = \frac{2(2x - 3)}{|2x - 3|} \text{ undefined at } x = \frac{3}{2}.$$

$$y'' = 0$$

	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		$-$	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		$+$	Increasing



39. $y = \sin x - \frac{1}{18} \sin 3x, 0 \leq x \leq 2\pi$

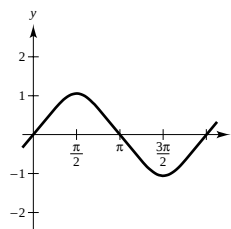
$$y' = \cos x - \frac{1}{6} \cos 3x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$y'' = -\sin x + \frac{1}{2} \sin 3x = 0 \text{ when } x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi.$$

Relative maximum: $\left(\frac{\pi}{2}, \frac{19}{18}\right)$

Relative minimum: $\left(\frac{3\pi}{2}, -\frac{19}{18}\right)$

Inflection points: $\left(\frac{\pi}{6}, \frac{4}{9}\right), \left(\frac{5\pi}{6}, \frac{4}{9}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{4}{9}\right), \left(\frac{11\pi}{6}, -\frac{4}{9}\right)$



41. $y = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$y' = 2 - \sec^2 x = 0 \text{ when } x = \pm \frac{\pi}{4}.$$

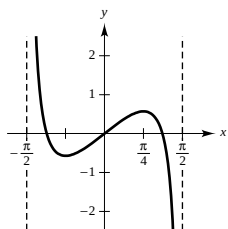
$$y'' = -2\sec^2 x \tan x = 0 \text{ when } x = 0.$$

Relative maximum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm \frac{\pi}{2}$

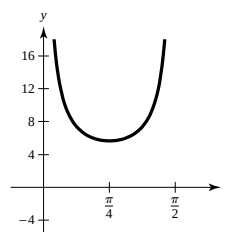


43. $y = 2(\csc x + \sec x), 0 < x < \frac{\pi}{2}$

$$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \Rightarrow x = \pi/4$$

Relative minimum: $\left(\frac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, x = \frac{\pi}{2}$



45. $g(x) = x \tan x, -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

$$g'(x) = \frac{x + \sin x \cos x}{\cos^2 x} = 0 \text{ when } x = 0$$

$$g''(x) = \frac{2(\cos x + x \sin x)}{\cos^3 x}$$

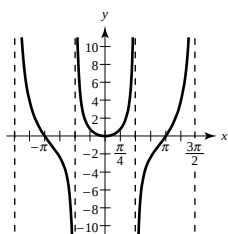
Vertical asymptotes: $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

Intercepts: $(-\pi, 0), (0, 0), (\pi, 0)$

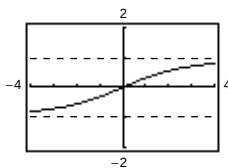
Symmetric with respect to y-axis.

Increasing on $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \frac{3\pi}{2})$

Points of inflection: $(\pm 2.80, 0)$

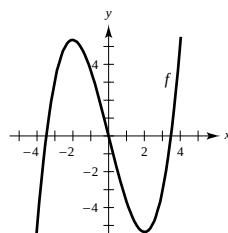


49. $y = \frac{x}{\sqrt{x^2 + 7}}$

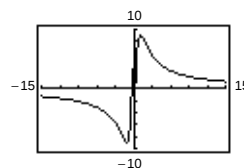

 $(0, 0)$ point of inflection

 $y = \pm 1$ horizontal asymptotes

53.


 (any vertical translate of f will do)

47. $f(x) = \frac{20x}{x^2 + 1} - \frac{1}{x} = \frac{19x^2 - 1}{x(x^2 + 1)}$


 $x = 0$ vertical asymptote

 $y = 0$ horizontal asymptote

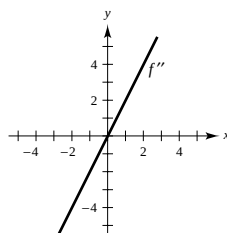
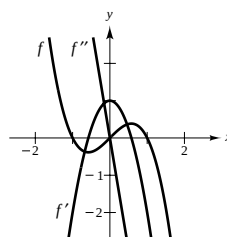
 Minimum: $(-1.10, -9.05)$

 Maximum: $(1.10, 9.05)$

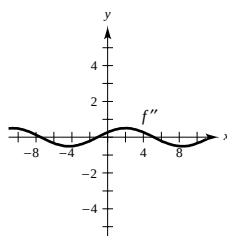
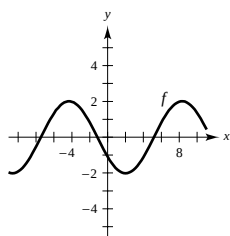
 Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$

 51. f is cubic.

 f' is quadratic.

 f'' is linear.


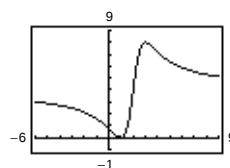
55.

(any vertical translate of f will do)

57. Since the slope is negative, the function is decreasing on $(2, 8)$, and hence $f(3) > f(5)$.

$$59. f(x) = \frac{4(x-1)^2}{x^2 - 4x + 5}$$

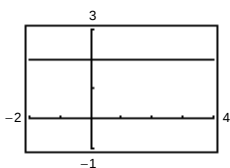
Vertical asymptote: none

Horizontal asymptote: $y = 4$ 

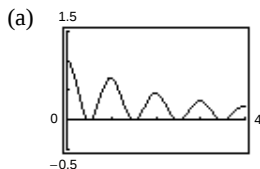
The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

$$61. h(x) = \frac{6-2x}{3-x} = \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined,} & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

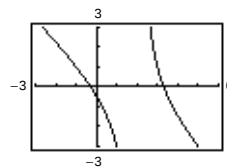
hole at $(3, 2)$

$$65. f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, (0, 4)$$

On $(0, 4)$ there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

$$63. f(x) = \frac{x^2 - 3x - 1}{x - 2} = -x + 1 + \frac{3}{x - 2}$$



The graph appears to approach the slant asymptote $y = -x + 1$.

$$(b) f'(x) = \frac{-\cos \pi x (x \cos \pi x + 2\pi(x^2 + 1)\sin \pi x)}{(x^2 + 1)^{3/2}} = 0$$

$$\text{Critical numbers} \approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}.$$

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

67. Vertical asymptote: $x = 5$

 Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 5}$$

71. $f(x) = \frac{ax}{(x - b)^2}$

- (a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches ∞ as $x \rightarrow b$. If $a < 0$, the graph approaches $-\infty$ as $x \rightarrow b$. The graph approaches its vertical asymptote faster as $|a| \rightarrow 0$.

73. $f(x) = \frac{3x^n}{x^4 + 1}$

- (a) For n even, f is symmetric about the y -axis. For n odd, f is symmetric about the origin.
- (b) The x -axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.
- (c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

 69. Vertical asymptote: $x = 5$

 Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 5} = \frac{3x^2 - 13x - 9}{x - 5}$$

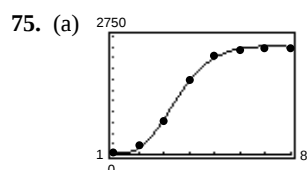
- (b) As b varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum ($a > 0$) or maximum ($a < 0$) are changed.

- (d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}$$

(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3



- (b) When $t = 10$, $N(10) \approx 2434$ bacteria.
- (c) N is a maximum when $t \approx 7.2$ (seventh day).
- (d) $N'(t) = 0$ for $t \approx 3.2$
- (e) $\lim_{t \rightarrow \infty} N(t) = \frac{13,250}{7} \approx 1892.86$

Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

—CONTINUED—

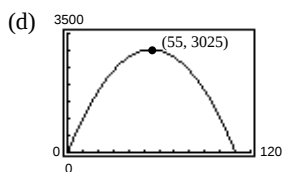
1. —CONTINUED—

(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60 .

(c) $P = x(110 - x) = 110x - x^2$



The solution appears to be $x = 55$.

(e) $\frac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$$\frac{d^2P}{dx^2} = -2 < 0$$

P is a maximum when $x = 110 - x = 55$.

The two numbers are 55 and 55.

3. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + y = x + \frac{192}{x}$$

$$\frac{dS}{dx} = 1 - \frac{192}{x^2} = 0 \text{ when } x = \sqrt{192}.$$

$$\frac{d^2S}{dx^2} = \frac{384}{x^3} > 0 \text{ when } x = \sqrt{192}.$$

S is a minimum when $x = y = \sqrt{192}$.

7. Let x be the length and y the width of the rectangle.

$$2x + 2y = 100$$

$$y = 50 - x$$

$$A = xy = x(50 - x)$$

$$\frac{dA}{dx} = 50 - 2x = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 25.$$

A is maximum when $x = y = 25$ meters.

5. Let x be a positive number.

$$S = x + \frac{1}{x}$$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0 \text{ when } x = 1.$$

$$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0 \text{ when } x = 1.$$

The sum is a minimum when $x = 1$ and $1/x = 1$.

9. Let x be the length and y the width of the rectangle.

$$xy = 64$$

$$y = \frac{64}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$$\frac{dP}{dx} = 2 - \frac{128}{x^2} = 0 \text{ when } x = 8.$$

$$\frac{d^2P}{dx^2} = \frac{256}{x^3} > 0 \text{ when } x = 8.$$

P is minimum when $x = y = 8$ feet.

$$11. d = \sqrt{(x-4)^2 + (\sqrt{x}-0)^2}$$

$$= \sqrt{x^2 - 7x + 16}$$

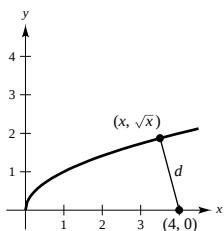
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $(7/2, \sqrt{7/2})$.



$$13. d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$$

$$= \sqrt{x^4 - 4x + (17/4)}$$

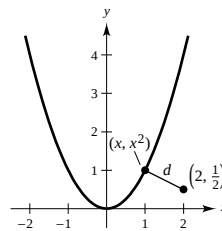
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \frac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $(2, \frac{1}{2})$ is $(1, 1)$.



$$15. \frac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$$

$$\frac{d^2Q}{dx^2} = kQ_0 - 2kx$$

$$= k(Q_0 - 2x) = 0 \text{ when } x = \frac{Q_0}{2}.$$

$$\frac{d^3Q}{dx^3} = -2k < 0 \text{ when } x = \frac{Q_0}{2}.$$

dQ/dx is maximum when $x = Q_0/2$.

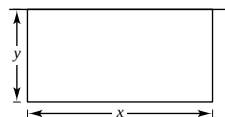
$$17. xy = 180,000 \text{ (see figure)}$$

$S = x + 2y = \left(x + \frac{360,000}{x}\right)$ where S is the length of fence needed.

$$\frac{dS}{dx} = 1 - \frac{360,000}{x^2} = 0 \text{ when } x = 600.$$

$$\frac{d^2S}{dx^2} = \frac{720,000}{x^3} > 0 \text{ when } x = 600.$$

S is a minimum when $x = 600$ meters and $y = 300$ meters.



$$19. (a) A = 4(\text{area of side}) + 2(\text{area of Top})$$

$$(a) A = 4(3)(11) + 2(3)(3) = 150 \text{ square inches}$$

$$(b) A = 4(5)(5) + 2(5)(5) = 150 \text{ square inches}$$

$$(c) A = 4(3.25)(6) + 2(6)(6) = 150 \text{ square inches}$$

$$(c) S = 4xy + 2x^2 = 150 \Rightarrow y = \frac{150 - 2x^2}{4x}$$

$$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$$

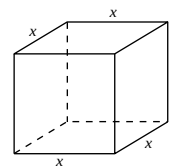
$$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Rightarrow x = \pm 5$$

$$(b) V = (\text{length})(\text{width})(\text{height})$$

$$(a) V = (3)(3)(11) = 99 \text{ cubic inches}$$

$$(b) V = (5)(5)(5) = 125 \text{ cubic inches}$$

$$(c) V = (6)(6)(3.25) = 117 \text{ cubic inches}$$



By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

21. (a) $V = x(s - 2x)^2, 0 < x < \frac{s}{2}$

$$\frac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$$

$$= (s - 2x)(s - 6x) = 0 \text{ when } x = \frac{s}{2}, \frac{s}{6} \text{ (} s/2 \text{ is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 24x - 8s$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = \frac{s}{6}.$$

$$V = \frac{2s^3}{27} \text{ is maximum when } x = \frac{s}{6}.$$

(b) If the length is doubled, $V = \frac{2}{27}(2s)^3 = 8\left(\frac{2}{27}s^3\right)$. Volume is increased by a factor of 8.

23. $16 = 2y + x + \pi\left(\frac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

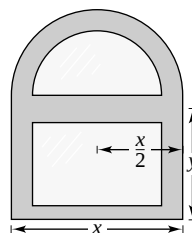
$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \frac{16}{4 + \pi}$ feet and $x = \frac{32}{4 + \pi}$ feet.



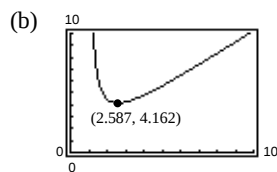
25. (a) $\frac{y-2}{0-1} = \frac{0-2}{x-1}$

$$y = 2 + \frac{2}{x-1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x-1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$$

—CONTINUED—



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

25. —CONTINUED—

$$(c) \text{ Area} = A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x-1}\right) = x + \frac{x}{x-1}$$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

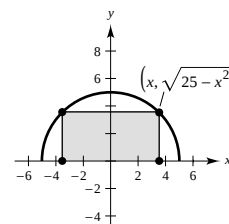
$$x = 0, 2 \text{ (select } x = 2\text{)}$$

Then $y = 4$ and $A = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

$$27. A = 2xy = 2x\sqrt{25-x^2} \text{ (see figure)}$$

$$\begin{aligned} \frac{dA}{dx} &= 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25-x^2}}\right) + 2\sqrt{25-x^2} \\ &= 2\left(\frac{25-2x^2}{\sqrt{25-x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54. \end{aligned}$$



By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm \frac{5\sqrt{2}}{2}, 0\right), \left(\pm \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

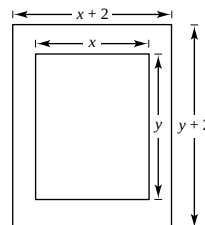
$$\text{Width: } \frac{5\sqrt{2}}{2}; \text{ Length: } 5\sqrt{2}$$

$$29. xy = 30 \Rightarrow y = \frac{30}{x}$$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$

$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2-30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$



By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $(2 + \sqrt{30})$ by $(2 + \sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

$$31. V = \pi r^2 h = 22 \text{ cubic inches or } h = \frac{22}{\pi r^2}$$

Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

—CONTINUED—

31. —CONTINUED—

(b)

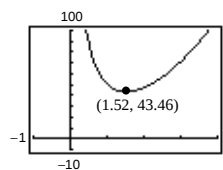
Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for $r = 1.6$.

$$(c) S = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

$$(e) \frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0 \text{ when } r = \sqrt[3]{11/\pi} \approx 1.52 \text{ in.}$$

$$h = \frac{22}{\pi r^2} \approx 3.04 \text{ in.}$$

Note: Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

33. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$

$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

35. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2(r + \sqrt{r^2 - x^2})$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi \left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x(r + \sqrt{r^2 - x^2}) \right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2) = 0$$

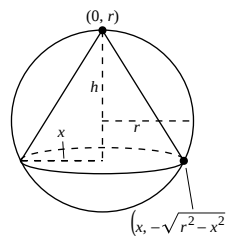
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi \left(\frac{8r^2}{9} \right) \left(\frac{4r}{3} \right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

37. No, there is no minimum area. If the sides are x and y , then $2x + 2y = 20 \Rightarrow y = 10 - x$.
The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

39. $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

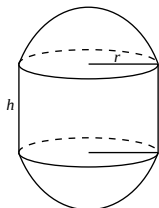
$$S = 4\pi r^2 + 2\pi r h = 4\pi r^2 + 2\pi r \left(\frac{12}{\pi r^2} - \frac{4}{3}r \right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.



41. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y \left(\frac{\sqrt{3}}{2}y \right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

43. Let S be the strength and k the constant of proportionality. Given $h^2 + w^2 = 24^2$, $h^2 = 24^2 - w^2$,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^2) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

47. $\sin \alpha = \frac{h}{s} \Rightarrow s = \frac{h}{\sin \alpha}, 0 < \alpha < \frac{\pi}{2}$

$$\tan \alpha = \frac{h}{2} \Rightarrow h = 2 \tan \alpha \Rightarrow s = \frac{2 \tan \alpha}{\sin \alpha} = 2 \sec \alpha$$

$$I = \frac{k \sin \alpha}{s^2} = \frac{k \sin \alpha}{4 \sec^2 \alpha} = \frac{k}{4} \sin \alpha \cos^2 \alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4} [\sin \alpha (-2 \sin \alpha \cos \alpha) + \cos^2 \alpha (\cos \alpha)]$$

$$= \frac{k}{4} \cos \alpha [\cos^2 \alpha - 2 \sin^2 \alpha]$$

$$= \frac{k}{4} \cos \alpha [1 - 3 \sin^2 \alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin \alpha = \pm \frac{1}{\sqrt{3}}.$$

Since α is acute, we have

$$\sin \alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2 \tan \alpha = 2 \left(\frac{1}{\sqrt{2}} \right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4) \sin \alpha (9 \sin^2 \alpha - 7) < 0$ when $\sin \alpha = 1/\sqrt{3}$, this yields a maximum.

49. $S = \sqrt{x^2 + 4}, L = \sqrt{1 + (3 - x)^2}$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

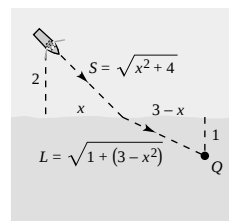
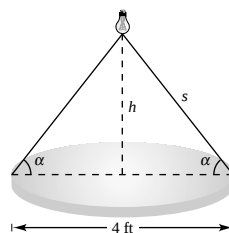
You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ($x = 1$). Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

45. $R = \frac{v_0^2}{g} \sin 2\theta$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g} \cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g} \sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, R is maximum when $\theta = \pi/4$.



$$51. T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{\sqrt{x^2 - 6x + 10}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin \theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin \theta_2$$

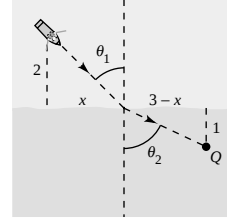
we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

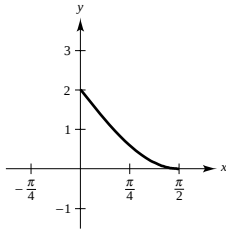
Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

this condition yields a minimum time.

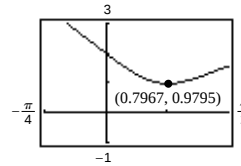


$$53. f(x) = 2 - 2 \sin x$$



- (a) Distance from origin to y-intercept is 2.
Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

$$(b) d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2 \sin x)^2}$$



Minimum distance = 0.9795 at $x = 0.7967$.

$$(c) \text{ Let } f(x) = d^2(x) = x^2 + (2 - 2 \sin x)^2.$$

$$f'(x) = 2x + 2(2 - 2 \sin x)(-2 \cos x)$$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

$$55. F \cos \theta = k(W - F \sin \theta)$$

$$F = \frac{kW}{\cos \theta + k \sin \theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k \cos \theta - \sin \theta)}{(\cos \theta + k \sin \theta)^2} = 0$$

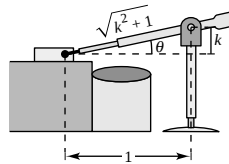
$$k \cos \theta = \sin \theta \Rightarrow k = \tan \theta \Rightarrow \theta = \arctan k$$

Since

$$\cos \theta + k \sin \theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos \theta + k \sin \theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$



57. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	≈ 22.1
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	≈ 42.5
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	≈ 59.7
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	≈ 72.7
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	≈ 80.5
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	≈ 22.1
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	≈ 42.5
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	≈ 59.7
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	≈ 72.7
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	≈ 80.5
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	≈ 83.1
8	$8 + 16 \cos 70^\circ$	$8 \sin 70^\circ$	≈ 80.7
8	$8 + 16 \cos 80^\circ$	$8 \sin 80^\circ$	≈ 74.0
8	$8 + 16 \cos 90^\circ$	$8 \sin 90^\circ$	≈ 64.0

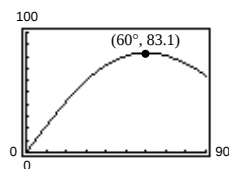
The maximum cross-sectional area is approximately 83.1 square feet.

$$(c) A = (a + b) \frac{h}{2}$$

$$= [8 + (8 + 16 \cos \theta)] \frac{8 \sin \theta}{2}$$

$$= 64(1 + \cos \theta) \sin \theta, 0^\circ < \theta < 90^\circ$$

(e)



$$(d) \frac{dA}{d\theta} = 64(1 + \cos \theta) \cos \theta + (-64 \sin \theta) \sin \theta$$

$$= 64(\cos \theta + \cos^2 \theta - \sin^2 \theta)$$

$$= 64(2 \cos^2 \theta + \cos \theta - 1)$$

$$= 64(2 \cos \theta - 1)(\cos \theta + 1)$$

$$= 0 \text{ when } \theta = 60^\circ, 180^\circ, 300^\circ.$$

The maximum occurs when $\theta = 60^\circ$.

$$59. C = 100 \left(\frac{200}{x^2} + \frac{x}{x + 30} \right), 1 \leq x$$

$$C' = 100 \left(-\frac{400}{x^3} + \frac{30}{(x + 30)^2} \right)$$

Approximation: $x \approx 40.45$ units, or 4045 units

$$61. S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

$$\text{Line: } y = \frac{64}{141}x$$

$$S = \left| 4 \left(\frac{64}{141} \right) - 1 \right| + \left| 5 \left(\frac{64}{141} \right) - 6 \right| + \left| 10 \left(\frac{64}{141} \right) - 3 \right|$$

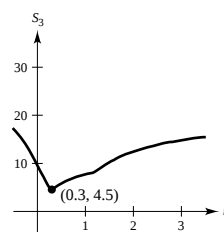
$$= \left| \frac{256}{141} - 1 \right| + \left| \frac{320}{141} - 6 \right| + \left| \frac{640}{141} - 3 \right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

$$63. S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}$$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

$$\text{Line: } y \approx 0.3x$$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$



Section 3.8 Newton's Method

1. $f(x) = x^2 - 3$

$$f'(x) = 2x$$

$$x_1 = 1.7$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	-0.1100	3.4000	-0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

3. $f(x) = \sin x$

$$f'(x) = \cos x$$

$$x_1 = 3$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	-0.9900	-0.1425	3.1425
2	3.1425	-0.0009	-1.0000	0.0009	3.1416

5. $f(x) = x^3 + x - 1$

$$f'(x) = 3x^2 + 1$$

Approximation of the zero of f is 0.682.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3750	1.7500	-0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

7. $f(x) = 3\sqrt{x-1} - x$

$$f'(x) = \frac{3}{2\sqrt{x-1}} - 1$$

Approximation of the zero of f is 1.146.

Similarly, the other zero is approximately 7.854.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	-0.0181	3.0118	-0.0060	1.1458
3	1.1458	-0.0003	2.9284	-0.0001	1.1459

9. $f(x) = x^3 + 3$

$$f'(x) = 3x^2$$

Approximation of the zero of f is -1.442.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	-0.3750	6.7500	-0.0556	-1.4444
2	-1.4444	-0.0134	6.2589	-0.0021	-1.4423
3	-1.4423	-0.0003	6.2407	-0.0001	-1.4422

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$$f'(x) = 3x^2 - 7.8x + 4.79$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3360	1.6400	-0.2049	0.7049
2	0.7049	-0.0921	0.7824	-0.1177	0.8226
3	0.8226	-0.0231	0.4037	-0.0573	0.8799
4	0.8799	-0.0045	0.2495	-0.0181	0.8980
5	0.8980	-0.0004	0.2048	-0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

—CONTINUED—

11. —CONTINUED—

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	-0.1600	-0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

13. $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of f is -0.489.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.0206	1.8776	-0.0110	-0.4890
2	-0.4890	0.0000	1.8723	0.0000	-0.4890

15. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	-0.0001	1.7661	0.0000	0.5687

17. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	-0.1373	-21.5048	0.0064	4.4936
2	4.4936	-0.0039	-20.2271	0.0002	4.4934

19. $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i}$

$= \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$

21. $x_{i+1} = \frac{x_i^2 + 7}{2x_i}$

i	1	2	3	4	5
x_i	2.0000	2.7500	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

23. $x_{i+1} = \frac{3x_i^4 + 6}{4x_i^3}$

i	1	2	3	4
x_i	1.5000	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

25. $f(x) = 1 + \cos x$

$f'(x) = -\sin x$

Approximation of the zero: 3.141

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.0100	-0.1411	-0.0709	3.0709
2	3.0709	0.0025	-0.0706	-0.0354	3.1063
3	3.1063	0.0006	-0.0353	-0.0176	3.1239
4	3.1239	0.0002	-0.0177	-0.0088	3.1327
5	3.1327	0.0000	-0.0089	-0.0044	3.1371
6	3.1371	0.0000	-0.0045	-0.0022	3.1393
7	3.1393	0.0000	-0.0023	-0.0011	3.1404
8	3.1404	0.0000	-0.0012	-0.0006	3.1410

27. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

 $f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

29. $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$y' = -3x^2 + 12x - 10 = f'(x)$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

 $x_4 = 1$ and so on.

Fails to converge

31. Answers will vary. See page 222.

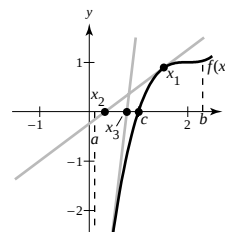
 Newton's Method uses tangent lines to approximate c such that $f(c) = 0$.

 First, estimate an initial x_1 close to c (see graph).

 Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

 Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

 Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

 Let x_{n+1} be the final approximation of c .


33. Let $g(x) = f(x) - x = \cos x - x$

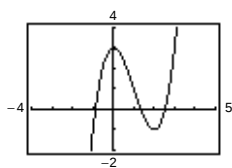
$g'(x) = -\sin x - 1$.

The fixed point is approximately 0.74.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.4597	-1.8415	0.2496	0.7504
2	0.7504	-0.0190	-1.6819	0.0113	0.7391
3	0.7391	0.0000	-1.6736	0.0000	0.7391

35. $f(x) = x^3 - 3x^2 + 3$, $f'(x) = 3x^2 - 6x$

(a)



(c) $x_1 = \frac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

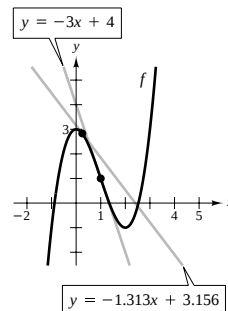
(e) If the initial guess x_1 is not “close to” the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

Continuing, the zero is 1.347.

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

37. $f(x) = \frac{1}{x} - a = 0$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2 \left(\frac{1}{x_n} - a \right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

39. $f(x) = x \cos x$, $(0, \pi)$

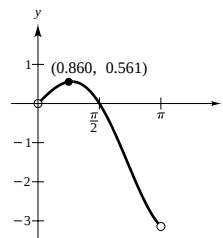
$$f'(x) = -x \sin x + \cos x = 0$$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$$[F'(x) = -2 \sin x + x \cos x]$$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	0.9000	-0.0834	-2.1261	0.0392	0.8608
2	0.8608	-0.0010	-2.0778	0.0005	0.8603

Approximation to the critical number: 0.860



41. $y = f(x) = 4 - x^2, (1, 0)$

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

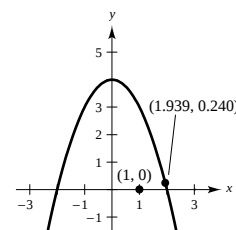
$$g(x) = D' = 4x^3 - 14x - 2$$

$$g'(x) = 12x^2 - 14$$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	-0.0012	31.0934	0.0000	1.9385

$$x \approx 1.939$$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.



43. Minimize: $T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x - 3)\sqrt{x^2 + 4}$$

$$16x^2(x^2 - 6x + 10) = 9(x - 3)^2(x^2 + 4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	-1.0480	150.2780	-0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ miles

45. $2,500,000 = -76x^3 + 4830x^2 - 320,000$

$$76x^3 - 4830x^2 + 2,820,000 = 0$$

Let $f(x) = 76x^3 - 4830x^2 + 2,820,000$

$$f'(x) = 228x^2 - 9660x.$$

From the graph, choose $x_1 = 40$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	-44000.0000	-21600.0000	2.0370	37.9630
2	37.9630	17157.6209	-38131.4039	-0.4500	38.4130
3	38.4130	780.0914	-34642.2263	-0.0225	38.4355
4	38.4355	2.6308	-34465.3435	-0.0001	38.4356

The zero occurs when $x \approx 38.4356$ which corresponds to \$384,356.

47. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in **reduced** form.

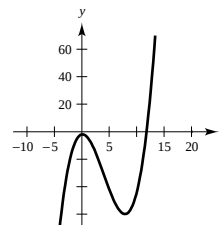
49. True

51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$$

Let $x_1 = 12$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	12.0000	7.0000	36.7500	0.1905	11.8095
2	11.8095	0.2151	34.4912	0.0062	11.8033
3	11.8033	0.0015	34.4186	0.0000	11.8033



Approximation: $x \approx 11.803$

Section 3.9 Differentials

1. $f(x) = x^2$

$$f'(x) = 2x$$

Tangent line at $(2, 4)$: $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

3. $f(x) = x^5$

$$f'(x) = 5x^4$$

Tangent line at $(2, 32)$: $y - f(2) = f'(2)(x - 2)$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

5. $f(x) = \sin x$

$$f'(x) = \cos x$$

Tangent line at $(2, \sin 2)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

7. $y = f(x) = \frac{1}{2}x^3$, $f'(x) = \frac{3}{2}x^2$, $x = 2$, $\Delta x = dx = 0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(2.1) - f(2)$$

$$= 0.6305$$

$$dy = f'(x)dx$$

$$= f'(2)(0.1)$$

$$= 6(0.1) = 0.6$$

9. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(-0.99) - f(-1) \\ &= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394\end{aligned}$$

$$\begin{aligned}dy &= f'(x) dx \\ &= f'(-1)(0.01) \\ &= (-4)(0.01) = -0.04\end{aligned}$$

11. $y = 3x^2 - 4$

$$dy = 6x dx$$

13. $y = \frac{x+1}{2x-1}$

$$dy = \frac{-3}{(2x-1)^2} dx$$

15. $y = x\sqrt{1-x^2}$

$$dy = \left(x \frac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right) dx = \frac{1-2x^2}{\sqrt{1-x^2}} dx$$

17. $y = 2x - \cot^2 x$

$$\begin{aligned}dy &= (2 + 2 \cot x \csc^2 x) dx \\ &= (2 + 2 \cot x + 2 \cot^3 x) dx\end{aligned}$$

19. $y = \frac{1}{3} \cos\left(\frac{6\pi x - 1}{2}\right)$

$$dy = -\pi \sin\left(\frac{6\pi x - 1}{2}\right) dx$$

21. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$
 $\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$
 $\approx 1 + (1)(0.04) = 1.04$

23. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$
 $\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$
 $\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

25. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$
 $\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$
 $\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

27. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$
 $\approx 8 + 0(-0.07) = 8$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$
 $\approx 8 + 0(0.1) = 8$

29. $A = x^2$

$$x = 12$$

$$\Delta x = dx = \pm \frac{1}{64}$$

$$dA = 2x dx$$

$$\begin{aligned}\Delta A &\approx dA = 2(12)\left(\pm \frac{1}{64}\right) \\ &= \pm \frac{3}{8} \text{ square inches}\end{aligned}$$

31. $A = \pi r^2$

$$r = 14$$

$$\Delta r = dr = \pm \frac{1}{4}$$

$$\begin{aligned}\Delta A &\approx dA = 2\pi r dr = \pi(28)\left(\pm \frac{1}{4}\right) \\ &= \pm 7\pi \text{ square inches}\end{aligned}$$

33. (a)
- $x = 15$
- centimeter

$$\Delta x = dx = \pm 0.05 \text{ centimeters}$$

$$A = x^2$$

$$dA = 2x dx = 2(15)(\pm 0.05)$$

$$= \pm 1.5 \text{ square centimeters}$$

Percentage error:

$$\frac{dA}{A} = \frac{\pm 1.5}{(15)^2} = 0.00666 \dots = \frac{2}{3}\%$$

$$(b) \frac{dA}{A} = \frac{2x dx}{x^2} = \frac{2 dx}{x} \leq 0.025$$

$$\frac{dx}{x} \leq \frac{0.025}{2} = 0.0125 = 1.25\%$$

- 37.
- $V = \pi r^2 h = 40\pi r^2$
- ,
- $r = 5$
- cm,
- $h = 40$
- cm,
- $dr = 0.2$
- cm

$$\Delta V \approx dV = 80\pi r dr = 80\pi(5)(0.2) = 80\pi \text{ cm}^3$$

39. (a)
- $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}} dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi dL)/(g\sqrt{L/g})}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2} (\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

$$\text{Percentage error: } \frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$$

- 41.
- $\theta = 26^\circ 45' = 26.75^\circ$

$$d\theta = \pm 15' = \pm 0.25^\circ$$

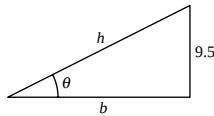
$$(a) h = 9.5 \csc \theta$$

$$dh = -9.5 \csc \theta \cot \theta d\theta$$

$$\frac{dh}{h} = -\cot \theta d\theta$$

$$\left| \frac{dh}{h} \right| = (\cot 26.75^\circ)(0.25^\circ)$$

Converting to radians, $(\cot 0.4669)(0.0044)$
 $\approx 0.0087 = 0.87\%$ (in radians).



- 35.
- $r = 6$
- inches

$$\Delta r = dr = \pm 0.02 \text{ inches}$$

$$(a) V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi \text{ cubic inches}$$

$$(b) S = 4\pi r^2$$

$$dS = 8\pi r dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi \text{ square inches}$$

$$(c) \text{Relative error: } \frac{dV}{V} = \frac{4\pi r^2 dr}{(4/3)\pi r^3} = \frac{3dr}{r}$$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

$$\text{Relative error: } \frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r}$$

$$= \frac{2(0.02)}{6} = 0.00666 \dots = \frac{2}{3}\%$$

$$(b) (0.0025)(3600)(24) = 216 \text{ seconds}$$

$$= 3.6 \text{ minutes}$$

$$(b) \left| \frac{dh}{h} \right| = \cot \theta d\theta \leq 0.02$$

$$\frac{d\theta}{\theta} \leq \frac{0.02}{\theta(\cot \theta)} = \frac{0.02 \tan \theta}{\theta}$$

$$\frac{d\theta}{\theta} \leq \frac{0.02 \tan 26.75^\circ}{26.75^\circ} \approx \frac{0.02 \tan 0.4669}{0.4669}$$

$$\approx 0.0216 = 2.16\% \text{ (in radians)}$$

43. $r = \frac{v_0^2}{32} (\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$dr = \frac{(2200)^2}{16} (\cos 2\theta) d\theta$

$\theta = 10\left(\frac{\pi}{180}\right)$

$d\theta = (11 - 10)\frac{\pi}{180}$

$\Delta r \approx dr$

$= \frac{(2200)^2}{16} \cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961$ feet

≈ 4961 feet

47. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt[4]{x} + \frac{1}{4^4 \sqrt{x^3}} dx$

$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[4]{625})^3}(-1)$

$= 5 - \frac{1}{500} = 4.998$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

45. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$f(x + \Delta x) \approx f(x) + f'(x) dx$

$= \sqrt{x} + \frac{1}{2\sqrt{x}} dx$

$f(x + \Delta x) = \sqrt{99.4}$

$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

49. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/(2\sqrt{x})$.

Then

$f(4.02) \approx f(4) + f'(4) dx$

$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$

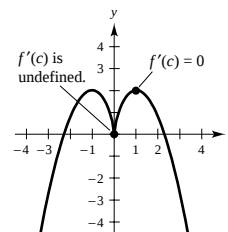
51. In general, when $\Delta x \rightarrow 0$, dy approaches Δy .

53. True

55. True

Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c .



3. $g(x) = 2x + 5 \cos x$, $[0, 2\pi]$

$g'(x) = 2 - 5 \sin x$

$= 0$ when $\sin x = \frac{2}{5}$.

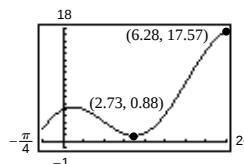
Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum



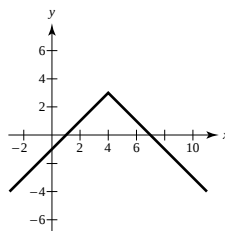
5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$$f'(x) = (x+3)(3x-1) = 0 \text{ for } x = \frac{1}{3}.$$

$$c = \frac{1}{3} \text{ satisfies } f'(c) = 0.$$

7. $f(x) = 3 - |x - 4|$

(a)



$$f(1) = f(7) = 0$$

(b) f is not differentiable at $x = 4$.

9. $f(x) = x^{2/3}, 1 \leq x \leq 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

11. $f(x) = x - \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

13. $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x-1)^2(x-3)$

$$f'(x) = (x-1)^2(1) + (x-3)(2)(x-1)$$

$$= (x-1)(3x-7)$$

$$\text{Critical numbers: } x = 1 \text{ and } x = \frac{7}{3}$$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

17. $h(x) = \sqrt{x}(x-3) = x^{3/2} - 3x^{1/2}$

$$\text{Domain: } (0, \infty)$$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}}$$

$$\text{Critical number: } x = 1$$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$

$v = y' = -4 \sin(12t) - 3 \cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4 \sin(12t) - 3 \cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x$, $0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

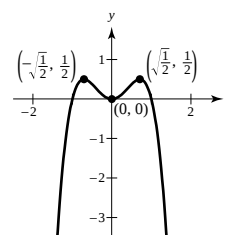
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm \frac{1}{\sqrt{2}}$

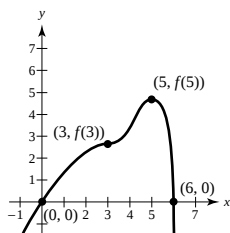
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm \frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

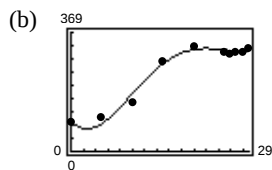


27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$



(c) Maximum at (21.9, 319.5) (≈ 1992)

Minimum at (2.6, 69.6) (≈ 1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) (≈ 1979)

33. $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2}{3 + 5/x^2} = \frac{2}{3}$

35. $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x} = 0$, since $|5 \cos x| \leq 5$.

37. $h(x) = \frac{2x + 3}{x - 4}$

Discontinuity: $x = 4$

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 4} = \lim_{x \rightarrow \infty} \frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \frac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x \rightarrow \infty} \left(\frac{3}{x} - 2 \right) = -2$$

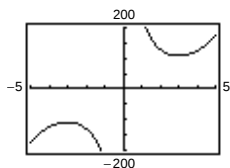
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: (-3, -108)

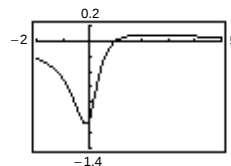


Vertical asymptote: $x = 0$

43. $f(x) = \frac{x - 1}{1 + 3x^2}$

Relative minimum: (-0.155, -1.077)

Relative maximum: (2.155, 0.077)



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4 - x)$

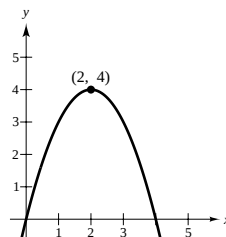
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$$f'(x) = 4 - 2x = 0 \text{ when } x = 2.$$

$$f''(x) = -2$$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f'(-2\sqrt{2}) > 0$$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

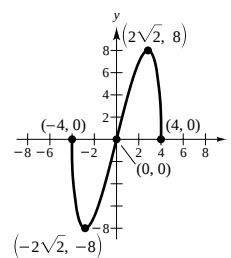
$$f'(2\sqrt{2}) < 0$$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin



49. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

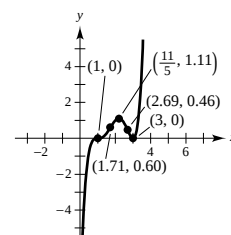
Therefore, $(3, 0)$ is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$



51. $f(x) = x^{1/3}(x + 3)^{2/3}$

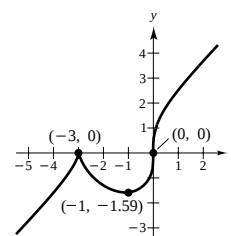
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$



53. $f(x) = \frac{x+1}{x-1}$

Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

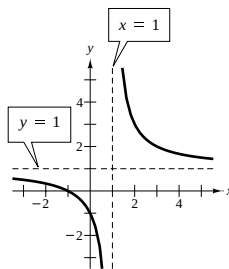
$$f'(x) = \frac{-2}{(x-1)^2} < 0 \text{ if } x \neq 1.$$

$$f''(x) = \frac{4}{(x-1)^3}$$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$



55. $f(x) = \frac{4}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$$f'(x) = \frac{-8x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{-8(1-3x^2)}{(1+x^2)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

$$f''(0) < 0$$

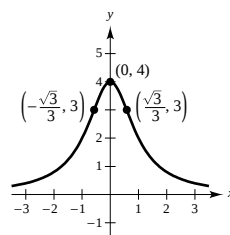
Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $(\pm \sqrt{3}/3, 3)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



57. $f(x) = x^3 + x + \frac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$$f''(-1) < 0$$

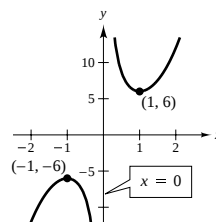
Therefore, $(-1, -6)$ is a relative maximum.

$$f''(1) > 0$$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



59. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$$f''(0) < 0$$

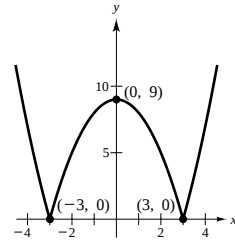
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis



61. $f(x) = x + \cos x$

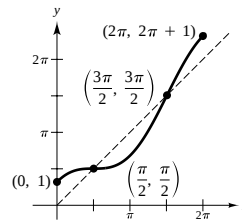
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$$f'(x) = 1 - \sin x \geq 0, f \text{ is increasing.}$$

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Points of inflection: $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{3\pi}{2})$

Intercept: $(0, 1)$



63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

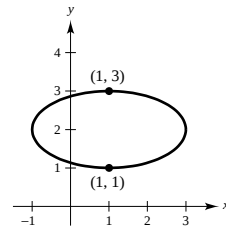
Minimum: $(1, 1)$

(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y \frac{dy}{dx} - 2 - 16 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$



The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

65. Let $t = 0$ at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10,000 - 2400t + 244t^2$$

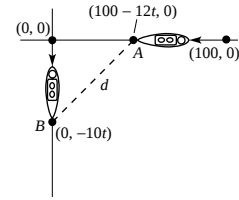
$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

Ship A at (40.98, 0); Ship B at (0, -49.18)

$$d^2 = 10,000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4:55 \text{ P.M..}$$

$$d \approx 64 \text{ km}$$



67. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,

$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

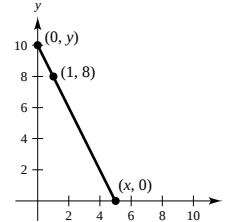
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{8x}{x - 1}\right)^2.$$

$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 5 \text{ (minimum).}$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$



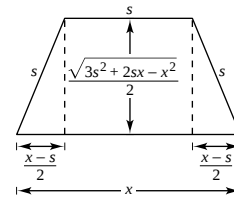
69. $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x + s}{2}\right) \frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4} \left[\frac{(s - x)(s + x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2} \right]$$

$$= \frac{2(2s - x)(s + x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.



71. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$.

Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y - 6}{0 - 4} = \frac{6 - 0}{4 - x} \text{ or } y = \frac{6x}{x - 4}$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x - 4}\right)^2.$$

$$f'(x) = 2x + 72\left(\frac{x}{x - 4}\right)\left[\frac{-4}{(x - 4)^2}\right] = 0$$

$$x[(x - 4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73. $\csc \theta = \frac{L_1}{6}$ or $L_1 = 6 \csc \theta$ (see figure)

$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right) = 6 \csc \theta + 9 \sec \theta$$

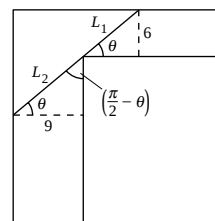
$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$



75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33,000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$$f'(x) = 3x^2 - 3$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

CHAPTER 3

Applications of Differentiation

Section 3.1	Extrema on an Interval	378
Section 3.2	Rolle's Theorem and the Mean Value Theorem	381
Section 3.3	Increasing and Decreasing Functions and the First Derivative Test	387
Section 3.4	Concavity and the Second Derivative Test	394
Section 3.5	Limits at Infinity	402
Section 3.6	A Summary of Curve Sketching	410
Section 3.7	Optimization Problems	419
Section 3.8	Newton's Method	429
Section 3.9	Differentials	434
Review Exercises		437
Problem Solving		445

CHAPTER 3

Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Even-Numbered Exercises

2. $f(x) = \cos \frac{\pi x}{2}$

$$f'(x) = -\frac{\pi}{2} \sin \frac{\pi x}{2}$$

$$f'(0) = 0$$

$$f'(2) = 0$$

4. $f(x) = -3x\sqrt{x+1}$

$$f'(x) = -3x \left[\frac{1}{2}(x+1)^{-1/2} \right] + \sqrt{x+1}(-3)$$

$$= -\frac{3}{2}(x+1)^{-1/2}[x+2(x+1)]$$

$$= -\frac{3}{2}(x+1)^{-1/2}(3x+2)$$

$$f'\left(-\frac{2}{3}\right) = 0$$

6. Using the limit definition of the derivative,

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{(4 - |x|) - 4}{x} = -1$$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

8. Critical number: $x = 0$.

$x = 0$: neither

10. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum

12. $g(x) = x^2(x^2 - 4) = x^4 - 4x^2$

$$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

14. $f(x) = \frac{4x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4(1 - x^2)}{(x^2 + 1)^2}$$

Critical numbers: $x = \pm 1$

16. $f(\theta) = 2 \sec \theta + \tan \theta, 0 < \theta < 2\pi$

$$f'(\theta) = 2 \sec \theta \tan \theta + \sec^2 \theta$$

$$= \sec \theta (2 \tan \theta + \sec \theta)$$

$$= \sec \theta \left[2 \left(\frac{\sin \theta}{\cos \theta} \right) + \frac{1}{\cos \theta} \right]$$

$$= \sec^2 \theta (2 \sin \theta + 1)$$

On $(0, 2\pi)$, critical numbers: $\theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$

18. $f(x) = \frac{2x+5}{3}, [0, 5]$

$$f'(x) = \frac{2}{3} \Rightarrow \text{No critical numbers}$$

Left endpoint: $\left(0, \frac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

22. $f(x) = x^3 - 12x, [0, 4]$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

20. $f(x) = x^2 + 2x - 4, [-1, 1]$

$$f'(x) = 2x + 2 = 2(x + 1)$$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

24. $g(x) = \sqrt[3]{x}, [-1, 1]$

$$g'(x) = \frac{1}{3x^{2/3}}$$

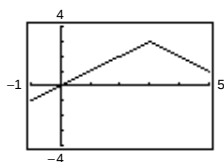
Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

26. $y = 3 - |t - 3|, [-1, 5]$

From the graph, you see that $t = 3$ is a critical number.



Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

28. $h(t) = \frac{t}{t-2}, [3, 5]$

$$h'(t) = \frac{-2}{(t-2)^2}$$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \frac{5}{3}\right)$ Minimum

30. $g(x) = \sec x, \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

$$g'(x) = \sec x \tan x$$

$$\text{Left endpoint: } \left(-\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right) \approx \left(-\frac{\pi}{6}, 1.1547\right)$$

$$\text{Right endpoint: } \left(\frac{\pi}{3}, 2\right) \text{ Maximum}$$

Critical number: $(0, 1)$ Minimum

32. $y = x^2 - 2 - \cos x, [-1, 3]$

$$y' = 2x - \sin x$$

Left endpoint: $(-1, -1.5403)$

Right endpoint: $(3, 7.99)$ Maximum

Critical number: $(0, -3)$ Minimum

34. (a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

36. (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

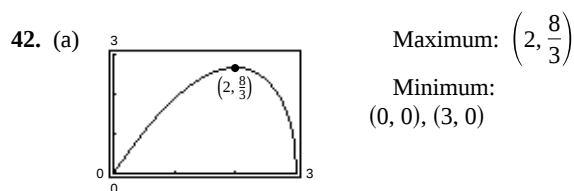
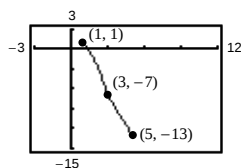
(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

$$38. f(x) = \begin{cases} 2 - x^2, & 1 \leq x < 3 \\ 2 - 3x, & 3 \leq x \leq 5 \end{cases}$$

Left endpoint: (1, 1) Maximum

Right endpoint: (5, -13) Minimum



$$(b) f(x) = \frac{4}{3}x\sqrt{3-x}, [0, 3]$$

$$\begin{aligned} f'(x) &= \frac{4}{3} \left[x \left(\frac{1}{2} \right) (3-x)^{-1/2} (-1) + (3-x)^{1/2} (1) \right] \\ &= \frac{4}{3} (3-x)^{-1/2} \left(\frac{1}{2} \right) [-x + 2(3-x)] \\ &= \frac{2(6-3x)}{3\sqrt{3-x}} = \frac{6(2-x)}{3\sqrt{3-x}} = \frac{2(2-x)}{\sqrt{3-x}} \end{aligned}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$$f(2) = \frac{8}{3}$$

Maximum: $\left(2, \frac{8}{3}\right)$

$$46. f(x) = \frac{1}{x^2 + 1}, [-1, 1]$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4} \quad (\text{See Exercise 44.})$$

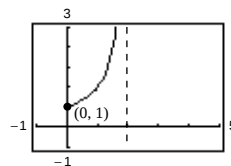
$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

$$40. f(x) = \frac{2}{2-x}, [0, 2)$$

Left endpoint: (0, 1) Minimum



$$44. f(x) = \frac{1}{x^2 + 1}, \left[\frac{1}{2}, 3\right]$$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

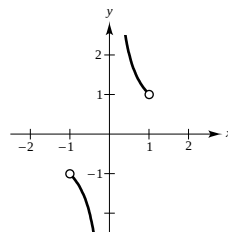
$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

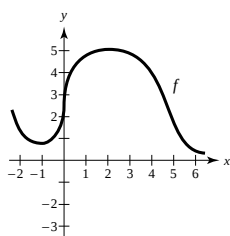
Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \frac{1}{2}$ is the maximum value.

48. Let $f(x) = 1/x$. f is continuous on $(0, 1)$ but does not have a maximum. f is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



50.



52. (a) No

(b) Yes

54. (a) No

(b) Yes

56. $x = \frac{v^2 \sin 2\theta}{32}, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

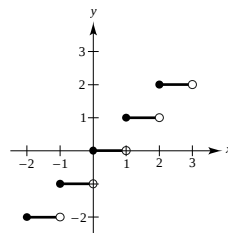
 $\frac{d\theta}{dt}$ is constant.

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{d\theta} \frac{d\theta}{dt} \quad (\text{by the Chain Rule}) \\ &= \frac{v^2 \cos 2\theta}{16} \frac{d\theta}{dt} \end{aligned}$$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt . This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the spinkler gets the most water.

60. $f(x) = \llbracket x \rrbracket$

The derivative of f is undefined at every integer and is zero at any noninteger real number. All real numbers are critical numbers.



62. True. This is stated in the Extreme Value Theorem.

 64. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f .

$$\begin{aligned} g(x) &= f(x - k) \\ &= (x - k)^2 \end{aligned}$$

$x = k$ is a critical number of g .

Section 3.2 Rolle's Theorem and the Mean Value Theorem

 2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since f is not continuous at $x = 2\pi$.

4. $f(x) = x(x - 3)$

x -intercepts: $(0, 0), (3, 0)$

$$f'(x) - 2x - 3 = 0 \text{ at } x = \frac{3}{2}.$$

6. $f(x) = -3x\sqrt{x+1}$

x -intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x \cdot \frac{1}{2}(x+1)^{-1/2} - 3(x+1)^{1/2} = -3(x+1)^{-1/2} \left(\frac{x}{2} + (x+1) \right)$$

$$f'(x) = -3(x+1)^{-1/2} \left(\frac{3}{2}x + 1 \right) = 0 \text{ at } x = -\frac{2}{3}.$$

8. $f(x) = x^2 - 5x + 4$, $[1, 4]$

$$f(1) = f(4) = 0$$

f is continuous on $[1, 4]$. f is differentiable on $(1, 4)$.

Rolle's Theorem applies.

$$f'(x) = 2x - 5$$

$$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$$

$$c \text{ value: } \frac{5}{2}$$

12. $f(x) = 3 - |x - 3|$, $[0, 6]$

$$f(0) = f(6) = 0$$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = \cos x$, $[0, 2\pi]$

$$f(0) = f(2\pi) = 1$$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$.

Rolle's Theorem applies.

$$f'(x) = -\sin x$$

$$c \text{ value: } \pi$$

20. $f(x) = \sec x$, $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

f is continuous on $[-\pi/4, \pi/4]$. f is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$f'(x) = \sec x \tan x$$

$$\sec x \tan x = 0$$

$$x = 0$$

$$c \text{ value: } 0$$

10. $f(x) = (x - 3)(x + 1)^2$, $[-1, 3]$

$$f(-1) = f(3) = 0$$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = (x - 3)(2)(x + 1) + (x + 1)^2$$

$$= (x + 1)[2x - 6 + x + 1]$$

$$= (x + 1)(3x - 5)$$

$$c \text{ value: } \frac{5}{3}$$

14. $f(x) = \frac{x^2 - 1}{x}$, $[-1, 1]$

$$f(-1) = f(1) = 0$$

f is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

18. $f(x) = \cos 2x$, $\left[-\frac{\pi}{12}, \frac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{6}\right)$$

Rolle's Theorem does not apply.

22. $f(x) = x - x^{1/3}$, $[0, 1]$

$$f(0) = f(1) = 0$$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$.

(Note: f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

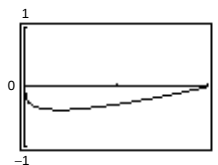
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

$$c \text{ value: } \frac{\sqrt{3}}{9} \approx 0.1925$$



$$24. f(x) = \frac{x}{2} - \sin \frac{\pi x}{6}, [-1, 0]$$

$$f(-1) = f(0) = 0$$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$.
Rolle's Theorem applies.

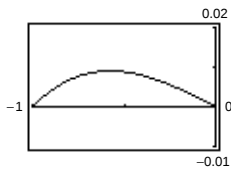
$$f'(x) = \frac{1}{2} - \frac{\pi}{6} \cos \frac{\pi x}{6} = 0$$

$$\cos \frac{\pi x}{6} = \frac{3}{\pi}$$

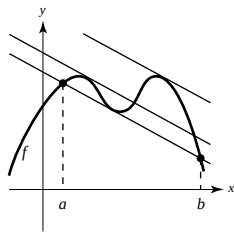
$$x = -\frac{6}{\pi} \arccos \frac{3}{\pi} \quad [\text{Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

c value: -0.5756



28.



$$26. C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$$

$$(a) C(3) = C(6) = \frac{25}{3}$$

$$(b) C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

$$\text{In the interval } (3, 6): c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098.$$

$$30. f(x) = |x - 3|, [0, 6]$$

f is not differentiable at $x = 3$.

32. $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

34. $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

36. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

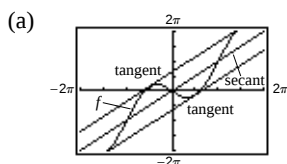
$$\frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$f'(x) = 3x^2 = 1$$

$$x = \pm \frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \frac{\sqrt{3}}{3}$.

40. $f(x) = x - 2 \sin x$ on $[-\pi, \pi]$



(b) Secant line:

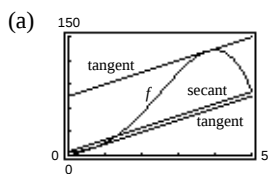
$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

42. $f(x) = -x^4 + 4x^3 + 8x^2 + 5$, $(0, 5)$, $(5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$



(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

38. $f(x) = 2 \sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = 2 \cos x + 2 \cos 2x = 0$$

$$2[\cos x + 2 \cos^2 x - 1] = 0$$

$$2(2 \cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2}$$

$$\cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \frac{\pi}{3}$.

- (c) $f'(x) = 1 - 2 \cos x = 1$

$$\cos x = 0$$

$$c = \pm \frac{\pi}{2}, \quad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

$$\text{Tangent lines: } y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$$

$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

- (c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

- Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

$$44. S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

$$(a) \frac{S(12) - S(0)}{12 - 0} = \frac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \frac{450}{7}$$

$$(b) S'(t) = 200\left(\frac{9}{(2+t)^2}\right) = \frac{450}{7}$$

$$\frac{1}{(2+t)^2} = \frac{1}{28}$$

$$2+t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

46. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b) .

$$(a) g(x) = f(x) + k$$

$$g(a) = g(b) = f(a) + k$$

$$g'(x) = f'(x) \Rightarrow g'(c) = 0$$

Interval: $[a, b]$

Critical number of g : c

$$(b) g(x) = f(x - k)$$

$$g(a+k) = g(b+k) = f(a)$$

$$g'(x) = f'(x - k)$$

$$g'(c+k) = f'(c) = 0$$

Interval: $[a+k, b+k]$

Critical number of g : $c+k$

$$(c) g(x) = f(kx)$$

$$g\left(\frac{a}{k}\right) = g\left(\frac{b}{k}\right) = f(a)$$

$$g'(x) = kf'(kx)$$

$$g'\left(\frac{c}{k}\right) = kf'(c) = 0$$

Interval: $\left[\frac{a}{k}, \frac{b}{k}\right]$

Critical number of g : $\frac{c}{k}$

48. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500^\circ$ and $T(5) = 390^\circ$. The average temperature over the interval $[0, 5]$ is

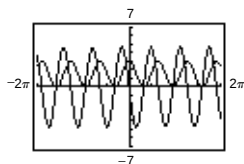
$$\frac{390 - 1500}{5 - 0} = -222^\circ \text{ F/hr.}$$

By the Mean Value Theorem, there exists a time t_0 , $0 < t_0 < 5$, such that $T'(t_0) = -222$.

$$50. f(x) = 3 \cos^2\left(\frac{\pi x}{2}\right), \quad f'(x) = 6 \cos\left(\frac{\pi x}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$$

$$= -3\pi \cos\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi x}{2}\right)$$

(a)



(b) f and f' are both continuous on the entire real line.

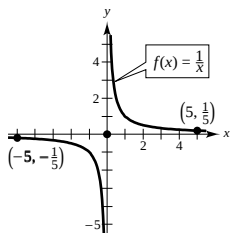
(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

$$(d) \lim_{x \rightarrow 3^-} f'(x) = 0$$

$$\lim_{x \rightarrow 3^+} f'(x) = 0$$

- 52.
- f
- is not continuous on
- $[-5, 5]$
- .

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



54. False.
- f
- must also be continuous
- and*
- differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

56. True

58. Suppose
- $f(x)$
- is not constant on
- (a, b)
- . Then there exists
- x_1
- and
- x_2
- in
- (a, b)
- such that
- $f(x_1) \neq f(x_2)$
- . Then by the Mean Value Theorem, there exists
- c
- in
- (a, b)
- such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b) .

60. Suppose
- $f(x)$
- has two fixed points
- c_1
- and
- c_2
- . Then, by the Mean Value Theorem, there exists
- c
- such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x .

62. Let
- $f(x) = \cos x$
- .
- f
- is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval
- $[a, b]$
- , there exists
- c
- in
- (a, b)
- such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c| |b - a|$$

$$|\cos b - \cos a| \leq |b - a| \text{ since } |-\sin c| \leq 1.$$

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

2. $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$ Decreasing on: $(-1, \infty)$

4. $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0), (1, \infty)$ Decreasing on: $(-\infty, -1), (0, 1)$

6. $y = \frac{x^2}{x + 1}$

$$y' = \frac{x(x + 2)}{(x + 1)^2}$$

Critical numbers: $x = 0, -2$ Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -2$	$-2 < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on $(-\infty, -2), (0, \infty)$ Decreasing on $(-2, -1), (-1, 0)$

8. $h(x) = 27x - x^3$

$$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$$

$$h'(x) = 0$$

Critical numbers: $x = \pm 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-3, 3)$ Decreasing on $(-\infty, -3), (3, \infty)$

10. $y = x + \frac{4}{x}$

$$y' = \frac{(x - 2)(x + 2)}{x^2}$$

Critical numbers: $x = \pm 2$ Discontinuity: 0

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of y' :	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing: $(-\infty, -2), (2, \infty)$ Decreasing: $(-2, 0), (0, 2)$

12. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-4, \infty)$ Decreasing on: $(-\infty, -4)$ Relative minimum: $(-4, -6)$

14. $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, -4)$ Decreasing on: $(-4, \infty)$ Relative maximum: $(-4, 4)$

16. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on $(-\infty, 0), (4, \infty)$ Decreasing on $(0, 4)$ Relative maximum: $(0, 15)$ Relative minimum: $(4, -17)$

18. $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (0, \infty)$ Decreasing on: $(-2, 0)$ Relative maximum: $(-2, 0)$ Relative minimum: $(0, -4)$

20. $f(x) = x^4 - 32x + 4$

$$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$ Decreasing on: $(-\infty, 2)$ Relative minimum: $(2, -44)$

22. $f(x) = x^{2/3} - 4$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$ Decreasing on: $(-\infty, 0)$ Relative minimum: $(0, -4)$

24. $f(x) = (x - 1)^{1/3}$

$$f'(x) = \frac{1}{3(x - 1)^{2/3}}$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

26. $f(x) = |x + 3| - 1$

$$f'(x) = \frac{x + 3}{|x + 3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$ Decreasing on: $(-\infty, -3)$ Relative minimum: $(-3, -1)$

28. $f(x) = \frac{x}{x + 1}$

$$f'(x) = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2}$$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

$$30. f(x) = \frac{x+3}{x^2} = \frac{1}{x} + \frac{3}{x^2}$$

$$f'(x) = -\frac{1}{x^2} - \frac{6}{x^3} = \frac{-(x+6)}{x^3}$$

Critical number: $x = -6$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\frac{1}{12}\right)$

$$32. f(x) = \frac{x^2 - 3x - 4}{x - 2}$$

$$f'(x) = \frac{(x-2)(2x-3) - (x^2-3x-4)(1)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}$$

Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

$$34. f(x) = \sin x \cos x = \frac{1}{2} \sin 2x, 0 < x < 2\pi$$

$$f'(x) = \cos 2x = 0$$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Relative maxima: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, -\frac{1}{2}\right), \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$

$$36. f(x) = \frac{\sin x}{1 + \cos^2 x}, \quad 0 < x < 2\pi$$

$$f'(x) = \frac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$$

$$\text{Critical numbers: } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

$$\text{Increasing on: } \left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$$

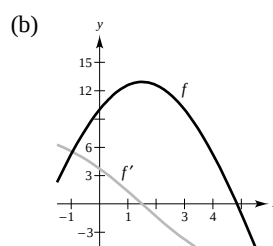
$$\text{Relative maximum: } \left(\frac{\pi}{2}, 1\right)$$

$$\text{Decreasing on: } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$\text{Relative minimum: } \left(\frac{3\pi}{2}, -1\right)$$

$$38. f(x) = 10(5 - \sqrt{x^2 - 3x + 16}), [0, 5]$$

$$(a) f'(x) = -\frac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$$



$$(c) -\frac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$$

$$\text{Critical number: } x = \frac{3}{2}$$

(d) Intervals:

$$\left(0, \frac{3}{2}\right) \quad \left(\frac{3}{2}, 5\right)$$

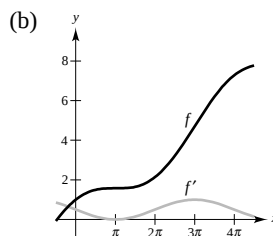
$$f'(x) > 0 \quad f'(x) < 0$$

Increasing Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

$$40. f(x) = \frac{x}{2} + \cos \frac{x}{2}, [0, 4\pi]$$

$$(a) f'(x) = \frac{1}{2} - \frac{1}{2} \sin \frac{x}{2}$$



$$(c) \frac{1}{2} - \frac{1}{2} \sin \frac{x}{2} = 0$$

$$\sin \frac{x}{2} = 1$$

$$\frac{x}{2} = \frac{\pi}{2}$$

$$\text{Critical number: } x = \pi$$

(d) Intervals:

$$(0, \pi) \quad (\pi, 4\pi)$$

$$f'(x) > 0 \quad f'(x) > 0$$

Increasing Increasing

f is increasing when f' is positive.

42. $f(t) = \cos^2 t - \sin^2 t = -2 \sin^2 t = g(t), -2 < t < 2$

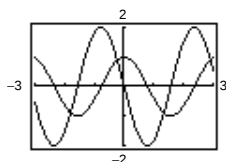
$$f'(t) = -4 \sin t \cos t = -2 \sin 2t$$

 f symmetric with respect to y -axis

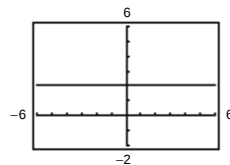
zeros of f : $\pm \frac{\pi}{4}$

Relative maximum: $(0, 1)$

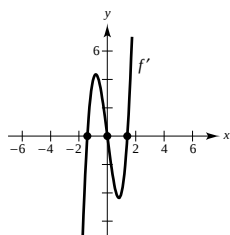
Relative minimum: $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right)$



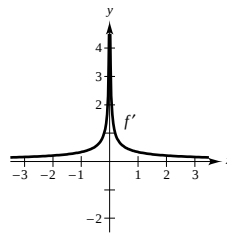
44. $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.



46. f is a 4th degree polynomial $\Rightarrow f'$ is a cubic polynomial.



48. f has positive slope

In Exercises 50–54, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

50. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

52. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

54. $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

56. Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

 $(5, f(5))$ is a relative minimum.

58. $s(t) = 4.9(\sin \theta)t^2$

(a) $v(t) = 9.8(\sin \theta)t$ speed = $|9.8(\sin \theta)t|$

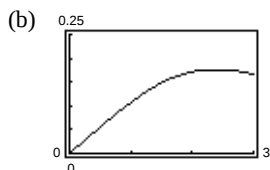
(b) If $\theta = \pi/2$, the speed is maximum,

$v(t) = 9.8t$.

60. $C = \frac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.The concentration is greatest when $t \approx 2.38$ hours.

$$\begin{aligned} \text{(c) } C' &= \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2} \\ &= \frac{3(27 - 2t^3)}{(27 + t^3)^2} \end{aligned}$$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

$$62. P = 2.44x - \frac{x^2}{20,000} - 5000, 0 \leq x \leq 35,000$$

$$P' = 2.44 - \frac{x}{10,000} = 0$$

$$x = 24,400$$

Increasing when $0 < x < 24,400$ hamburgers.

Decreasing when $24,400 < x < 35,000$ hamburgers.

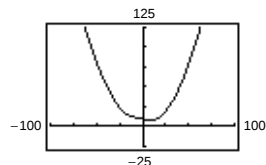
Test intervals:	$0 < x < 24,400$	$24,400 < x < 35,000$
Sign of P' :	$P' > 0$	$P' < 0$

$$64. R = \sqrt{0.001T^4 - 4T + 100}$$

$$(a) R' = \frac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$$

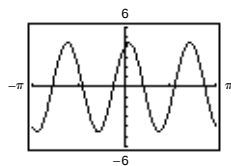
$$T = 10^\circ, R \approx 8.3666\Omega$$

(b)



The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10^\circ$.

$$66. f(x) = 2 \sin(3x) + 4 \cos(3x)$$



The maximum value is approximately 4.472. You could use calculus by finding $f'(x)$ and then observing that the maximum value of f occurs at a point where $f'(x) = 0$. For instance, $f'(0.154) \approx 0$, and $f(0.154) = 4.472$.

68. (a) Use a cubic polynomial

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

$$(b) f'(x) = 3a_3x^2 + 2a_2x + a_1$$

$$(0, 0): \quad 0 = a_0 \quad (f(0) = 0)$$

$$0 = a_1 \quad (f'(0) = 0)$$

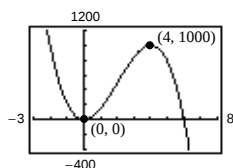
$$(4, 1000): \quad 1000 = 64a_3 + 16a_2 \quad (f(4) = 1000)$$

$$0 = 48a_3 + 8a_2 \quad (f'(4) = 0)$$

$$(c) \text{ The solution is } a_0 = a_1 = 0, a_2 = \frac{375}{2}, a_3 = \frac{-125}{4}$$

$$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$$

(d)



70. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

(1, 2): $2 = a_4 + a_3 + a_2 + a_1 + a_0$ $(f(1) = 2)$

$0 = 4a_4 + 3a_3 + 2a_2 + a_1$ $(f'(1) = 0)$

(-1, 4): $4 = a_4 - a_3 + a_2 - a_1 + a_0$ $(f(-1) = 4)$

$0 = -4a_4 + 3a_3 - 2a_2 + a_1$ $(f'(-1) = 0)$

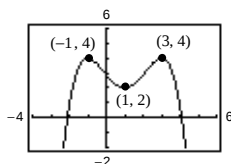
(3, 4): $4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ $(f(3) = 4)$

$0 = 108a_4 + 27a_3 + 6a_2 + a_1$ $(f'(3) = 0)$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}.$$

(d)



72. False

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

74. True

If $f(x)$ is an n th-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

76. False.

The function might not be continuous.

78. Suppose $f'(x)$ changes from positive to negative at c . Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b) . By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b) . Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f .

Section 3.4 Concavity and the Second Derivative Test

2. $y = -x^3 + 3x^2 - 2, y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

4. $f(x) = \frac{x^2 - 1}{2x + 1}, y'' = \frac{-6}{(2x + 1)^3}$

Concave upward: $(-\infty, -\frac{1}{2})$

Concave downward: $(-\frac{1}{2}, \infty)$

6. $y = \frac{1}{270}(-3x^5 + 40x^3 + 135x), y'' = \frac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

10. $y = x + 2 \csc x, \quad (-\pi, \pi)$

$$y' = 1 - 2 \csc x \cot x$$

$$y'' = -2 \csc x(-\csc^2 x) - 2 \cot x(-\csc x \cot x)$$

$$= 2(\csc^3 x + \csc x \cot^2 x)$$

Concave upward: $(0, \pi)$ Concave downward: $(-\pi, 0)$

12. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$$f'(x) = 6x^2 - 6x - 12$$

$$f''(x) = 12x - 6$$

$$f''(x) = 12x - 6 = 0 \text{ when } x = \frac{1}{2}.$$

Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $(\frac{1}{2}, -\frac{13}{2})$

14. $f(x) = 2x^4 - 8x + 3$

$$f'(x) = 8x^3 - 8$$

$$f''(x) = 24x^2 = 0 \text{ when } x = 0.$$

However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all x .Concave upward on $(-\infty, \infty)$

16. $f(x) = x^3(x - 4)$

$$f'(x) = x^3 + 3x^2(x - 4)$$

$$= x^2[x + 3(x - 4)] = 4x^2(x - 3)$$

$$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$$

$$f''(x) = 12x(x - 2) = 0 \text{ when } x = 0, 2.$$

Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0), (2, -16)$

18. $f(x) = x\sqrt{x+1}$, Domain: $[-1, \infty)$

$$f'(x) = (x) \frac{1}{2}(x+1)^{-1/2} + \sqrt{x+1} = \frac{3x+2}{2\sqrt{x+1}}$$

$$f''(x) = \frac{6\sqrt{x+1} - (3x+2)(x+1)^{-1/2}}{4(x+1)} = \frac{3x+4}{4(x+1)^{3/2}}$$

 $f''(x) > 0$ on the entire domain of f (except for $x = -1$, for which $f''(x)$ is undefined).

There are no points of inflection.

Concave upward on $(-1, \infty)$

20. $f(x) = \frac{x+1}{\sqrt{x}}$ Domain: $x > 0$

$$f'(x) = \frac{x-1}{2x^{3/2}}$$

$$f''(x) = \frac{3-x}{4x^{5/2}}$$

Point of inflection: $(3, \frac{4}{\sqrt{3}}) = (3, \frac{4\sqrt{3}}{3})$

Test intervals	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

22. $f(x) = 2 \csc \frac{3x}{2}, 0 < x < 2\pi$

$$f'(x) = -3 \csc \frac{3x}{2} \cot \frac{3x}{2}$$

$$f''(x) = \frac{9}{2} \left(\csc^3 \frac{3x}{2} + \csc \frac{3x}{2} \cot^2 \frac{3x}{2} \right) \neq 0 \text{ for any } x \text{ in the domain of } f.$$

Concave upward: $\left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

No points of inflection

24. $f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

$$f''(x) = 0 \text{ when } x = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Test interval:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

26. $f(x) = x + 2 \cos x, [0, 2\pi]$

$$f'(x) = 1 - 2 \sin x$$

$$f''(x) = -2 \cos x$$

$$f''(x) = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

28. $f(x) = x^2 + 3x - 8$

$$f'(x) = 2x + 3$$

$$f''(x) = 2$$

Critical number: $x = -\frac{3}{2}$

$$f''\left(-\frac{3}{2}\right) > 0$$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

30. $f(x) = -(x - 5)^2$

$$f'(x) = -2(x - 5)$$

$$f''(x) = -2$$

Critical number: $x = 5$

$$f''(5) < 0$$

Therefore, $(5, 0)$ is a relative maximum.

32. $f(x) = x^3 - 9x^2 + 27x$

$$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$$

$$f''(x) = 6(x - 3)$$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all x and, therefore, there are no relative extrema.

34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

$$g'(x) = \frac{-(x - 4)(x - 1)(x + 2)}{2}$$

$$g''(x) = 3 + 3x - \frac{3}{2}x^2$$

Critical numbers: $x = -2, 1, 4$

$$g''(-2) = -9 < 0$$

$(-2, 0)$ is a relative maximum.

$$g''(1) = 9/2 > 0$$

$(1, -10.125)$ is a relative minimum.

$$g''(4) = -9 < 0$$

$(4, 0)$ is a relative maximum.

36. $f(x) = \sqrt{x^2 + 1}$

$$f'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Critical number: $x = 0$

$$f''(x) = \frac{1}{(x^2 + 1)^{3/2}}$$

$$f''(0) = 1 > 0$$

Therefore, $(0, 1)$ is a relative minimum.

38. $f(x) = \frac{x}{x - 1}$

$$f'(x) = \frac{-1}{(x - 1)^2}$$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

40. $f(x) = 2 \sin x + \cos 2x, 0 \leq x \leq 2\pi$

$$f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x - 4 \sin x \cos x = 2 \cos x(1 - 2 \sin x) = 0 \text{ when } x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}.$$

$$f''(x) = -2 \sin x - 4 \cos 2x$$

$$f''\left(\frac{\pi}{6}\right) < 0$$

$$f''\left(\frac{\pi}{2}\right) > 0$$

$$f''\left(\frac{5\pi}{6}\right) < 0$$

$$f''\left(\frac{3\pi}{2}\right) > 0$$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

42. $f(x) = x^2\sqrt{6-x^2}, [-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \frac{3x(4-x^2)}{\sqrt{6-x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

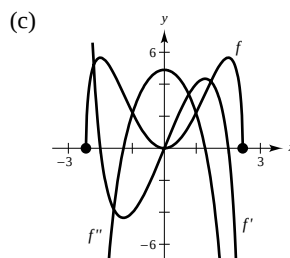
$f''(x) = \frac{6(x^4 - 9x^2 + 12)}{(6-x^2)^{3/2}}$

$f''(x) = 0$ when $x = \pm \sqrt{\frac{9-\sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow (\pm 2, 4\sqrt{2})$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

44. $f(x) = \sqrt{2x} \sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x} \cos x + \frac{\sin x}{\sqrt{2x}}$

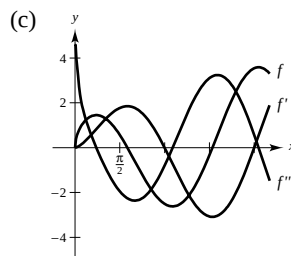
Critical numbers: $x \approx 1.84, 4.82$

$$\begin{aligned} f''(x) &= -\sqrt{2x} \sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}} \\ &= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}} \\ &= \frac{4x \cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}} \end{aligned}$$

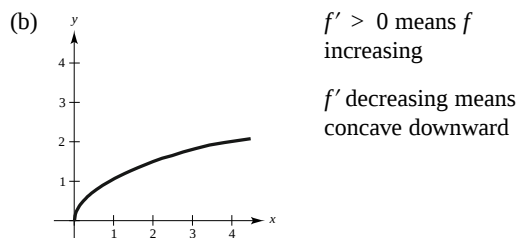
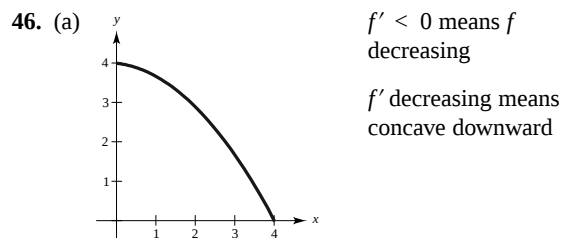
(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

Points of inflection: $(0.75, 0.83), (3.42, -0.72)$



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.



48. (a) The rate of change of sales is increasing.
 $S'' > 0$

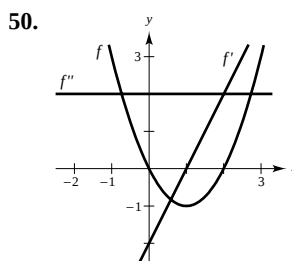
(b) The rate of change of sales is decreasing.
 $S' > 0, S'' < 0$

(c) The rate of change of sales is constant.
 $S' = C, S'' = 0$

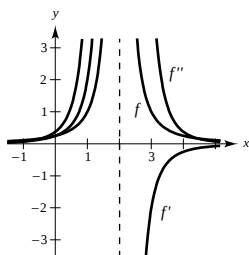
(d) Sales are steady.
 $S = C, S' = 0, S'' = 0$

(e) Sales are declining, but at a lower rate.
 $S' < 0, S'' > 0$

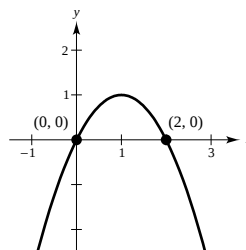
(f) Sales have bottomed out and have started to rise.
 $S' > 0$



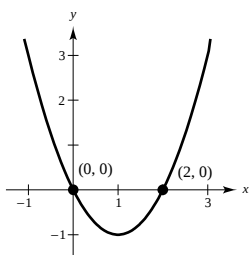
52.



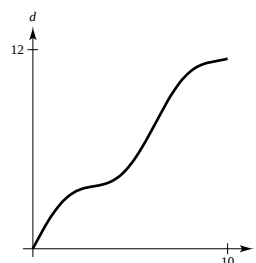
54.



56.



58. (a)



(b) Since the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

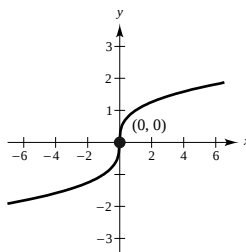
60. (a) $f(x) = \sqrt[3]{x}$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3}$$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$$

$$\left. \begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned} \right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$$f'(2) = 12a + 4b + c = 0, f'(4) = 48a + 8b + c = 0, f''(3) = 18a + 2b = 0$$

$$28a + 6b + c = -1 \quad 18a + 2b = 0$$

$$12a + 4b + c = 0 \quad 16a + 2b = -1$$

$$16a + 2b = -1 \quad 2a = 1$$

$$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$$

$$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$$

64. (a) line
- OA
- :
- $y = -0.06x$
- slope:
- -0.06

line CB : $y = 0.04x + 50$ slope: 0.04

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$(-1000, 60): 60 = (-1000)^3a + (1000)^2b - 1000c + d$$

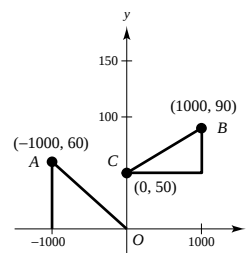
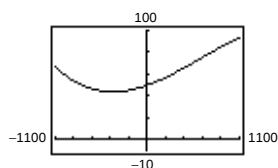
$$-0.06 = (1000)^2 3a - 2000b + c$$

$$(1000, 90): 90 = (1000)^3a + (1000)^2b + 1000c + d$$

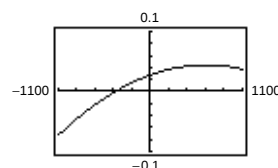
$$0.04 = (1000)^2 3a + 2000b + c$$

The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

- (b)
- $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



- (c)



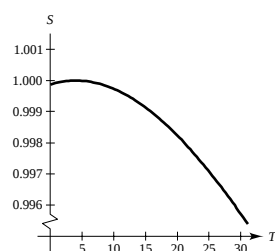
- (d) The steepest part of the road is 6% at the point A.

66. $S = \frac{5.755T^3}{10^8} - \frac{8.521T^2}{10^6} + \frac{0.654T}{10^4} + 0.99987, \quad 0 < T < 25$

- (a) The maximum occurs when
- $T \approx 4^\circ$
- and
- $S \approx 0.999999$
- .

(c) $S(20^\circ) \approx 0.9982$

- (b)



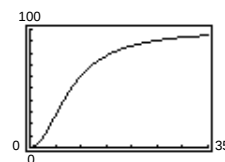
68. $C = 2x + \frac{300,000}{x}$

$$C' = 2 - \frac{300,000}{x^2} = 0 \text{ when } x = 100\sqrt{15} \approx 387$$

By the First Derivative Test, C is minimized when $x \approx 387$ units.

70. $S = \frac{100t^2}{65 + t^2}, \quad t > 0$

- (a)



(b) $S'(t) = \frac{13,000t}{(65 + t^2)^2}$

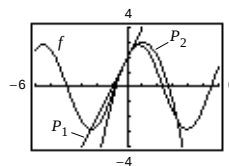
$$S''(t) = \frac{13,000(65 - 3t^2)}{(65 + t^2)^3} = 0 \Rightarrow t = 4.65$$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

- (c)
- $S'(t) > 0$
- for
- $t > 0$
- .

As t increases, the speed increases, but at a slower rate.

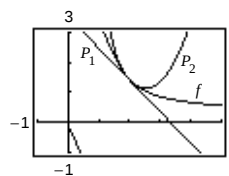
$$\begin{aligned}
 72. \quad f(x) &= 2(\sin x + \cos x), & f(0) &= 2 \\
 f'(x) &= 2(\cos x - \sin x), & f'(0) &= 2 \\
 f''(x) &= 2(-\sin x - \cos x), & f''(0) &= -2 \\
 P_1(x) &= 2 + 2(x - 0) = 2(1 + x)
 \end{aligned}$$



$$\begin{aligned}
 P_1'(x) &= 2 \\
 P_2(x) &= 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2 \\
 P_2'(x) &= 2 - 2x \\
 P_2''(x) &= -2
 \end{aligned}$$

The values of f , P_1 , P_2 , and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

$$\begin{aligned}
 74. \quad f(x) &= \frac{\sqrt{x}}{x-1}, & f(2) &= \sqrt{2} \\
 f'(x) &= \frac{-(x+1)}{2\sqrt{x}(x-1)^2}, & f'(2) &= -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4} \\
 f''(x) &= \frac{3x^2 + 6x - 1}{4x^{3/2}(x-1)^3}, & f''(2) &= \frac{23}{8\sqrt{2}} = \frac{23\sqrt{2}}{16} \\
 P_1(x) &= \sqrt{2} + \left(-\frac{3\sqrt{2}}{4}\right)(x-2) = -\frac{3\sqrt{2}}{4}x + \frac{5\sqrt{2}}{2} \\
 P_1'(x) &= -\frac{3\sqrt{2}}{4}
 \end{aligned}$$



$$\begin{aligned}
 P_2(x) &= \sqrt{2} + \left(-\frac{3\sqrt{2}}{4}\right)(x-2) + \frac{1}{2}\left(\frac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \frac{3\sqrt{2}}{4}(x-2) + \frac{23\sqrt{2}}{32}(x-2)^2 \\
 P_2'(x) &= -\frac{3\sqrt{2}}{4} + \frac{23\sqrt{2}}{16}(x-2) \\
 P_2''(x) &= \frac{23\sqrt{2}}{16}
 \end{aligned}$$

The values of f , P_1 , P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

$$\begin{aligned}
 76. \quad f(x) &= x(x-6)^2 = x^3 - 12x^2 + 36x \\
 f'(x) &= 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0 \\
 f''(x) &= 6x - 24 = 6(x-4) = 0
 \end{aligned}$$

Relative extrema: (2, 32) and (6, 0)

Point of inflection (4, 16) is midway between the relative extrema of f .

78. $p(x) = ax^3 + bx^2 + cx + d$

$$p'(x) = 3ax^2 + 2bx + c$$

$$p''(x) = 6ax + 2b$$

$$6ax + 2b = 0$$

$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

80. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

82. True

$$y = \sin(bx)$$

$$\text{Slope: } y' = b \cos(bx)$$

$$-b \leq y' \leq b \quad (\text{Assume } b > 0)$$

84. False. For example, let $f(x) = (x - 2)^4$.

Section 3.5 Limits at Infinity

2. $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

4. $f(x) = 2 + \frac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

6. $f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1}$

No vertical asymptotes

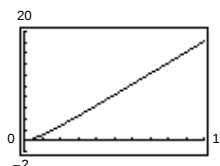
Horizontal asymptote: $y = 2$

Matches (e)

8. $f(x) = \frac{2x^2}{x + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998

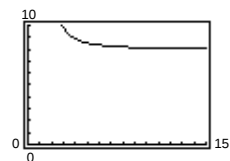
$\lim_{x \rightarrow \infty} f(x) = \infty$ (Limit does not exist.)



10. $f(x) = \frac{8x}{\sqrt{x^2 - 3}}$

x	10^1	10^2	10^3	10^4	10^5	10^6	10^7
$f(x)$	8.12	8.001	8	8	8	8	8

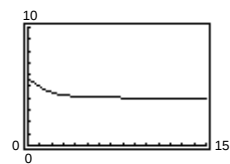
$$\lim_{x \rightarrow \infty} f(x) = 8$$



12. $f(x) = 4 + \frac{3}{x^2 + 2}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4.0	4.0	4	4

$$\lim_{x \rightarrow \infty} f(x) = 4$$



14. (a) $h(x) = \frac{f(x)}{x} = \frac{5x^2 - 3x + 7}{x} = 5x - 3 + \frac{7}{x}$

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad (\text{Limit does not exist})$$

(b) $h(x) = \frac{f(x)}{x^2} = \frac{5x^2 - 3x + 7}{x^2} = 5 - \frac{3}{x} + \frac{7}{x^2}$

$$\lim_{x \rightarrow \infty} h(x) = 5$$

(c) $h(x) = \frac{f(x)}{x^3} = \frac{5x^2 - 3x + 7}{x^3} = \frac{5}{x} - \frac{3}{x^2} + \frac{7}{x^3}$

$$\lim_{x \rightarrow \infty} h(x) = 0$$

16. (a) $\lim_{x \rightarrow \infty} \frac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x \rightarrow \infty} \frac{3 - 2x}{3x - 1} = -\frac{2}{3}$

(c) $\lim_{x \rightarrow \infty} \frac{3 - 2x^2}{3x - 1} = -\infty \quad (\text{Limit does not exist})$

18. (a) $\lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4x^{3/2} + 1} = \frac{5}{4}$

(c) $\lim_{x \rightarrow \infty} \frac{5x^{3/2}}{4\sqrt{x} + 1} = \infty \quad (\text{Limit does not exist})$

20. $\lim_{x \rightarrow \infty} \frac{3x^3 + 2}{9x^3 - 2x^2 + 7} = \frac{3}{9} = \frac{1}{3}$

22. $\lim_{x \rightarrow \infty} \left(4 + \frac{3}{x}\right) = 4 + 0 = 4$

24. $\lim_{x \rightarrow \infty} \left(\frac{1}{2}x - \frac{4}{x^2}\right) = -\infty \quad (\text{Limit does not exist})$

26. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow -\infty} \frac{1}{\left(\frac{\sqrt{x^2 + 1}}{-\sqrt{x^2}}\right)} \quad (\text{for } x < 0, x = -\sqrt{x^2})$

$$= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{x + (1/x)}} = -1$$

$$\begin{aligned}
 28. \lim_{x \rightarrow -\infty} \frac{-3x + 1}{\sqrt{x^2 + x}} &= \lim_{x \rightarrow -\infty} \frac{-3 + (1/x)}{\frac{\sqrt{x^2 + x}}{-\sqrt{x^2}}}, \text{ (for } x < 0 \text{ we have } -\sqrt{x^2} = x) \\
 &= \lim_{x \rightarrow -\infty} \frac{3 - (1/x)}{\sqrt{1 + (1/x)}} = 3
 \end{aligned}$$

$$\begin{aligned}
 30. \lim_{x \rightarrow \infty} \frac{x - \cos x}{x} &= \lim_{x \rightarrow \infty} \left(1 - \frac{\cos x}{x} \right) \\
 &= 1 - 0 = 1
 \end{aligned}$$

$$32. \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$$

Note:

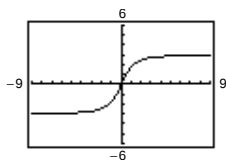
$$\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0 \text{ by the Squeeze Theorem since}$$

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}.$$

$$34. f(x) = \frac{3x}{\sqrt{x^2 + 2}}$$

$$\lim_{x \rightarrow \infty} f(x) = 3$$

$$\lim_{x \rightarrow -\infty} f(x) = -3$$



$$\begin{aligned}
 36. \lim_{x \rightarrow \infty} x \tan \frac{1}{x} &= \lim_{t \rightarrow 0^+} \frac{\tan t}{t} = \lim_{t \rightarrow 0^+} \left[\frac{\sin t}{t} \cdot \frac{1}{\cos t} \right] \\
 &= (1)(1) = 1
 \end{aligned}$$

(Let $x = 1/t$.)

Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

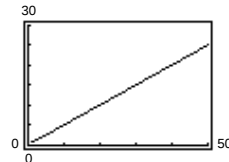
$$38. \lim_{x \rightarrow \infty} (2x - \sqrt{4x^2 + 1}) = \lim_{x \rightarrow \infty} \left[(2x - \sqrt{4x^2 + 1}) \cdot \frac{2x + \sqrt{4x^2 + 1}}{2x + \sqrt{4x^2 + 1}} \right] = \lim_{x \rightarrow \infty} \frac{-1}{2x + \sqrt{4x^2 + 1}} = 0$$

$$\begin{aligned}
 40. \lim_{x \rightarrow -\infty} (3x + \sqrt{9x^2 - x}) &= \lim_{x \rightarrow -\infty} \left[(3x + \sqrt{9x^2 - x}) \cdot \frac{3x - \sqrt{9x^2 - x}}{3x - \sqrt{9x^2 - x}} \right] \\
 &= \lim_{x \rightarrow -\infty} \frac{x}{3x - \sqrt{9x^2 - x}} \\
 &= \lim_{x \rightarrow -\infty} \frac{1}{3 - \frac{\sqrt{9x^2 - x}}{x}} \text{ (for } x < 0 \text{ we have } x = -\sqrt{x^2}) \\
 &= \lim_{x \rightarrow -\infty} \frac{1}{3 + \sqrt{9 - (1/x)}} = \frac{1}{6}
 \end{aligned}$$

42.	x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
	$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

$$\lim_{x \rightarrow \infty} \frac{x^2 - x\sqrt{x^2 - x}}{1} \cdot \frac{x^2 + x\sqrt{x^2 - x}}{x^2 + x\sqrt{x^2 - x}} = \lim_{x \rightarrow \infty} \frac{x^3}{x^2 + x\sqrt{x^2 - x}} = \infty$$

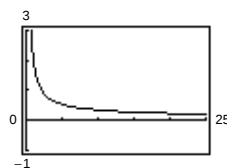
Limit does not exist.



44.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001

$$\lim_{x \rightarrow \infty} \frac{x+1}{x\sqrt{x}} = 0$$



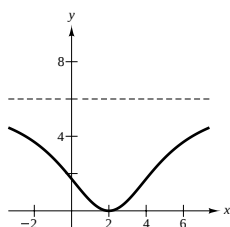
46. $x = 2$ is a critical number.

$$f'(x) < 0 \text{ for } x < 2.$$

$$f'(x) > 0 \text{ for } x > 2.$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 6$$

For example, let $f(x) = \frac{-6}{0.1(x-2)^2 + 1} + 6$.



48. (a) The function is even: $\lim_{x \rightarrow -\infty} f(x) = 5$

(b) The function is odd: $\lim_{x \rightarrow -\infty} f(x) = -5$

50. $y = \frac{x-3}{x-2}$

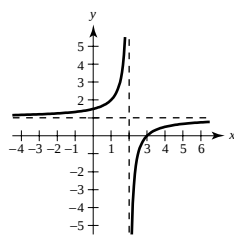
Intercepts: $(3, 0)$, $(0, \frac{3}{2})$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x-3}{x-2} = 1 = \lim_{x \rightarrow \infty} \frac{x-3}{x-2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



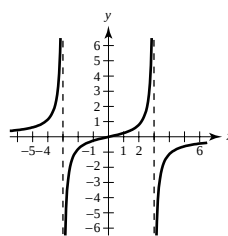
52. $y = \frac{2x}{9-x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



54. $y = \frac{x^2}{x^2-9}$

Intercept: $(0, 0)$

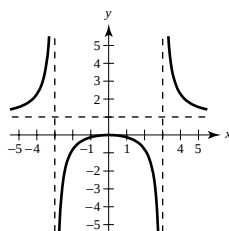
Symmetry: y -axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2-9} = 1 = \lim_{x \rightarrow \infty} \frac{x^2}{x^2-9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



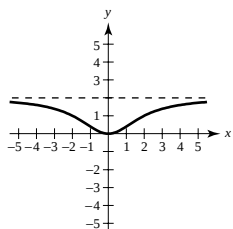
56. $y = \frac{2x^2}{x^2 + 4}$

Intercept: (0, 0)

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Relative minimum: (0, 0)



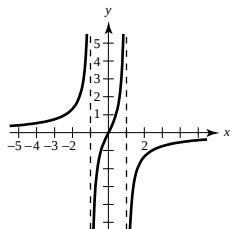
60. $y = \frac{2x}{1 - x^2}$

Intercept: (0, 0)

Symmetry: origin

Horizontal asymptote: $y = 0$ since

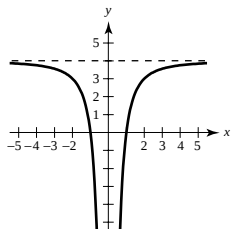
$$\lim_{x \rightarrow -\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x \rightarrow \infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)

64. $y = 4\left(1 - \frac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 4$ Vertical asymptote: $x = 0$ 

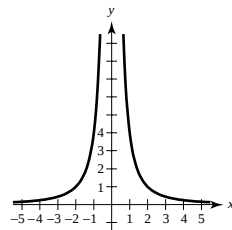
58. $x^2y = 4$

Intercepts: none

Symmetry: y-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \rightarrow -\infty} \frac{4}{x^2} = 0 = \lim_{x \rightarrow \infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)

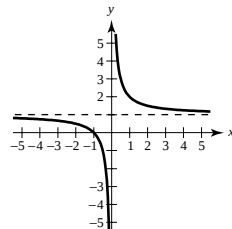
62. $y = 1 + \frac{1}{x}$

Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right) = 1 = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)

66. $y = \frac{x}{\sqrt{x^2 - 4}}$

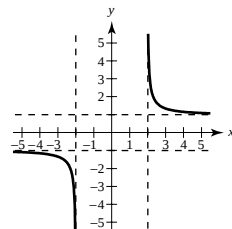
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)

68. $f(x) = \frac{x^2}{x^2 - 1}$

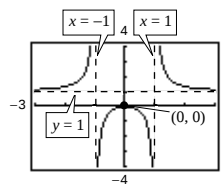
$$f'(x) = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{(x^2 - 1)^2(-2) + 2x(2)(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}$$

Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f , there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$



70. $f(x) = \frac{1}{x^2 - x - 2} = \frac{1}{(x + 1)(x - 2)}$

$$f'(x) = \frac{-(2x - 1)}{(x^2 - x - 2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

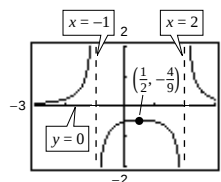
$$f''(x) = \frac{(x^2 - x - 2)^2(-2) + (2x - 1)(2)(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4}$$

$$= \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$$

Since $f''(\frac{1}{2}) < 0$, then $(\frac{1}{2}, -\frac{4}{9})$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f , there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$



72. $f(x) = \frac{x + 1}{x^2 + x + 1}$

$$f'(x) = \frac{-x(x + 2)}{(x^2 + x + 1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3 + 3x^2 - 1)}{(x^2 + x + 1)^3} = 0 \text{ when } x \approx 0.5321, -0.6527, -2.8794.$$

$$f''(0) < 0$$

Therefore, $(0, 1)$ is a relative maximum.

$$f''(-2) > 0$$

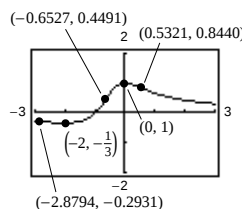
Therefore,

$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$



$$74. g(x) = \frac{2x}{\sqrt{3x^2 + 1}}$$

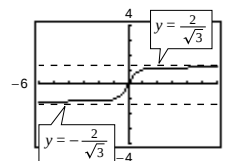
$$g'(x) = \frac{2}{(3x^2 + 1)^{3/2}}$$

$$g''(x) = \frac{-18x}{(3x^2 + 1)^{5/2}}$$

No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm \frac{2}{\sqrt{3}}$

No vertical asymptotes



$$76. f(x) = \frac{2 \sin 2x}{x} \quad \text{Hole at } (0, 4)$$

$$f'(x) = \frac{4x \cos 2x - 2 \sin 2x}{x^2}$$

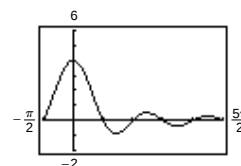
There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

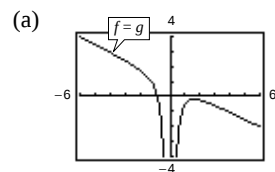
Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes



$$78. f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}, g(x) = -\frac{1}{2}x + 1 - \frac{1}{x^2}$$

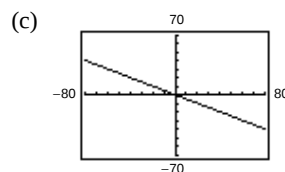


(b)

$$f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}$$

$$= -\left[\frac{x^3}{2x^2} - \frac{2x^2}{2x^2} + \frac{2}{2x^2} \right]$$

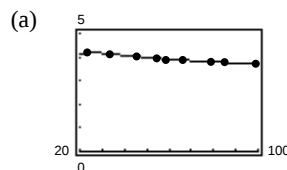
$$= -\frac{1}{2}x + 1 - \frac{1}{x^2} = g(x)$$



The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

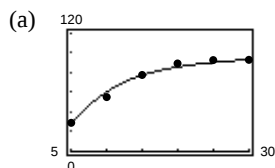
$$80. \lim_{v_1/v_2 \rightarrow \infty} 100 \left[1 - \frac{1}{(v_1/v_2)^c} \right] = 100[1 - 0] = 100\%$$

$$82. y = \frac{3.351t^2 + 42.461t - 543.730}{t^2}$$



(b) Yes. $\lim_{t \rightarrow \infty} y = 3.351$

84. $S = \frac{100t^2}{65 + t^2}, t > 0$



(b) Yes, $\lim_{t \rightarrow \infty} S = \frac{100}{1} = 100$

86. $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m .

Case 1: If $n < m$: $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{\frac{a_n}{x^{m-n}} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{0}{b_m} = 0$

Case 2: If $m = n$: $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{a_n + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{a_n}{b_m}$.

Case 3: If $n > m$: $\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{a_n x^{n-m} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty$.

88. False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. Thus, $y_1' = 1/(2\sqrt{x+1})$ and $y_1'(0) = 1/2$. Finally,

$$y_1'' = -\frac{1}{4(x+1)^{3/2}} \text{ and } y_1''(0) = -\frac{1}{4}.$$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \geq 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/(2\sqrt{x+1}), & x \geq 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x \geq 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x , but $f(x)$ increases without bound.

Section 3.6 A Summary of Curve Sketching

2. The slope of f approaches ∞ as $x \rightarrow 0^-$, and approaches $-\infty$ as $x \rightarrow 0^+$. Matches (C)

6. (a) x_0, x_2, x_4

(c) x_1

(e) x_2, x_3

4. The slope is positive up to approximately $x = 1.5$. Matches (B)

(b) x_2, x_3

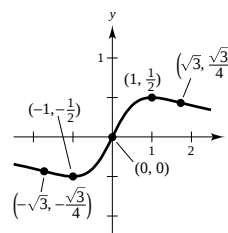
(d) x_1

8. $y = \frac{x}{x^2 + 1}$

$$y' = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0 \text{ when } x = \pm 1.$$

$$y'' = -\frac{2x(3 - x^2)}{(x^2 + 1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

Horizontal asymptote: $y = 0$



	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		-	-	Decreasing, concave down
$x = -\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	-	0	Point of inflection
$-\sqrt{3} < x < -1$		-	+	Decreasing, concave up
$x = -1$	$-\frac{1}{2}$	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 1$		+	-	Increasing, concave down
$x = 1$	$\frac{1}{2}$	0	-	Relative maximum
$1 < x < \sqrt{3}$		-	-	Decreasing, concave down
$x = \sqrt{3}$	$\frac{\sqrt{3}}{4}$	-	0	Point of inflection
$\sqrt{3} < x < \infty$		-	+	Decreasing, concave up

10. $y = \frac{x^2 + 1}{x^2 - 9}$

$$y' = \frac{-20x}{(x^2 - 9)^2} = 0 \quad \text{when } x = 0$$

$$y'' = \frac{60(x^2 + 3)}{(x^2 - 9)^3} < 0 \quad \text{when } x = 0$$

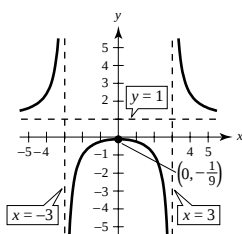
Therefore, $(0, -\frac{1}{9})$ is a relative maximum.

Intercept: $(0, -\frac{1}{9})$

Vertical asymptotes: $x = \pm 3$

Horizontal asymptote: $y = 1$

Symmetric about y-axis



12. $f(x) = \frac{x+2}{x} = 1 + \frac{2}{x}$

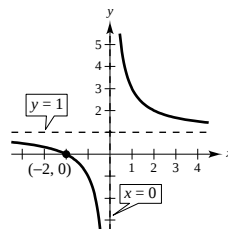
$$f'(x) = \frac{-2}{x^2} < 0 \quad \text{when } x \neq 0.$$

$$f''(x) = \frac{4}{x^3} \neq 0$$

Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$



14. $f(x) = x + \frac{32}{x^2}$

$$f'(x) = 1 - \frac{64}{x^3} = \frac{(x-4)(x^2+4x+16)}{x^3} = 0 \quad \text{when } x = 4.$$

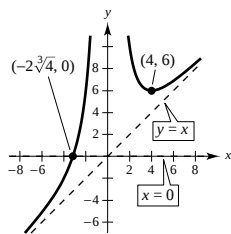
$$f''(x) = \frac{192}{x^4} > 0 \quad \text{if } x \neq 0.$$

Therefore, $(4, 6)$ is a relative minimum.

Intercept: $(-2\sqrt[3]{4}, 0)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



16. $f(x) = \frac{x^3}{x^2 - 4} = x + \frac{4x}{x^2 - 4}$

$$f'(x) = \frac{x^2(x^2 - 12)}{(x^2 - 4)^2} = 0 \quad \text{when } x = 0, \pm 2\sqrt{3}$$

$$f''(x) = \frac{8x(x^2 + 12)}{(x^2 - 4)^3} = 0 \quad \text{when } x = 0$$

Intercept: $(0, 0)$

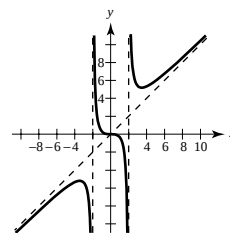
Relative maximum: $(-2\sqrt{3}, -3\sqrt{3})$

Relative minimum: $(2\sqrt{3}, 3\sqrt{3})$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$



$$18. y = \frac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \frac{3}{x - 2}$$

$$y' = 2 - \frac{3}{(x - 2)^2} = \frac{2x^2 - 8x + 5}{(x - 2)^2} = 0 \text{ when } x = \frac{4 \pm \sqrt{6}}{2}.$$

$$y'' = \frac{6}{(x - 2)^3} \neq 0$$

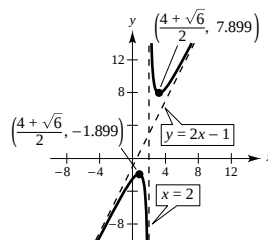
$$\text{Relative maximum: } \left(\frac{4 - \sqrt{6}}{2}, -1.8990 \right)$$

$$\text{Relative minimum: } \left(\frac{4 + \sqrt{6}}{2}, 7.8990 \right)$$

$$\text{Intercept: } (0, -5/2)$$

$$\text{Vertical asymptote: } x = 2$$

$$\text{Slant asymptote: } y = 2x - 1$$



$$20. g(x) = x\sqrt{9 - x} \quad \text{Domain: } x \leq 9$$

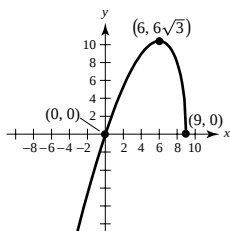
$$g'(x) = \frac{3(6 - x)}{2\sqrt{9 - x}} = 0 \text{ when } x = 6$$

$$g''(x) = \frac{3(x - 12)}{4(9 - x)^{3/2}} < 0 \text{ when } x = 6$$

$$\text{Relative maximum: } (6, 6\sqrt{3})$$

$$\text{Intercepts: } (0, 0), (9, 0)$$

$$\text{Concave downward on } (-\infty, 9)$$



$$22. y = x\sqrt{16 - x^2} \quad \text{Domain: } -4 \leq x \leq 4$$

$$y' = \frac{2(8 - x^2)}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2}$$

$$y'' = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}} = 0 \text{ when } x = 0$$

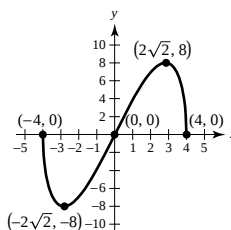
$$\text{Relative maximum: } (2\sqrt{2}, 8)$$

$$\text{Relative minimum: } (-2\sqrt{2}, -8)$$

$$\text{Intercepts: } (0, 0), (\pm 4, 0)$$

$$\text{Symmetric with respect to the origin}$$

$$\text{Point of inflection: } (0, 0)$$



$$24. y = 3(x - 1)^{2/3} - (x - 1)^2$$

$$y' = \frac{2}{(x - 1)^{1/3}} - 2(x - 1) = \frac{2 - 2(x - 1)^{4/3}}{(x - 1)^{1/3}} = 0 \text{ when } x = 0, 2$$

$$(y' \text{ undefined for } x = 1)$$

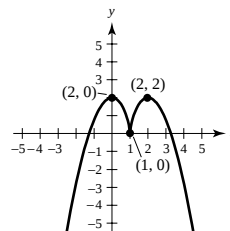
$$y'' = \frac{-2}{3(x - 1)^{4/3}} - 2 < 0 \text{ for all } x \neq 1$$

$$\text{Concave downward on } (-\infty, 1) \text{ and } (1, \infty)$$

$$\text{Relative maximum: } (0, 2), (2, 2)$$

$$\text{Relative minimum: } (1, 0)$$

$$\text{Intercepts: } (0, 2), (1, 0), (-1.280, 0), (3.280, 0)$$

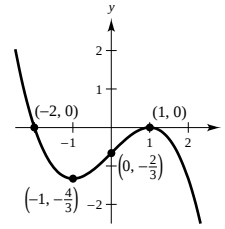


26. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$

$y'' = -2x = 0$ when $x = 0$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	+	0	Point of inflection
$0 < x < 1$		+	-	Increasing, concave down
$x = 1$	0	0	-	Relative maximum
$1 < x < \infty$		-	-	Decreasing, concave down

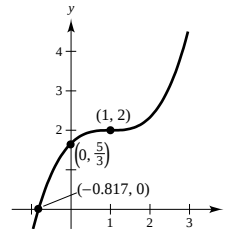


28. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

$f'(x) = (x - 1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x - 1) = 0$ when $x = 1$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		+	-	Increasing, concave down
$x = 1$	2	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up



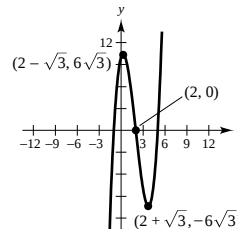
30. $f(x) = (x + 1)(x - 2)(x - 5)$

$f'(x) = (x + 1)(x - 2) + (x + 1)(x - 5) + (x - 2)(x - 5)$

$= 3(x^2 - 4x + 1) = 0$ when $x = 2 \pm \sqrt{3}$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 2 - \sqrt{3}$		+	-	Increasing, concave down
$x = 2 - \sqrt{3}$	$6\sqrt{3}$	0	-	Relative maximum
$2 - \sqrt{3} < x < 2$		-	-	Decreasing, concave down
$x = 2$	0	-	0	Point of inflection
$2 < x < 2 + \sqrt{3}$		-	+	Decreasing, concave up
$x = 2 + \sqrt{3}$	$-6\sqrt{3}$	0	+	Relative minimum
$2 + \sqrt{3} < x < \infty$		+	+	Increasing, concave up



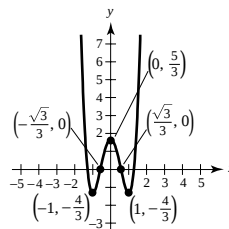
Intercepts: (0, 10), (-1, 0), (2, 0), (5, 0)

32. $y = 3x^4 - 6x^2 + \frac{5}{3}$

$$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0 \text{ when } x = 0, x = \pm 1.$$

$$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	$-4/3$	0	+	Relative minimum
$-1 < x < -\frac{\sqrt{3}}{3}$		+	+	Increasing, concave up
$x = -\frac{\sqrt{3}}{3}$	0	+	0	Point of inflection
$-\frac{\sqrt{3}}{3} < x < 0$		+	-	Increasing, concave down
$x = 0$	$5/3$	0	-	Relative maximum
$0 < x < \frac{\sqrt{3}}{3}$		-	-	Decreasing, concave down
$x = \frac{\sqrt{3}}{3}$	0	-	0	Point of inflection
$\frac{\sqrt{3}}{3} < x < 1$		-	+	Decreasing, concave up
$x = 1$	$-4/3$	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

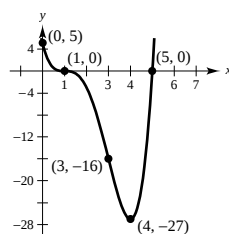


34. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

$$f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2 = 0 \text{ when } x = 1, x = 4.$$

$$f''(x) = 12x^2 - 48x + 36 = 12(x - 3)(x - 1) = 0 \text{ when } x = 3, x = 1.$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		-	+	Decreasing, concave up
$x = 1$	0	0	0	Point of inflection
$1 < x < 3$		-	-	Decreasing, concave down
$x = 3$	-16	-	0	Point of inflection
$3 < x < 4$		-	+	Decreasing, concave up
$x = 4$	-27	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave up

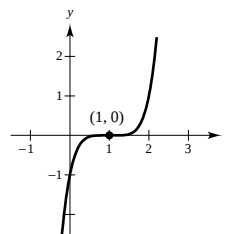


36. $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.

	y	y'	y''	Conclusion
$-\infty < x < 1$		+	-	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up

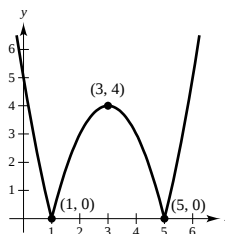


38. $y = |x^2 - 6x + 5|$

$$y' = \frac{2(x - 3)(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \frac{2(x - 3)(x - 5)(x - 1)}{|(x - 5)(x - 1)|}$$

$= 0$ when $x = 3$ and undefined when $x = 1, x = 5$.

$$y'' = \frac{2(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \frac{2(x - 5)(x - 1)}{|(x - 5)(x - 1)|} \text{ undefined when } x = 1, x = 5.$$



	y	y'	y''	Conclusion
$-\infty < x < 1$		-	+	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		+	-	Increasing, concave down
$x = 3$	4	0	-	Relative maximum
$3 < x < 5$		-	-	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		+	+	Increasing, concave up

40. $y = \cos x - \frac{1}{2} \cos 2x, 0 \leq x \leq 2\pi$

$y' = -\sin x + \sin 2x = -\sin x(1 - 2 \cos x) = 0$ when $x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$.

$y'' = -\cos x + 2 \cos 2x = -\cos x + 2(2 \cos^2 x - 1)$

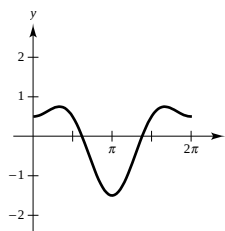
$= 4 \cos^2 x - \cos x - 2 = 0$ when $\cos x = \frac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931$.

Therefore, $x \approx 0.5678$ or $5.7154, x \approx 2.2057$ or 4.0775 .

Relative maxima: $\left(\frac{\pi}{3}, \frac{3}{4}\right), \left(\frac{5\pi}{3}, \frac{3}{4}\right)$

Relative minimum: $\left(\pi, -\frac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323), (2.2057, -0.4449), (5.7154, 0.6323), (4.0775, -0.4449)$



42. $y = 2(x - 2) + \cot x, 0 < x < \pi$

$$y' = 2 - \csc^2 x = 0 \text{ when } x = \frac{\pi}{4}, \frac{3\pi}{4}$$

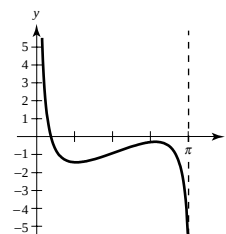
$$y'' = 2 \csc^2 x \cot x = 0 \text{ when } x = \frac{\pi}{2}$$

Relative maximum: $\left(\frac{3\pi}{4}, \frac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 3\right)$

Point of inflection: $\left(\frac{\pi}{2}, \pi - 4\right)$

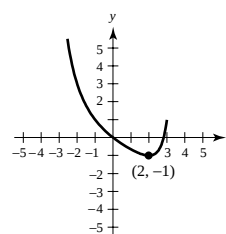
Vertical asymptotes: $x = 0, \pi$



44. $y = \sec^2\left(\frac{\pi x}{8}\right) - 2 \tan\left(\frac{\pi x}{8}\right) - 1, -3 < x < 3$

$$y' = 2 \sec^2\left(\frac{\pi x}{8}\right) \tan\left(\frac{\pi x}{8}\right) \left(\frac{\pi}{8}\right) - 2 \sec^2\left(\frac{\pi x}{8}\right) \left(\frac{\pi}{8}\right) = 0 \Rightarrow x = 2$$

Relative minimum: $(2, -1)$



46. $g(x) = x \cot x, -2\pi < x < 2\pi$

$$g'(x) = \frac{\sin x \cos x - x}{\sin^2 x}$$

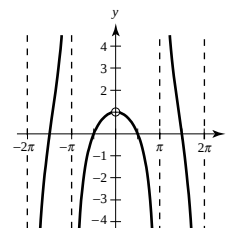
$$g'(0) \text{ does not exist. But } \lim_{x \rightarrow 0} x \cot x = \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1.$$

Vertical asymptotes: $x = \pm 2\pi, \pm \pi$

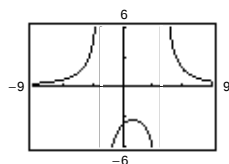
Intercepts: $\left(-\frac{3\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

Symmetric with respect to y-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$



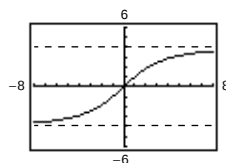
48. $f(x) = 5\left(\frac{1}{x-4} - \frac{1}{x+2}\right)$



$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

50. $f(x) = \frac{4x}{\sqrt{x^2 + 15}}$



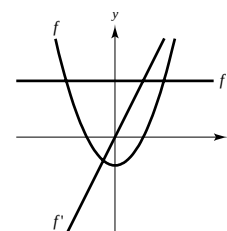
$y = \pm 4$ horizontal asymptotes

$(0, 0)$ point of inflection

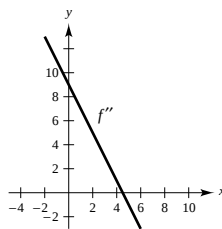
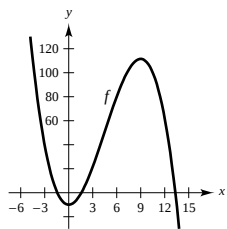
52. f'' is constant.

f' is linear.

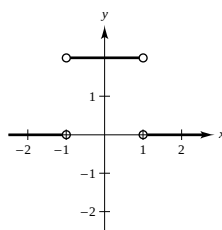
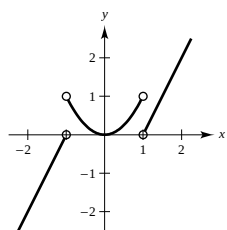
f is quadratic.



54.


 (any vertical translate of f will do)

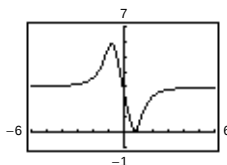
56.


 (any vertical translate of the 3 segments of f will do)

58. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

$$60. g(x) = \frac{3x^4 - 5x + 3}{x^4 + 1}$$

Vertical asymptote: none

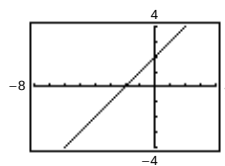
 Horizontal asymptote: $y = 3$


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

$$62. g(x) = \frac{x^2 + x - 2}{x - 1}$$

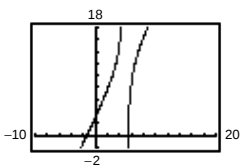
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined,} & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



hole at $(1, 3)$

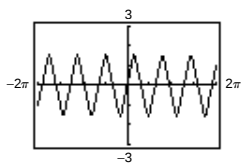
$$64. g(x) = \frac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \frac{5}{x - 5}$$



The graph appears to approach the slant asymptote $y = 2x + 2$.

66. $f(x) = \tan(\sin \pi x)$

(a)



(c) Periodic with period 2

(e) On $(0, 1)$, the graph of f is concave downward.

68. Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x+3}$$

72. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2), a \neq 0$

$$f'(x) = a^2x - a = a(ax - 1) = 0 \text{ when } x = \frac{1}{a}.$$

$$f''(x) = a^2 > 0 \text{ for all } x.$$

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

74. Tangent line at P : $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0) f'(x_0)$$

x-intercept: $(x_0 + f(x_0) f'(x_0), 0)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(g) $|AB| = |x_0 - (x_0 + f(x_0) f'(x_0))| = |f(x_0) f'(x_0)|$

(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$$= -\tan(\sin \pi x) = -f(x)$$

Symmetry with respect to the origin

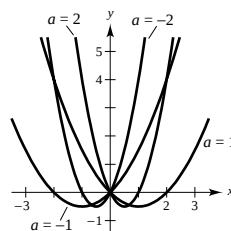
(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

70. Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1-x^2}{x}$$

(b)



(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $(0, f(x_0) - x_0 f'(x_0))$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(f) $|PC|^2 = y_0^2 + \left(\frac{f(x_0)}{f'(x_0)}\right)^2 = \frac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC|^2 = \left|\frac{f(x_0) \sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|^2$$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)| \sqrt{1 + [f'(x_0)]^2}$$

Section 3.7 Optimization Problems

2. Let x and y be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$

$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

P is a maximum when $x = y = S/2$.

6. Let x and y be two positive numbers such that $x + 2y = 100$.

$$P = xy = y(100 - 2y) = 100y - 2y^2$$

$$\frac{dP}{dy} = 100 - 4y = 0 \text{ when } y = 25.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 25.$$

P is a maximum when $x = 50$ and $y = 25$.

8. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$

$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)

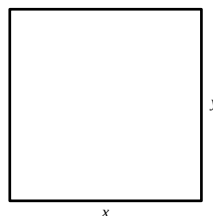
4. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + 3y = \frac{192}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{192}{y^2} = 0 \text{ when } y = 8.$$

$$\frac{d^2S}{dy^2} = \frac{384}{y^3} > 0 \text{ when } y = 8.$$

S is minimum when $y = 8$ and $x = 24$.



10. Let x be the length and y the width of the rectangle.

$$xy = A$$

$$y = \frac{A}{x}$$

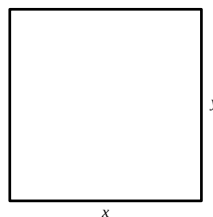
$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

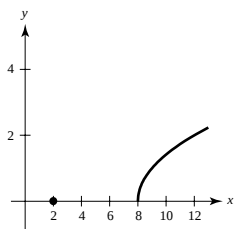
P is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)



12. $f(x) = \sqrt{x-8}$, $(2, 0)$

From the graph, it is clear that $(8, 0)$ is the closest point on the graph of f to $(2, 0)$.



16. $F = \frac{v}{22 + 0.02v^2}$

$$\frac{dF}{dv} = \frac{22 - 0.02v^2}{(22 + 0.02v^2)^2}$$

$$= 0 \text{ when } v = \sqrt{1100} \approx 33.166.$$

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

14. $f(x) = (x+1)^2$, $(5, 3)$

$$\begin{aligned} d &= \sqrt{(x-5)^2 + [(x+1)^2 - 3]^2} \\ &= \sqrt{x^2 - 10x + 25 + (x^2 + 2x - 2)^2} \\ &= \sqrt{x^2 - 10x + 25 + x^4 + 4x^3 - 8x + 4} \\ &= \sqrt{x^4 + 4x^3 + x^2 - 18x + 29} \end{aligned}$$

Since d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 + 4x^3 + x^2 - 18x + 29$$

$$g'(x) = 4x^3 + 12x^2 + 2x - 18$$

$$= 2(x-1)(2x^2 + 8x + 9)$$

By the First Derivative Test, $x = 1$ yields a minimum. Hence, $(1, 4)$ is closest to $(5, 3)$.

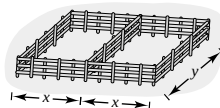
18. $4x + 3y = 200$ is the perimeter. (see figure)

$$A = 2xy = 2x\left(\frac{200 - 4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(50 - 2x) = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 25.$$

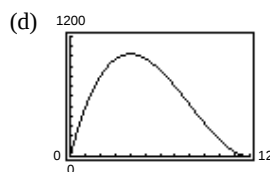
A is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.



20. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

(b) $V = x(24 - 2x)^2$, $0 < x < 12$



The maximum volume seems to be 1024.

$$(c) \quad \frac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$$

$$= 12(12 - x)(4 - x) = 0 \text{ when } x = 12, 4 \text{ (12 is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 12(2x - 16)$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = 4.$$

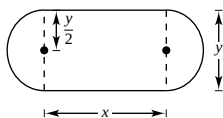
When $x = 4$, $V = 1024$ is maximum.

22. (a) $P = 2x + 2\pi r$

$$= 2x + 2\pi\left(\frac{y}{2}\right)$$

$$= 2x + \pi y = 200$$

$$\Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$$

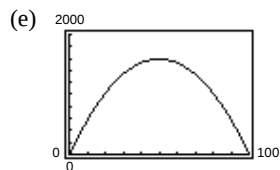


(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) \approx 1528$

The maximum area of the rectangle is approximately 1592 m².

(c) $A = xy = x \frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$



Maximum area is approximately

$$1591.55 \text{ m}^2 \text{ (} x = 50 \text{ m).}$$

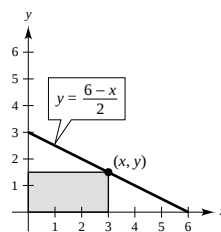
24. You can see from the figure that $A = xy$ and $y = \frac{6 - x}{2}$.

$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

A is a maximum when $x = 3$ and $y = 3/2$.



26. (a) $A = \frac{1}{2} \text{ base} \times \text{height}$

$$= \frac{1}{2}(2\sqrt{16-h^2})(4+h)$$

$$= \sqrt{16-h^2}(4+h)$$

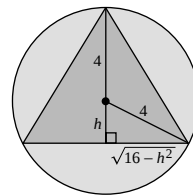
$$\frac{dA}{dh} = \frac{1}{2}(16-h^2)^{-1/2}(-2h)(4+h) + (16-h^2)^{1/2}$$

$$= (16-h^2)^{-1/2}[-h(4+h) + (16-h^2)]$$

$$= \frac{-2(h^2+2h-8)}{\sqrt{16-h^2}} = \frac{-2(h+4)(h-2)}{\sqrt{16-h^2}}$$

$$\frac{dA}{dh} = 0 \text{ when } h = 2, \text{ which is a maximum by the First Derivative Test.}$$

Hence, the sides are $2\sqrt{16-h^2} = 4\sqrt{3}$, an equilateral triangle. Area = $12\sqrt{3}$ sq. units.



(b) $\cos \alpha = \frac{4+h}{\sqrt{8}\sqrt{4+h}} = \frac{\sqrt{4+h}}{\sqrt{8}}$

$$\tan \alpha = \frac{\sqrt{16-h^2}}{4+h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)(\sqrt{16-h^2})(4+h)$$

$$= (4+h)^2 \tan \alpha$$

$$= 64 \cos^4 \alpha \tan \alpha$$

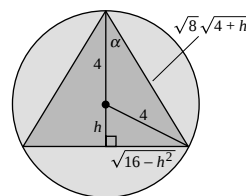
$$A'(\alpha) = 64[\cos^4 \alpha \sec^2 \alpha + 4 \cos^3 \alpha (-\sin \alpha) \tan \alpha] = 0$$

$$\Rightarrow \cos^4 \alpha \sec^2 \alpha = 4 \cos^3 \alpha \sin \alpha \tan \alpha$$

$$1 = 4 \cos \alpha \sin \alpha \tan \alpha$$

$$\frac{1}{4} = \sin^2 \alpha$$

$$\sin \alpha = \frac{1}{2} \Rightarrow \alpha = 30^\circ \text{ and } A = 12\sqrt{3}.$$

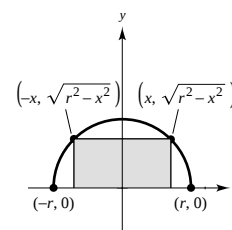


(c) Equilateral triangle

28. $A = 2xy = 2x\sqrt{r^2-x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.

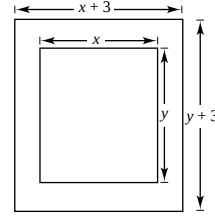


$$30. \quad xy = 36 \Rightarrow y = \frac{36}{x}$$

$$\begin{aligned} A &= (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right) \\ &= 36 + \frac{108}{x} + 3x + 9 \end{aligned}$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

Dimensions: 9×9



$$32. \quad V = \pi r^2 h = V_0 \text{ cubic units or } h = \frac{V_0}{\pi r^2}$$

$$S = 2\pi r^2 + 2\pi r h = 2\left(\pi r^2 + \frac{V_0}{r}\right)$$

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V_0}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ units.}$$

$$h = \frac{V_0}{\pi\left(\sqrt[3]{V_0/2\pi}\right)^2} = \frac{V_0(2\pi)^{2/3}}{\pi V_0^{2/3}} = \frac{2V_0^{1/3}}{(2\pi)^{1/3}} = 2r$$

By the First Derivative Test, this will yield the minimum surface area.

$$34. \quad V = \pi r^2 x$$

$$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r \quad (\text{see figure})$$

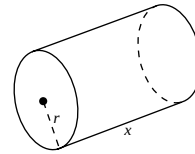
$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.



$$36. \quad V = \pi x^2 h = \pi x^2(2\sqrt{r^2 - x^2}) = 2\pi x^2\sqrt{r^2 - x^2} \quad (\text{see figure})$$

$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

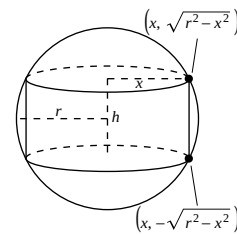
$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \Rightarrow x = \frac{\sqrt{6}r}{3}.$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$



38. No. The volume will change because the shape of the container changes when squeezed.

$$40. V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi r h) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12,000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

42. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot \frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot \frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, \quad 0 \leq x \leq \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot \frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot \frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot \frac{\pi}{5}\right)x^2 + \frac{3}{2}(\sqrt{3})\left(\frac{20 - 5x}{6}\right)^2, \quad 0 \leq x \leq 4.$$

$$A' = \frac{5}{2}\left(\cot \frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot \frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot \frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, \quad 0 \leq x \leq 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot \frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, \quad 0 \leq x \leq \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

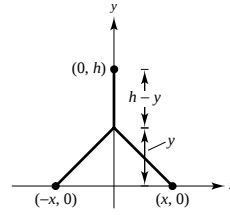
44. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

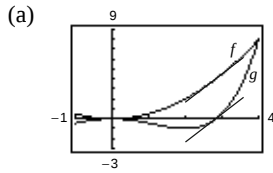
$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



46. $f(x) = \frac{1}{2}x^2$ $g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$



(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$
 $d'(x) = 2x - \frac{1}{4}x^3 = 0 \Rightarrow 8x = x^3$

$$\Rightarrow x = 0, 2\sqrt{2} \text{ (in } [0, 4])$$

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

- (c) $f'(x) = x$, Tangent line at $(2\sqrt{2}, 4)$ is

$$y - 4 = 2\sqrt{2}(x - 2\sqrt{2})$$

$$y = 2\sqrt{2}x - 4.$$

$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $(2\sqrt{2}, 0)$ is

$$y - 0 = \left(\frac{1}{4}(2\sqrt{2})^3 - 2\sqrt{2}\right)(x - 2\sqrt{2})$$

$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

- (d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point x where the distance is maximum.

48. Let F be the illumination at point P which is x units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d-x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d-x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d-x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d-x}$$

$$(d-x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

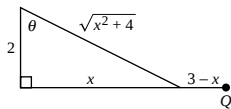
$$d\sqrt[3]{I_1} = x(\sqrt[3]{I_1} + \sqrt[3]{I_2})$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d-x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

50. (a) $T = \frac{\sqrt{x^2 + 4}}{2} + \frac{(3-x)}{4}$



(b) $\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} - \frac{1}{4} = 0$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{2}$$

$$2x^2 = x^2 + 4$$

$$x^2 = 4$$

$$x = 2$$

$$T(2) = \sqrt{2} + \frac{1}{4} \text{ hours}$$

—CONTINUED—

50. —CONTINUED—

$$(c) \quad T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{(3-x)}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} - \frac{1}{v_2} = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{v_1}{v_2}$$

$$\sin \theta = \frac{v_1}{v_2}$$

θ depends on $\frac{v_1}{v_2}$ only.

$$(d) \text{ Cost} = \sqrt{x^2 + 4} C_1 + (3-x)C_2$$

$$= \frac{\sqrt{x^2 + 4}}{(1/C_1)} + \frac{(3-x)}{(1/C_2)}$$

$$\text{From above, } \sin \theta = \frac{1/C_1}{1/C_2} = \frac{C_2}{C_1}$$

$$52. \quad T = \frac{\sqrt{x^2 + d_1^2}}{v_1} + \frac{\sqrt{d_2^2 + (a-x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2 + (a-x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + d_1^2}} = \sin \theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2 + (a-x)^2}} = -\sin \theta_2$$

we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2 + d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2 + (a-x)^2]^{3/2}} > 0$$

this condition yields a minimum time.

$$54. \quad C(x) = 2k\sqrt{x^2 + 4} + k(4-x)$$

$$C'(x) = \frac{2xk}{\sqrt{x^2 + 4}} - k = 0$$

$$2x = \sqrt{x^2 + 4}$$

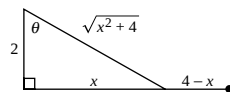
$$4x^2 = x^2 + 4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

$$\text{Or, use Exercise 50(d): } \sin \theta = \frac{C_2}{C_1} = \frac{1}{2} \Rightarrow \theta = 30^\circ.$$

$$\text{Thus, } x = \frac{2}{\sqrt{3}}.$$



$$56. \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{144 - r^2}$$

$$\frac{dV}{dr} = \frac{1}{3}\pi \left[r^2 \left(\frac{1}{2} \right) (144 - r^2)^{-1/2} (-2r) + 2r \sqrt{144 - r^2} \right]$$

$$= \frac{1}{3}\pi \left[\frac{288r - 3r^3}{\sqrt{144 - r^2}} \right] = \pi \left[\frac{r(96 - r^2)}{\sqrt{144 - r^2}} \right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

$$\text{Area of circle: } A = \pi(12)^2 = 144\pi$$

$$\text{Lateral surface area of cone: } S = \pi(4\sqrt{6})\sqrt{(4\sqrt{6})^2 + (4\sqrt{3})^2} = 48\sqrt{6}\pi$$

$$\text{Area of sector: } 144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}(3 - \sqrt{6}) \approx 1.153 \text{ radians or } 66^\circ$$

58. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$$P = (0.12)d - id$$

$$d = ki^2 \text{ (since } d \text{ is proportional to } i^2\text{)}$$

$$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$$

$$\frac{dP}{di} = k(0.24i - 3i^2) = 0 \text{ when } i = \frac{0.24}{3} = 0.08.$$

$$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0 \text{ when } i = 0.08 \text{ (Note: } k > 0\text{)}.$$

The profit is a maximum when $i = 8\%$.

60.
$$P = -\frac{1}{10}s^3 + 6s^2 + 400$$

(a)
$$\frac{dP}{ds} = -\frac{3}{10}s^2 + 12s = -\frac{3}{10}s(s - 40) = 0 \text{ when } s = 0, s = 40.$$

$$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12$$

$$\frac{d^2P}{ds^2}(0) > 0 \Rightarrow s = 0 \text{ yields a minimum.}$$

$$\frac{d^2P}{ds^2}(40) < 0 \Rightarrow s = 40 \text{ yields a maximum.}$$

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3,600,000$).

(b)
$$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12 = 0 \text{ when } s = 20.$$

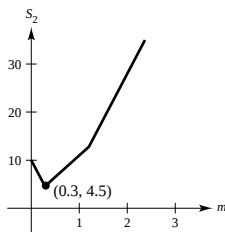
The point of diminishing returns occurs when $s = 20$, which corresponds to \$20,000 being spent on advertising.

62.
$$S_2 = |4m - 1| + |5m - 6| + |10m - 3|$$

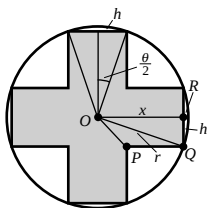
Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7 \text{ mi.}$$



64. (a) Label the figure so that $r^2 = x^2 + h^2$.
Then, the area A is 8 times the area of the region given by $OPQR$:



$$\begin{aligned} A &= 8 \left[\frac{1}{2} h^2 + (x - h)h \right] \\ &= 8 \left[\frac{1}{2} (r^2 - x^2) + (x - \sqrt{r^2 - x^2}) \sqrt{r^2 - x^2} \right] \\ &= 8x \sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\ A'(x) &= 8 \sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0 \end{aligned}$$

$$\begin{aligned} \frac{8x^2}{\sqrt{r^2 - x^2}} &= 8x + 8 \sqrt{r^2 - x^2} \\ x^2 &= x \sqrt{r^2 - x^2} + (r^2 - x^2) \\ 2x^2 - r^2 &= x \sqrt{r^2 - x^2} \end{aligned}$$

$$4x^4 - 4x^2 r^2 + r^4 = x^2 (r^2 - x^2)$$

$$5x^4 - 5x^2 r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10} [5 \pm \sqrt{5}].$$

Take positive value.

$$x = r \sqrt{\frac{5 + \sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

- (c) Note that $x^2 = \frac{r^2}{10}(5 + \sqrt{5})$ and $r^2 - x^2 = \frac{r^2}{10}(5 - \sqrt{5})$.

$$\begin{aligned} A(x) &= 8x \sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\ &= 8 \left[\frac{r^2}{10}(5 + \sqrt{5}) \frac{r^2}{10}(5 - \sqrt{5}) \right]^{1/2} + 4 \frac{r^2}{10}(5 + \sqrt{5}) - 4r^2 \\ &= 8 \left[\frac{r^4}{10}(20) \right]^{1/2} + 2r^2 + \frac{2}{5} \sqrt{5} r^2 - 4r^2 \\ &= \frac{8}{5} r^2 \sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5} r^2 \\ &= 2r^2 \left[\frac{4}{5} \sqrt{5} - 1 + \frac{\sqrt{5}}{5} \right] = 2r^2 (\sqrt{5} - 1) \end{aligned}$$

Using the angle approach, note that $\tan \theta = 2$, $\sin \theta = \frac{2}{\sqrt{5}}$ and $\sin^2 \left(\frac{\theta}{2} \right) = \frac{1}{2} (1 - \cos \theta) = \frac{1}{2} \left(1 - \frac{1}{\sqrt{5}} \right)$.

$$\begin{aligned} \text{Thus, } A(\theta) &= 4r^2 \left(\sin \theta - \sin^2 \frac{\theta}{2} \right) \\ &= 4r^2 \left(\frac{2}{\sqrt{5}} - \frac{1}{2} \left(1 - \frac{1}{\sqrt{5}} \right) \right) \\ &= \frac{4r^2 (\sqrt{5} - 1)}{2} = 2r^2 (\sqrt{5} - 1) \end{aligned}$$

- (b) Note that $\sin \frac{\theta}{2} = \frac{h}{r}$ and $\cos \frac{\theta}{2} = \frac{x}{r}$. The area A of the cross equals the sum of two large rectangles minus the common square in the middle.

$$\begin{aligned} A &= 2(2x)(2h) - 4h^2 = 8xh - 4h^2 \\ &= 8r^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} - 4r^2 \sin^2 \frac{\theta}{2} \\ &= 4r^2 \left(\sin \theta - \sin^2 \frac{\theta}{2} \right) \end{aligned}$$

$$A'(\theta) = 4r^2 \left(\cos \theta - \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) = 0$$

$$\cos \theta = \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin \theta$$

$$\tan \theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4^\circ$$

Section 3.8 Newton's Method

2. $f(x) = 2x^2 - 3$

$$f'(x) = 4x$$

$$x_1 = 1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	4	$-\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{5}{4} = 1.25$	0.125	5.0	0.025	1.225

4. $f(x) = \tan x$

$$f'(x) = \sec^2 x$$

$$x_1 = 0.1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

6. $f(x) = x^5 + x - 1$

$$f'(x) = 5x^4 + 1$$

Approximation of the zero of f is 0.755.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.4688	1.3125	-0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

8. $f(x) = x - 2\sqrt{x+1}$

$$f'(x) = 1 - \frac{1}{\sqrt{x+1}}$$

Approximation of the zero of f is 4.8284.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	5	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	.00085	4.8284

10. $f(x) = 1 - 2x^3$

$$f'(x) = -6x^2$$

Approximation of the zero of f is 0.7937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	-6	0.1667	0.8333
2	0.8333	-0.1573	-4.1663	0.0378	0.7955
3	0.7955	-0.0068	-3.7969	0.0018	0.7937
4	0.7937	0.0000	-3.7798	0.0000	0.7937

12. $f(x) = \frac{1}{2}x^4 - 3x - 3$

$$f'(x) = 2x^3 - 3$$

Approximation of the zero of f is -0.8937 .

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	0.5	-5	-0.1	-0.9
2	-0.9	0.0281	-4.458	-0.0063	-0.8937
3	-0.8937	0.0001	-4.4276	0.0000	-0.8937

Approximation of the zero of f is 2.0720.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	-1	13	-0.0769	2.0769
2	2.0769	0.0725	14.9175	0.0049	2.0720
3	2.0720	-0.0003	14.7910	0.0000	2.0720

14. $f(x) = x^3 - \cos x$

$$f'(x) = 3x^2 + \sin x$$

Approximation of the zero of f is 0.866.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0001	3.0087	0.0000	0.8655

16. $h(x) = f(x) - g(x) = 3 - x - \frac{1}{x^2 + 1}$

$$h'(x) = -1 + \frac{2x}{(x^2 + 1)^2}$$

Point of intersection of the graphs of f and g occurs when $x \approx 2.893$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	2.9000	-0.0063	-0.9345	0.0067	2.8933
2	2.8933	0.0000	-0.9341	0.0000	2.8933

18. $h(x) = f(x) - g(x) = x^2 - \cos x$

$$h'(x) = 2x + \sin x$$

One point of intersection of the graphs of f and g occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the y -axis, the other point of intersection occurs when $x \approx -0.824$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.8000	-0.0567	2.3174	-0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

20. $f(x) = x^n - a = 0$

$$f'(x) = nx^{n-1}$$

$$\begin{aligned} x_{i+1} &= x_i - \frac{x_i^n - a}{nx_i^{n-1}} \\ &= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}} \end{aligned}$$

22. $x_{i+1} = \frac{x_i^2 + 5}{2x_i}$

i	1	2	3	4
x_i	2.0000	2.2500	2.2361	2.2361

$$\sqrt{5} \approx 2.236$$

26. $f(x) = \tan x$
 $f'(x) = \sec^2 x$

Approximation of the zero: 3.142

28. $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$$y' = 12x^2 - 24x + 12 = f'(x)$$

$$x_1 = \frac{3}{2}$$

 $f'(x_2) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{1}{2}$	1
2	1	1	0	—	—

24. $x_{i+1} = \frac{2x_i^3 + 15}{3x_i^2}$

i	1	2	3	4
x_i	2.5000	2.4667	2.4662	2.4662

$$\sqrt[3]{15} \approx 2.466$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	-0.1425	1.0203	-0.1397	3.1397
2	3.1397	-0.0019	1.0000	-0.0019	3.1416
3	3.1416	0.0000	1.0000	0.0000	3.1416

30. $f(x) = 2 \sin x + \cos 2x$

$$f'(x) = 2 \cos x - 2 \sin 2x$$

$$x_1 = \frac{3\pi}{2}$$

 Fails because $f'(x_1) = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
1	$\frac{3\pi}{2}$	-3	0

 32. Newton's Method could fail if $f'(c) \approx 0$, or if the initial value x_1 is far from c .

34. Let $g(x) = f(x) - x = \cot x - x$

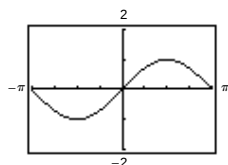
$$g'(x) = -\csc^2 x - 1.$$

The fixed point is approximately 0.86.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.3579	-2.4123	0.1484	0.8516
2	0.8516	0.0240	-2.7668	-0.0087	0.8603
3	0.8603	0.0001	-2.7403	0.0000	0.8603

36. $f(x) = \sin x$, $f'(x) = \cos x$

(a)



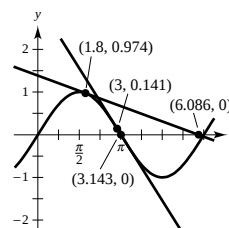
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)


 The x -intercepts correspond to the values resulting from the first iteration of Newton's Method.

 (e) If the initial guess x_1 is not "close to" the desired zero of the function, the x -intercept of the tangent line may approximate another zero of the function.

38. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$$\frac{1}{3} \approx 0.333$$

(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$$\frac{1}{11} \approx 0.091$$

40. $f(x) = x \sin x, (0, \pi)$

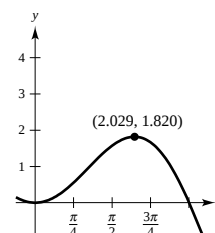
$$f'(x) = x \cos x + \sin x = 0$$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$$[F'(x) = 2 \cos x - x \sin x]$$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	2.0000	0.0770	-2.6509	-0.0290	2.0290
2	2.0290	-0.0007	-2.7044	0.0002	2.0288

Approximation to the critical number: 2.029



42. $y = f(x) = x^2, (4, -3)$

$$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$$

d is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

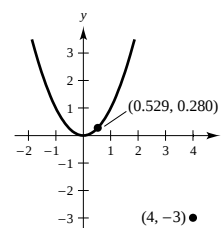
$$g(x) = D' = 4x^3 + 14x - 8$$

$$g'(x) = 12x^2 + 14$$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	0.5000	-0.5000	17.0000	-0.0294	0.5294
2	0.5294	0.0051	17.3632	0.0003	0.5291
3	0.5291	-0.0001	17.3594	0.0000	0.5291

$$x \approx 0.529$$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.



44. Maximize: $C = \frac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$$f'(x) = 12t^3 + 6t^2 - 300.$$

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

46. $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843$, $1 \leq x \leq 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248$.

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	-5.0750	37.7240	-0.1345	1.1345
2	1.1345	-0.2805	33.5849	-0.0084	1.1429
3	1.1429	0.0006	33.3293	0.0000	1.1429

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.5000	4.6725	-24.8760	-0.1878	3.6878
2	3.6878	-0.3286	-28.3550	0.0116	3.6762
3	3.6762	-0.0009	-28.1450	0.0000	3.6762

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

48. True

50. True

52. $f(x) = \sqrt{4 - x^2} \sin(x - 2)$

Domain: $[-2, 2]$

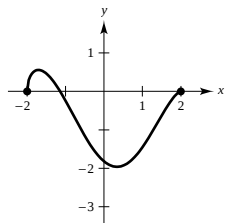
$x = -2$ and $x = 2$ are both zeros.

$$f'(x) = \sqrt{4 - x^2} \cos(x - 2) - \frac{x}{\sqrt{4 - x^2}} \sin(x - 2)$$

Let $x_1 = -1$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.0000	-0.2444	-1.7962	0.1361	-1.1361
2	-1.1361	-0.0090	-1.6498	0.0055	-1.1416
3	-1.1416	0.0000	-1.6422	0.0000	-1.1416

Zeros: $x = \pm 2$, $x \approx -1.142$



Section 3.9 Differentials

2. $f(x) = \frac{6}{x^2} = 6x^{-2}$

$$f'(x) = -12x^{-3} = \frac{-12}{x^3}$$

Tangent line at $\left(2, \frac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \frac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\frac{3}{2}x + \frac{9}{2}$	1.65	1.515	1.5	1.485	1.35

4. $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Tangent line at $(2, \sqrt{2})$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

6. $f(x) = \csc x$

$$f'(x) = -\csc x \cot x$$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1.0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

8. $y = f(x) = 1 - 2x^2, f'(x) = -4x, x = 0, \Delta x = dx = -0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(-0.1) - f(0)$$

$$= [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02$$

$$dy = f'(x) dx$$

$$= f'(0)(-0.1)$$

$$= (0)(-0.1) = 0$$

10. $y = f(x) = 2x + 1, f'(x) = 2, x = 2, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(2.01) - f(2)$$

$$= [2(2.01) + 1] - [2(2) + 1] = 0.02$$

$$dy = f'(x) dx$$

$$= f'(2)(0.01)$$

$$= 2(0.01) = 0.02$$

12. $y = 3x^{2/3}$

$$dy = 2x^{-1/3}dx = \frac{2}{x^{1/3}}dx$$

16. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$dy = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} \right) dx = \frac{x-1}{2x\sqrt{x}} dx$$

20. $y = \frac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{(x^2 + 1)2 \sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2} \right] dx$$

$$= \left[\frac{2 \sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2} \right] dx$$

24. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + 0(-0.1) = 1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + 0(0.04) = 1$$

28. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + 5(-0.07) = 7.65$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + 5(0.1) = 8.5$$

32. $x = 12$ inches

$$\Delta x = dx = \pm 0.03 \text{ inch}$$

(a) $V = x^3$

$$dV = 3x^2 dx = 3(12)^2(\pm 0.03)$$

$$= \pm 12.96 \text{ cubic inches}$$

(b) $S = 6x^2$

$$dS = 12x dx = 12(12)(\pm 0.03)$$

$$= \pm 4.32 \text{ square inches}$$

14. $y = \sqrt{9 - x^2}$

$$dy = \frac{1}{2}(9 - x^2)^{-1/2}(-2x)dx = \frac{-x}{\sqrt{9 - x^2}} dx$$

18. $y = x \sin x$

$$dy = (x \cos x + \sin x) dx$$

22. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + (-1)(-0.1) = 1.1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + (-1)(0.04) = 0.96$$

26. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + (3)(-0.07) = 7.79$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + (3)(0.1) = 8.3$$

30. $A = \frac{1}{2}bh$, $b = 36$, $h = 50$

$$db = dh = \pm 0.25$$

$$dA = \frac{1}{2}b dh + \frac{1}{2}h db$$

$$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$$

$$= \pm 10.75 \text{ square centimeters}$$

34. (a) $C = 56$ centimeters

$$\Delta C = dC = \pm 1.2 \text{ centimeters}$$

$$C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi \left(\frac{C}{2\pi} \right)^2 = \frac{1}{4\pi} C^2$$

$$dA = \frac{1}{2\pi} C dC = \frac{1}{2\pi} (56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $\frac{dA}{A} = \frac{(1/2\pi)C dC}{(1/4\pi)C^2} = \frac{2dC}{C} \leq 0.03$

$$\frac{dC}{C} \leq \frac{0.03}{2} = 0.015 = 1.5\%$$

36. $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, x changes from 115 to 120

$$dP = (500 - 2x - x + 77)dx = (577 - 3x)dx = [577 - 3(115)](120 - 115) = 1160$$

Approximate percentage change: $\frac{dP}{P}(100) = \frac{1160}{43517.50}(100) \approx 2.7\%$

38. $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

$$\Delta V \approx dV = 4\pi r^2 dr = 4\pi(100)^2(0.2) = 8000\pi \text{ cm}^3$$

40. $E = IR$

$$R = \frac{E}{I}$$

$$dR = -\frac{E}{I^2}dI$$

$$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

42. See Exercise 41.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5 \cot \theta)(9.5) = 45.125 \cot \theta$$

$$dA = -45.125 \csc^2 \theta d\theta$$

$$\left|\frac{dA}{A}\right| = \frac{\csc^2 \theta d\theta}{\cot \theta} = \frac{d\theta}{\sin \theta \cos \theta}$$

$$= \frac{0.25^\circ}{(\sin 26.75^\circ)(\cos 26.75^\circ)}$$

$$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$$

$$\approx 0.0109 = 1.09\% \text{ (in radians)}$$

44. $h = 50 \tan \theta$

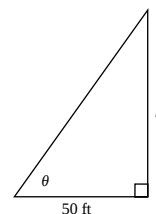
$$\theta = 71.5^\circ = 1.2479 \text{ radians}$$

$$dh = 50 \sec^2 \theta \cdot d\theta$$

$$\left|\frac{dh}{h}\right| = \left|\frac{50 \sec^2(1.2479)}{50 \tan(1.2479)}d\theta\right| \leq 0.06$$

$$\left|\frac{9.9316}{2.9886}d\theta\right| \leq 0.06$$

$$|d\theta| \leq 0.018$$



46. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$

$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

48. Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)dx = x^3 + 3x^2dx$$

$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

50. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

54. True, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a$

52. Propagated error $= f(x + \Delta x) - f(x)$,

$$\text{relative error} = \left| \frac{dy}{y} \right|, \text{ and the percent error} = \left| \frac{dy}{y} \right| \times 100.$$

56. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

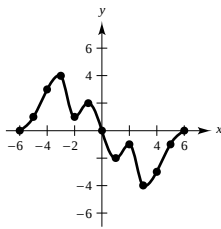
$$dy = f'(x) dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

Thus, $dy > \Delta y$ in this example.

Review Exercises for Chapter 3

2. (a) $f(4) = -f(-4) = -3$

(c)



At least six critical numbers on $(-6, 6)$.

4. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$\begin{aligned} f'(x) &= x \left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2} \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 2/\sqrt{5})$ Maximum

- (b) $f(-3) = -f(3) = -(-4) = 4$

- (d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

- (e) No, $\lim_{x \rightarrow 0} f(x)$ exists because f is continuous at $(0, 0)$.

- (f) Yes, f is differentiable at $x = 2$.

6. No. f is not differentiable at $x = 2$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

10. $f(x) = \frac{1}{x}$, $1 \leq x \leq 4$

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$

$$c = 2$$

12. $f(x) = \sqrt{x} - 2x$, $0 \leq x \leq 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$

$$c = 1$$

14. $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

18. $f(x) = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

20. $g(x) = \frac{3}{2} \sin\left(\frac{\pi x}{2} - 1\right), [0, 4]$

$g'(x) = \frac{3}{2} \left(\frac{\pi}{2}\right) \cos\left(\frac{\pi x}{2} - 1\right)$

$= 0 \text{ when } x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

22. (a) $y = A \sin(\sqrt{k/m}t) + B \cos(\sqrt{k/m}t)$

$y' = A\sqrt{k/m} \cos(\sqrt{k/m}t) - B\sqrt{k/m} \sin(\sqrt{k/m}t)$

$= 0 \text{ when } \frac{\sin \sqrt{k/m}t}{\cos \sqrt{k/m}t} = \frac{A}{B} \Rightarrow \tan(\sqrt{k/m}t) = \frac{A}{B}.$

Therefore,

$$\sin(\sqrt{k/m}t) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\cos(\sqrt{k/m}t) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi} \sqrt{k/m}$

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x = 0 \text{ when } x = 0.$$

Point of inflection: $(0, -16)$

26. $h(t) = t - 4\sqrt{t+1}$ Domain: $[-1, \infty)$

$$h'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0 \Rightarrow t = 3$$

$$h''(t) = \frac{1}{(t+1)^{3/2}}$$

$$h''(3) = \frac{1}{8} > 0 \quad (3, -5) \text{ is a relative minimum.}$$

30. $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

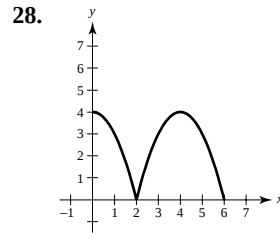
34. $\lim_{x \rightarrow \infty} \frac{2x}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2/x}{3 + 5/x^2} = 0$

38. $g(x) = \frac{5x^2}{x^2 + 2}$

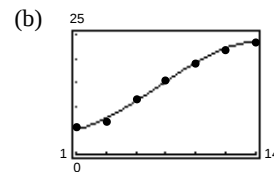
$$\lim_{x \rightarrow \infty} \frac{5x^2}{x^2 + 2} = \lim_{x \rightarrow \infty} \frac{5}{1 + (2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward



32. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

36. $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 4}} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

40. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow \infty} \frac{3x/x}{\sqrt{x^2 + 2}/\sqrt{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3$$

$$\lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow -\infty} \frac{3x/x}{\sqrt{x^2 + 2}/(-\sqrt{x^2})}$$

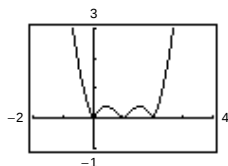
$$= \lim_{x \rightarrow -\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

42. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0)$, $(1, 0)$, $(2, 0)$

Relative maxima: $(1.577, 0.38)$, $(0.423, 0.38)$



46. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0 \text{ when } x = 0, 3.$$

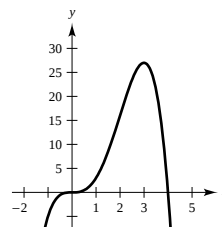
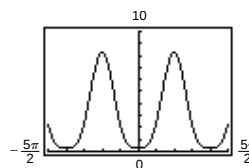
$$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \text{ when } x = 0, 2.$$

$$f''(3) < 0$$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0)$, $(2, 16)$

Intercepts: $(0, 0)$, $(4, 0)$



48. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = 4x(x^2 - 4) = 0 \text{ when } x = 0, \pm 2.$$

$$f''(x) = 4(3x^2 - 4) = 0 \text{ when } x = \pm \frac{2\sqrt{3}}{3}.$$

$$f''(0) < 0$$

Therefore, $(0, 16)$ is a relative maximum.

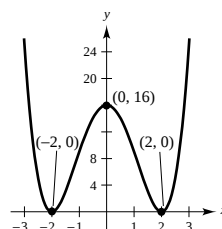
$$f''(\pm 2) > 0$$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0)$, $(0, 16)$, $(2, 0)$

Symmetry with respect to y-axis



50. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $[-\frac{16,875}{256}, \infty)$

$$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$$

$$= (4x - 7)(x + 2)^2 = 0 \text{ when } x = -2, \frac{7}{4}.$$

$$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$$

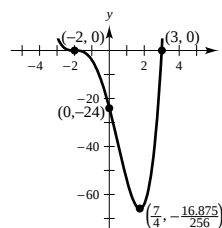
$$= 6(2x - 1)(x + 2) = 0 \text{ when } x = -2, \frac{1}{2}.$$

$$f''(\frac{7}{4}) > 0$$

Therefore, $(\frac{7}{4}, -\frac{16,875}{256})$ is a relative minimum.

Points of inflection: $(-2, 0)$, $(\frac{1}{2}, -\frac{625}{16})$

Intercepts: $(-2, 0)$, $(0, -24)$, $(3, 0)$



52. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

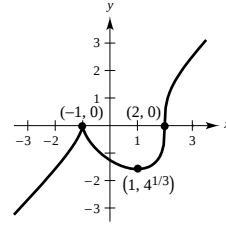
Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

$(-1, 0)$ is a relative maximum.

$(1, -\sqrt[3]{4})$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$



54. $f(x) = \frac{2x}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-2x(3-x^2)}{(1+x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$$f''(1) < 0$$

Therefore, $(1, 1)$ is a relative maximum.

$$f''(-1) > 0$$

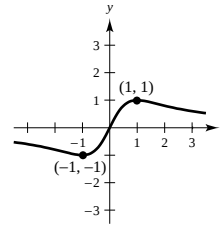
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $(-\sqrt{3}, -\sqrt{3}/2), (0, 0), (\sqrt{3}, \sqrt{3}/2)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



56. $f(x) = \frac{x^2}{1+x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \frac{1}{2}\right]$

$$f'(x) = \frac{(1+x^4)(2x) - x^2(4x^3)}{(1+x^4)^2} = \frac{2x(1-x)(1+x)(1+x^2)}{(1+x^4)^2} = 0 \text{ when } x = 0, \pm 1.$$

$$f''(x) = \frac{(1+x^4)^2(2-10x^4) - (2x-2x^5)(2)(1+x^4)(4x^3)}{(1+x^4)^4} = \frac{2(1-12x^4+3x^8)}{(1+x^4)^3} = 0 \text{ when } x = \pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}.$$

$$f''(\pm 1) < 0$$

Therefore, $\left(\pm 1, \frac{1}{2}\right)$ are relative maxima.

$$f''(0) > 0$$

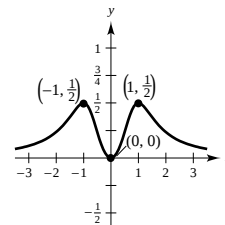
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm \sqrt[4]{\frac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm \sqrt[4]{\frac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



58. $f(x) = x^2 + \frac{1}{x} = \frac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = 0 \text{ when } x = \frac{1}{\sqrt[3]{2}}.$$

$$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3 + 1)}{x^3} = 0 \text{ when } x = -1.$$

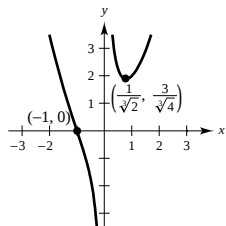
$$f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$$

Therefore, $\left(\frac{1}{\sqrt[3]{2}}, \frac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

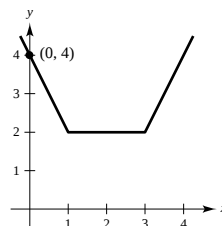


60. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \leq 1 \\ 2, & 1 < x \leq 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



62. $f(x) = \frac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[-\frac{3\sqrt{3}}{2\pi}, \frac{3\sqrt{3}}{2\pi}\right]$

$$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$$

Critical Numbers: $x = \pm \frac{2}{3}, 0$

$$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0 \text{ when } x = 0, \pm 1, \pm 0.420.$$

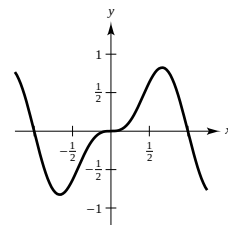
By the First Derivative Test: $\left(-\frac{2}{3}, -\frac{3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\frac{2}{3}, \frac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin



64. $f(x) = x^n$, n is a positive integer.

(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

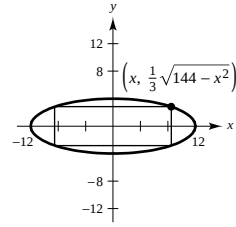
(b) $f''(x) = n(n-1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \geq 3$.

66. Ellipse: $\frac{x^2}{144} + \frac{y^2}{16} = 1$, $y = \frac{1}{3}\sqrt{144 - x^2}$

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\begin{aligned}\frac{dA}{dx} &= \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right] \\ &= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.\end{aligned}$$



The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \frac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

68. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,

$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

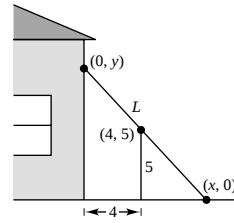
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{5x}{x - 4}\right)^2$$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4} \sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}} \sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$



70. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c) . The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$

$$(b - a)x = b(\bar{x} - a)$$

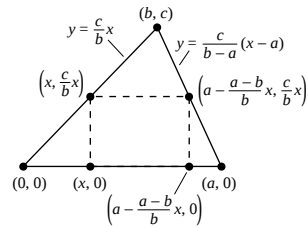
$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$

Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \frac{a - b}{b}x - x$

Height of rectangle: $\frac{c}{b}x$ (see figure)



$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

72. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y - b}{0 - a} = \frac{b - 0}{a - x} \Rightarrow y = \frac{-bx}{a - x}.$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a - x}\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{-bx}{a - x}\right)\left[\frac{-ab}{(a - x)^2}\right]$$

$$\frac{2x[(a - x)^3 + ab^2]}{(a - x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$\begin{aligned} L &= \sqrt{(a + \sqrt[3]{ab^2})^2 + (b + \sqrt[3]{a^2b})^2} \\ &= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2} \\ &= (a^{2/3} + b^{2/3})^{3/2} \text{ meters} \end{aligned}$$

74. Using Exercise 73 as a guide we have $L_1 = a \csc \theta$ and $L_2 = b \sec \theta$. Then $dL/d\theta = -a \csc \theta \cot \theta + b \sec \theta \tan \theta = 0$ when

$$\tan \theta = \sqrt[3]{a/b}, \sec \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}, \csc \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a \csc \theta + b \sec \theta = a \frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b \frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

76. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\begin{aligned} \frac{dT}{dv} &= \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41,250}{50v^2} \\ &= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.} \end{aligned}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

78. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$$f'(x) = 3x^2 + 2$$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

43. $r = \frac{v_0^2}{32} (\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$dr = \frac{(2200)^2}{16} (\cos 2\theta) d\theta$

$\theta = 10\left(\frac{\pi}{180}\right)$

$d\theta = (11 - 10)\frac{\pi}{180}$

$\Delta r \approx dr$

$= \frac{(2200)^2}{16} \cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961$ feet

≈ 4961 feet

47. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt[4]{x} + \frac{1}{4^4 \sqrt{x^3}} dx$

$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[4]{625})^3}(-1)$

$= 5 - \frac{1}{500} = 4.998$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

45. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$f(x + \Delta x) \approx f(x) + f'(x) dx$

$= \sqrt{x} + \frac{1}{2\sqrt{x}} dx$

$f(x + \Delta x) = \sqrt{99.4}$

$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

49. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/(2\sqrt{x})$.

Then

$f(4.02) \approx f(4) + f'(4) dx$

$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$

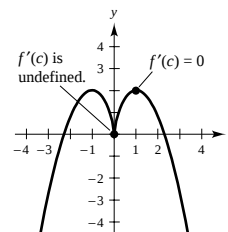
51. In general, when $\Delta x \rightarrow 0$, dy approaches Δy .

53. True

55. True

Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c .



3. $g(x) = 2x + 5 \cos x$, $[0, 2\pi]$

$g'(x) = 2 - 5 \sin x$

$= 0$ when $\sin x = \frac{2}{5}$.

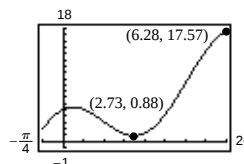
Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

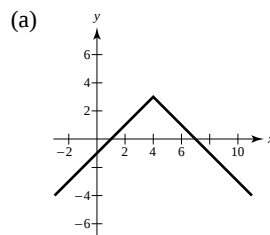


5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$$f'(x) = (x+3)(3x-1) = 0 \text{ for } x = \frac{1}{3}.$$

$$c = \frac{1}{3} \text{ satisfies } f'(c) = 0.$$

7. $f(x) = 3 - |x - 4|$



$$f(1) = f(7) = 0$$

- (b) f is not differentiable at $x = 4$.

9. $f(x) = x^{2/3}, 1 \leq x \leq 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

11. $f(x) = x - \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

13. $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x-1)^2(x-3)$

$$f'(x) = (x-1)^2(1) + (x-3)(2)(x-1)$$

$$= (x-1)(3x-7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

17. $h(x) = \sqrt{x}(x-3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$

$v = y' = -4 \sin(12t) - 3 \cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4 \sin(12t) - 3 \cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x$, $0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

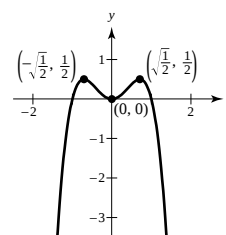
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm \frac{1}{\sqrt{2}}$

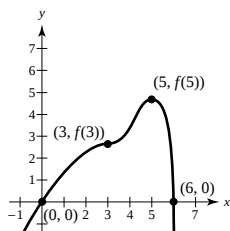
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm \frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

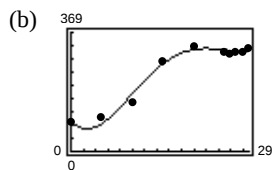


27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$



(c) Maximum at (21.9, 319.5) (≈ 1992)

Minimum at (2.6, 69.6) (≈ 1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) (≈ 1979)

33. $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2}{3 + 5/x^2} = \frac{2}{3}$

35. $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x} = 0$, since $|5 \cos x| \leq 5$.

37. $h(x) = \frac{2x + 3}{x - 4}$

Discontinuity: $x = 4$

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 4} = \lim_{x \rightarrow \infty} \frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \frac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x \rightarrow \infty} \left(\frac{3}{x} - 2 \right) = -2$$

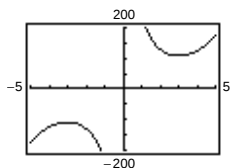
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: (-3, -108)

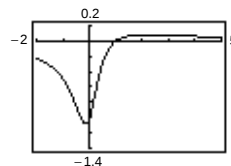


Vertical asymptote: $x = 0$

43. $f(x) = \frac{x - 1}{1 + 3x^2}$

Relative minimum: (-0.155, -1.077)

Relative maximum: (2.155, 0.077)



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4 - x)$

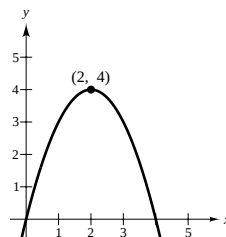
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$$f'(x) = 4 - 2x = 0 \text{ when } x = 2.$$

$$f''(x) = -2$$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f'(-2\sqrt{2}) > 0$$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

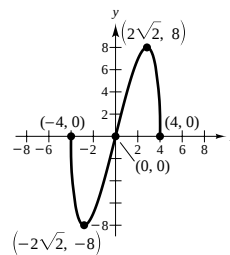
$$f'(2\sqrt{2}) < 0$$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin



49. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

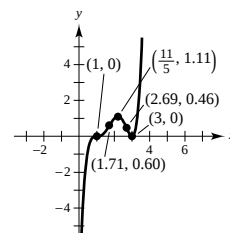
Therefore, $(3, 0)$ is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$



51. $f(x) = x^{1/3}(x + 3)^{2/3}$

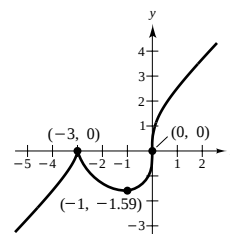
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$



53. $f(x) = \frac{x+1}{x-1}$

Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

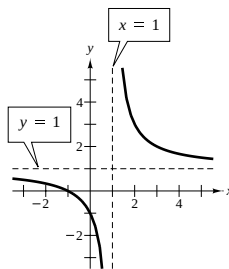
$$f'(x) = \frac{-2}{(x-1)^2} < 0 \text{ if } x \neq 1.$$

$$f''(x) = \frac{4}{(x-1)^3}$$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$



55. $f(x) = \frac{4}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$$f'(x) = \frac{-8x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{-8(1-3x^2)}{(1+x^2)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

$$f''(0) < 0$$

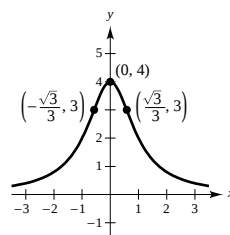
Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $(\pm \sqrt{3}/3, 3)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



57. $f(x) = x^3 + x + \frac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$$f''(-1) < 0$$

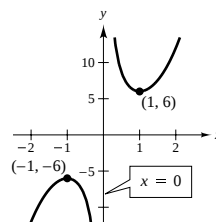
Therefore, $(-1, -6)$ is a relative maximum.

$$f''(1) > 0$$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



59. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$$f''(0) < 0$$

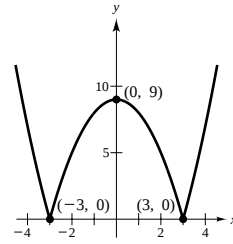
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis



61. $f(x) = x + \cos x$

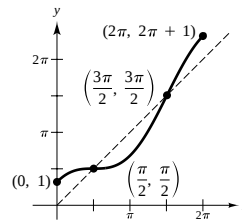
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$$f'(x) = 1 - \sin x \geq 0, f \text{ is increasing.}$$

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Points of inflection: $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{3\pi}{2})$

Intercept: $(0, 1)$



63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

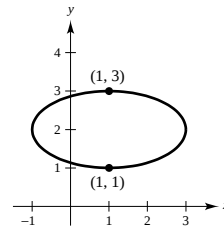
$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$



(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y \frac{dy}{dx} - 2 - 16 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

65. Let
- $t = 0$
- at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10,000 - 2400t + 244t^2$$

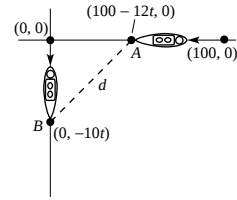
$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

Ship A at (40.98, 0); Ship B at (0, -49.18)

$$d^2 = 10,000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4:55 \text{ P.M..}$$

$$d \approx 64 \text{ km}$$



67. We have points
- $(0, y)$
- ,
- $(x, 0)$
- , and
- $(1, 8)$
- . Thus,

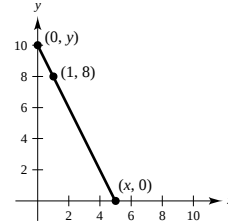
$$m = \frac{y-8}{0-1} = \frac{0-8}{x-1} \text{ or } y = \frac{8x}{x-1}.$$

$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{8x}{x-1}\right)^2.$$

$$f'(x) = 2x + 128\left(\frac{x}{x-1}\right)\left[\frac{(x-1)-x}{(x-1)^2}\right] = 0$$

$$x - \frac{64x}{(x-1)^3} = 0$$

$$x[(x-1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

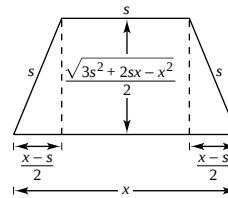
Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$ 

- 69.
- $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x+s}{2}\right) \frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4} \left[\frac{(s-x)(s+x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2} \right]$$

$$= \frac{2(2s-x)(s+x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

 A is a maximum when $x = 2s$.

71. You can form a right triangle with vertices
- $(0, 0)$
- ,
- $(x, 0)$
- and
- $(0, y)$
- .

Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y-6}{0-4} = \frac{6-0}{4-x} \text{ or } y = \frac{6x}{x-4}$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x-4}\right)^2.$$

$$f'(x) = 2x + 72\left(\frac{x}{x-4}\right)\left[\frac{-4}{(x-4)^2}\right] = 0$$

$$x[(x-4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

73. $\csc \theta = \frac{L_1}{6}$ or $L_1 = 6 \csc \theta$ (see figure)

$$\csc\left(\frac{\pi}{2} - \theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9 \csc\left(\frac{\pi}{2} - \theta\right)$$

$$L = L_1 + L_2 = 6 \csc \theta + 9 \csc\left(\frac{\pi}{2} - \theta\right) = 6 \csc \theta + 9 \sec \theta$$

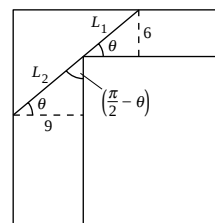
$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$



75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33,000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$$f'(x) = 3x^2 - 3$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

79. Find the zeros of
- $f(x) = x^4 - x - 3$
- .

$$f'(x) = 4x^3 - 1$$

From the graph you can see that $f(x)$ has two real zeros.

f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.2000	0.2736	-7.9120	-0.0346	-1.1654
2	-1.1654	0.0100	-7.3312	-0.0014	-1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	-0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$: $x \approx 1.453$.

- 81.
- $y = x(1 - \cos x) = x - x \cos x$

$$\frac{dy}{dx} = 1 + x \sin x - \cos x$$

$$dy = (1 + x \sin x - \cos x) dx$$

- 83.
- $S = 4\pi r^2$
- ,
- $dr = \Delta r = \pm 0.025$

$$dS = 8\pi r dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\begin{aligned} \frac{dS}{S}(100) &= \frac{8\pi r dr}{4\pi r^2}(100) = \frac{2 dr}{r}(100) \\ &= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\% \end{aligned}$$

$$V = \frac{4}{3}\pi r^3$$

$$\begin{aligned} dV &= 4\pi r^2 dr = 4\pi(9)^2(\pm 0.025) \\ &= \pm 8.1\pi \text{ cubic cm} \end{aligned}$$

$$\begin{aligned} \frac{dV}{V}(100) &= \frac{4\pi r^2 dr}{(4/3)\pi r^3}(100) = \frac{3 dr}{r}(100) \\ &= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\% \end{aligned}$$

Problem Solving for Chapter 3

1. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$

$$g'(b) = f'(b) - d = y_2 - d > 0$$

Hence, $a < c < b$ and $g'(c) = 0 \Rightarrow f'(c) = d$.

3. (a) For $a = -3, -2, -1, 0$, p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, p has a relative maximum at $(0, 0)$ and 2 relative minima.

$$(b) \quad p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{3}{a}}$$

$$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$$

For $x = 0$, $p''(0) = -12 < 0 \Rightarrow p$ has a relative maximum at $(0, 0)$.

- (c) If $a > 0$, $x = \pm\sqrt{\frac{3}{a}}$ are the remaining critical numbers.

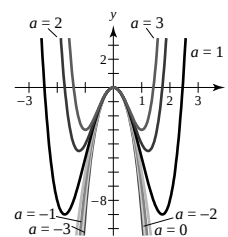
$$p''\left(\pm\sqrt{\frac{3}{a}}\right) = 12\left(\frac{3}{a}\right) - 12 = 24 > 0 \Rightarrow p \text{ has relative minima for } a > 0.$$

- (d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\frac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\frac{9}{a} = -3\left(\pm\sqrt{\frac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p .



5. $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$

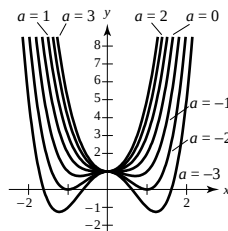
$$p''(x) = 16x^2 + 2a$$

For $a \geq 0$, there is one relative minimum at $(0, 0)$.

- (b) For $a < 0$, there is a relative maximum at $(0, 1)$.

- (c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

- (d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.



7. $f(x) = \frac{c}{x} + x^2$

$$f'(x) = -\frac{c}{x^2} + 2x = 0 \Rightarrow \frac{c}{x^2} = 2x \Rightarrow x^3 = \frac{c}{2} \Rightarrow x = \sqrt[3]{\frac{c}{2}}$$

$$f''(x) = \frac{2c}{x^3} + 2$$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\frac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum too.

Answer: all c .

9. Set $\frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k$.

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b) .

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0$.

There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$k = \frac{f''(c_2)}{2}$.

Thus, $k = \frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \frac{f''(c_2)}{2} \Rightarrow f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c_2)(b - a)^2$.

11. $E(\phi) = \frac{\tan \phi(1 - 0.1 \tan \phi)}{0.1 + \tan \phi} = \frac{10 \tan \phi - \tan^2 \phi}{1 + 10 \tan \phi}$

$E'(\phi) = \frac{(1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) - (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi}{(1 + 10 \tan \phi)^2} = 0$

$\Rightarrow (1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) = (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi$

$\Rightarrow 10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi + 100 \tan \phi \sec^2 \phi - 20 \tan^2 \phi \sec^2 \phi$

$= 100 \tan \phi \sec^2 \phi - 10 \tan^2 \phi \sec^2 \phi$

$\Rightarrow 10 - 2 \tan \phi = 10 \tan^2 \phi$

$\Rightarrow 10 \tan^2 \phi + 2 \tan \phi - 10 = 0$

$\tan \phi = \frac{-2 \pm \sqrt{4 + 400}}{20} \approx 0.90499, -1.10499$

Using the positive value, $\phi \approx 0.7356$, or 42.14° .

13. $v = -2400\pi \sin \theta$

$v' = -2400\pi \cos \theta = 0$

$\theta = \frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi, n \text{ an integer}$

15. The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Rightarrow \frac{8}{3}x = 4 \Rightarrow x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be r :

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12|7r - 12|}{5|12|} = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \Rightarrow r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\frac{r}{3} + \frac{r}{4} = 1 \Rightarrow \frac{7}{12}r = 1 \Rightarrow r = \frac{12}{7}$. No calculus necessary.

17. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$$

$$y'': \begin{array}{c} + + + \quad - - - \quad - - - \quad + + + \\ \hline -\frac{\sqrt{3}}{3} \quad 0 \quad \frac{\sqrt{3}}{3} \end{array}$$

The tangent line has greatest slope at $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$ and least slope at $\left(\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$.

19. (a)

x	0.1	0.2	0.3	0.4	0.5	1.0
$\sin x$	0.09983	0.19867	0.29552	0.38942	0.47943	0.84147

$$\sin x \leq x$$

- (b) Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

$$\text{Hence, } \left|\frac{\sin x}{x}\right| = |\cos(c)| \leq 1$$

$$\Rightarrow |\sin x| \leq |x|$$

$$\Rightarrow \sin x \leq x$$

50. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

54. True, $\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a$

52. Propagated error $= f(x + \Delta x) - f(x)$,

$$\text{relative error} = \left| \frac{dy}{y} \right|, \text{ and the percent error} = \left| \frac{dy}{y} \right| \times 100.$$

56. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

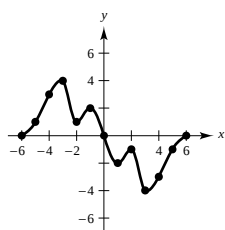
$$dy = f'(x) dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

Thus, $dy > \Delta y$ in this example.

Review Exercises for Chapter 3

2. (a) $f(4) = -f(-4) = -3$

(c)



At least six critical numbers on $(-6, 6)$.

4. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$\begin{aligned} f'(x) &= x \left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x) \right] + (x^2 + 1)^{-1/2} \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 2/\sqrt{5})$ Maximum

- (b) $f(-3) = -f(3) = -(-4) = 4$

- (d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

- (e) No, $\lim_{x \rightarrow 0} f(x)$ exists because f is continuous at $(0, 0)$.

- (f) Yes, f is differentiable at $x = 2$.

6. No. f is not differentiable at $x = 2$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

10. $f(x) = \frac{1}{x}$, $1 \leq x \leq 4$

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$

$$c = 2$$

12. $f(x) = \sqrt{x} - 2x$, $0 \leq x \leq 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$

$$c = 1$$

14. $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

18. $f(x) = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

20. $g(x) = \frac{3}{2} \sin\left(\frac{\pi x}{2} - 1\right), [0, 4]$

$g'(x) = \frac{3}{2} \left(\frac{\pi}{2}\right) \cos\left(\frac{\pi x}{2} - 1\right)$

$= 0 \text{ when } x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

22. (a) $y = A \sin(\sqrt{k/m}t) + B \cos(\sqrt{k/m}t)$

$y' = A\sqrt{k/m} \cos(\sqrt{k/m}t) - B\sqrt{k/m} \sin(\sqrt{k/m}t)$

$= 0 \text{ when } \frac{\sin \sqrt{k/m}t}{\cos \sqrt{k/m}t} = \frac{A}{B} \Rightarrow \tan(\sqrt{k/m}t) = \frac{A}{B}.$

Therefore,

$$\sin(\sqrt{k/m}t) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\cos(\sqrt{k/m}t) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi} \sqrt{k/m}$

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x = 0 \text{ when } x = 0.$$

Point of inflection: $(0, -16)$

26. $h(t) = t - 4\sqrt{t+1}$ Domain: $[-1, \infty)$

$$h'(t) = 1 - \frac{2}{\sqrt{t+1}} = 0 \Rightarrow t = 3$$

$$h''(t) = \frac{1}{(t+1)^{3/2}}$$

$$h''(3) = \frac{1}{8} > 0 \quad (3, -5) \text{ is a relative minimum.}$$

30. $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

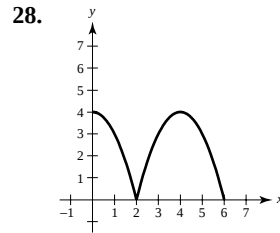
34. $\lim_{x \rightarrow \infty} \frac{2x}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2/x}{3 + 5/x^2} = 0$

38. $g(x) = \frac{5x^2}{x^2 + 2}$

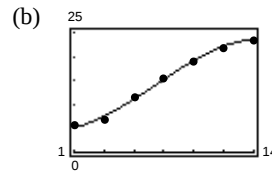
$$\lim_{x \rightarrow \infty} \frac{5x^2}{x^2 + 2} = \lim_{x \rightarrow \infty} \frac{5}{1 + (2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward



32. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

36. $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 4}} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

40. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow \infty} \frac{3x/x}{\sqrt{x^2 + 2}/\sqrt{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3$$

$$\lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow -\infty} \frac{3x/x}{\sqrt{x^2 + 2}/(-\sqrt{x^2})}$$

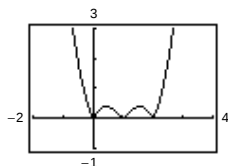
$$= \lim_{x \rightarrow -\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

42. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0)$, $(1, 0)$, $(2, 0)$

Relative maxima: $(1.577, 0.38)$, $(0.423, 0.38)$



46. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0 \text{ when } x = 0, 3.$$

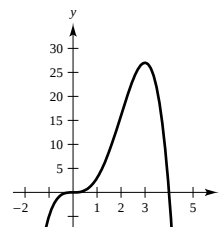
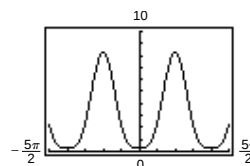
$$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \text{ when } x = 0, 2.$$

$$f''(3) < 0$$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0)$, $(2, 16)$

Intercepts: $(0, 0)$, $(4, 0)$



48. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = 4x(x^2 - 4) = 0 \text{ when } x = 0, \pm 2.$$

$$f''(x) = 4(3x^2 - 4) = 0 \text{ when } x = \pm \frac{2\sqrt{3}}{3}.$$

$$f''(0) < 0$$

Therefore, $(0, 16)$ is a relative maximum.

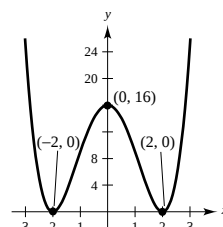
$$f''(\pm 2) > 0$$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0)$, $(0, 16)$, $(2, 0)$

Symmetry with respect to y-axis



50. $f(x) = (x-3)(x+2)^3$

Domain: $(-\infty, \infty)$; Range: $[-\frac{16,875}{256}, \infty)$

$$f'(x) = (x-3)(3)(x+2)^2 + (x+2)^3$$

$$= (4x-7)(x+2)^2 = 0 \text{ when } x = -2, \frac{7}{4}.$$

$$f''(x) = (4x-7)(2)(x+2) + (x+2)^2(4)$$

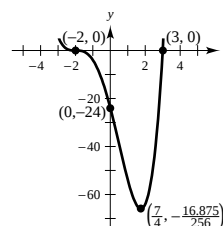
$$= 6(2x-1)(x+2) = 0 \text{ when } x = -2, \frac{1}{2}.$$

$$f''(\frac{7}{4}) > 0$$

Therefore, $(\frac{7}{4}, -\frac{16,875}{256})$ is a relative minimum.

Points of inflection: $(-2, 0)$, $(\frac{1}{2}, -\frac{625}{16})$

Intercepts: $(-2, 0)$, $(0, -24)$, $(3, 0)$



44. $g(x) = \frac{\pi^2}{3} - 4 \cos x + \cos 2x$

Relative minima: $(2\pi k, 0.29)$ where k is any integer.

Relative maxima: $((2k-1)\pi, 8.29)$ where k is any integer.

52. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

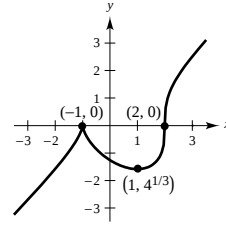
Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

$(-1, 0)$ is a relative maximum.

$(1, -\sqrt[3]{4})$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$



54. $f(x) = \frac{2x}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-2x(3-x^2)}{(1+x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$$f''(1) < 0$$

Therefore, $(1, 1)$ is a relative maximum.

$$f''(-1) > 0$$

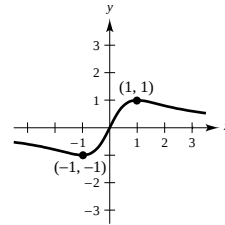
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $(-\sqrt{3}, -\sqrt{3}/2), (0, 0), (\sqrt{3}, \sqrt{3}/2)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



56. $f(x) = \frac{x^2}{1+x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \frac{1}{2}\right]$

$$f'(x) = \frac{(1+x^4)(2x) - x^2(4x^3)}{(1+x^4)^2} = \frac{2x(1-x)(1+x)(1+x^2)}{(1+x^4)^2} = 0 \text{ when } x = 0, \pm 1.$$

$$f''(x) = \frac{(1+x^4)^2(2-10x^4) - (2x-2x^5)(2)(1+x^4)(4x^3)}{(1+x^4)^4} = \frac{2(1-12x^4+3x^8)}{(1+x^4)^3} = 0 \text{ when } x = \pm \sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}.$$

$$f''(\pm 1) < 0$$

Therefore, $\left(\pm 1, \frac{1}{2}\right)$ are relative maxima.

$$f''(0) > 0$$

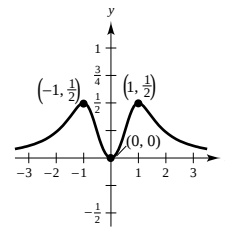
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm \sqrt[4]{\frac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm \sqrt[4]{\frac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



58. $f(x) = x^2 + \frac{1}{x} = \frac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = 0 \text{ when } x = \frac{1}{\sqrt[3]{2}}.$$

$$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3 + 1)}{x^3} = 0 \text{ when } x = -1.$$

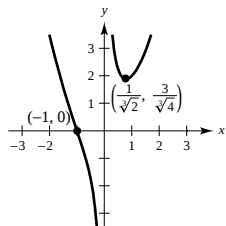
$$f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$$

Therefore, $\left(\frac{1}{\sqrt[3]{2}}, \frac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

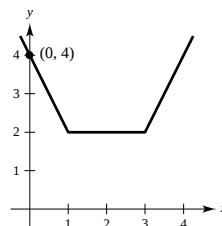


60. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \leq 1 \\ 2, & 1 < x \leq 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



62. $f(x) = \frac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\frac{-3\sqrt{3}}{2\pi}, \frac{3\sqrt{3}}{2\pi}\right]$

$$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$$

Critical Numbers: $x = \pm \frac{2}{3}, 0$

$$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0 \text{ when } x = 0, \pm 1, \pm 0.420.$$

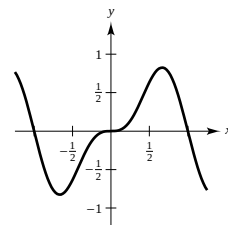
By the First Derivative Test: $\left(-\frac{2}{3}, \frac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\frac{2}{3}, \frac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin



64. $f(x) = x^n$, n is a positive integer.

(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

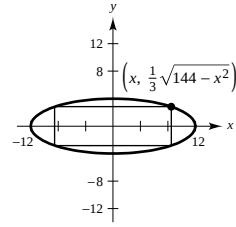
(b) $f''(x) = n(n-1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \geq 3$.

66. Ellipse: $\frac{x^2}{144} + \frac{y^2}{16} = 1$, $y = \frac{1}{3}\sqrt{144 - x^2}$

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\begin{aligned}\frac{dA}{dx} &= \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right] \\ &= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.\end{aligned}$$



The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \frac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

68. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,

$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

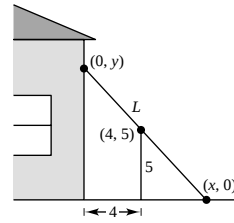
$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{5x}{x - 4}\right)^2$$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4} \sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}} \sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$



70. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c) . The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$

$$(b - a)x = b(\bar{x} - a)$$

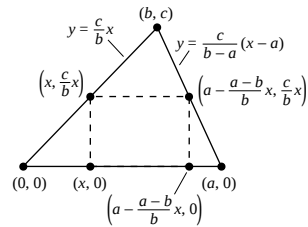
$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$

Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \frac{a - b}{b}x - x$

Height of rectangle: $\frac{c}{b}x$ (see figure)



$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

72. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y - b}{0 - a} = \frac{b - 0}{a - x} \Rightarrow y = \frac{-bx}{a - x}.$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a - x}\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{-bx}{a - x}\right)\left[\frac{-ab}{(a - x)^2}\right]$$

$$\frac{2x[(a - x)^3 + ab^2]}{(a - x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$\begin{aligned} L &= \sqrt{(a + \sqrt[3]{ab^2})^2 + (b + \sqrt[3]{a^2b})^2} \\ &= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2} \\ &= (a^{2/3} + b^{2/3})^{3/2} \text{ meters} \end{aligned}$$

74. Using Exercise 73 as a guide we have $L_1 = a \csc \theta$ and $L_2 = b \sec \theta$. Then $dL/d\theta = -a \csc \theta \cot \theta + b \sec \theta \tan \theta = 0$ when

$$\tan \theta = \sqrt[3]{a/b}, \sec \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}, \csc \theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a \csc \theta + b \sec \theta = a \frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b \frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

76. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\begin{aligned} \frac{dT}{dv} &= \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41,250}{50v^2} \\ &= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.} \end{aligned}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

78. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$$f'(x) = 3x^2 + 2$$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

80. Find the zeros of $f(x) = \sin \pi x + x - 1$.

$$f'(x) = \pi \cos \pi x + 1$$

From the graph you can see that $f(x)$ has three real zeros.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.2000	-0.2122	3.5416	-0.0599	0.2599
2	0.2599	-0.0113	3.1513	-0.0036	0.2635
3	0.2635	0.0000	3.1253	0.0000	0.2635

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	0.0000	-2.1416	0.0000	1.0000

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.8000	0.2122	3.5416	0.0599	1.7401
2	1.7401	0.0113	3.1513	0.0036	1.7365
3	1.7365	0.0000	3.1253	0.0000	1.7365

The three real zeros of $f(x)$ are $x \approx 0.264$, $x = 1$, and $x \approx 1.737$.

82. $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}} dx$$

84. $p = 75 - \frac{1}{4}x$

$$\Delta p = p(8) - p(7)$$

$$= \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$$

$$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$$

$[\Delta p = dp \text{ because } p \text{ is linear}]$

Problem Solving for Chapter 3

2. (a) $dV = 3x^2 dx = 3x^2 \Delta x$

$$\Delta V = (x + \Delta x)^3 - x^3 = 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\Delta V - dV = 3x(\Delta x)^2 + (\Delta x)^3 = \underbrace{[3x\Delta x + (\Delta x)^2]}_{\varepsilon} \Delta x$$

$$= \varepsilon \Delta x, \text{ where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0.$$

- (b) Let $\varepsilon = \frac{\Delta y}{\Delta x} - f'(x)$. Then $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.

$$\text{Furthermore, } \Delta y - dy = \Delta y - f'(x)dx = \varepsilon \Delta x.$$

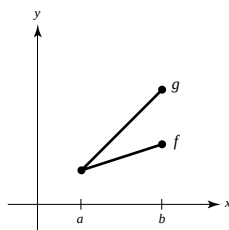
4. Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on (a, b) . $h(a) = 0$ and $h(b) = g(b) - f(b)$.

By the Mean Value Theorem, there exists c in (a, b) such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Since $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$,

$$g(b) - f(b) > 0 \Rightarrow g(b) > f(b).$$



6. (a) $f' = 2ax + b$, $f'' = 2a \neq 0$. No points of inflection.

(b) $f' = 3ax^2 - 2bx + c$, $f'' = 6ax + 2b = 0 \Rightarrow x = \frac{-b}{3a}$. One point of inflection.

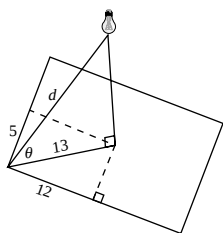
(c) $y' = ky(L - y) = kLy - ky^2$

$$y'' = kLy' - 2kyy' = ky'(L - 2y)$$

If $y = \frac{L}{2}$, then $y'' = 0$ and this is a point of inflection because of the analysis below.

$$y'': \begin{array}{c} ++++++ \\ y = \frac{L}{2} \end{array}$$

8.



$$d = \sqrt{13^2 + x^2}, \sin \theta = \frac{x}{d}.$$

Let A be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{(13^2 + x^2)} \sin \theta = \frac{kIx}{(13^2 + x^2)^{3/2}}$$

$$A'(x) = kI \frac{(x^2 + 169)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2 + 169)^{1/2}(2x)}{(169 + x^2)^3} = 0$$

$$\Rightarrow (x^2 + 169)^{3/2} = 3x^2(x^2 + 169)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ feet}$$

By the First Derivative Test, this is a maximum.

10. Let T be the intersection of PQ and RS . Let MN be the perpendicular to SQ and PR passing through T .

Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \Rightarrow SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \Rightarrow NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x - 2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

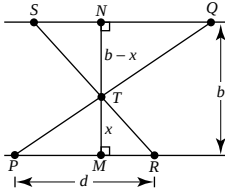
$$A'(x) = 0 \Rightarrow 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$

$$\text{Hence, we have } SQ = \frac{b-x}{x}d = \frac{b - (b/\sqrt{2})}{b/\sqrt{2}}d = (\sqrt{2} - 1)d.$$

Using the Second Derivative Test, this is a minimum. There is no maximum.



12. (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have

$$f(x) = x^2 > M.$$

- (b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\frac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\frac{1}{\varepsilon}}$, you have

$$x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left| \frac{1}{x^2} - 0 \right| < \varepsilon.$$

- (c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

$$\text{Let } \delta = \frac{1}{N}. \text{ Let } x = \frac{1}{y}.$$

$$\text{If } 0 < y < \delta = \frac{1}{N}, \text{ then } \frac{1}{x} < \frac{1}{N} \Rightarrow x > N \text{ and}$$

$$|f(x) - L| = \left| f\left(\frac{1}{y}\right) - L \right| < \varepsilon.$$

14. Distance = $\sqrt{4^2 + x^2} + \sqrt{(4-x)^2 + 4^2} = f(x)$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} + \frac{4-x}{\sqrt{(4-x)^2 + 4^2}} = 0$$

$$x\sqrt{(4-x)^2 + 4^2} = (x-4)\sqrt{4^2 + x^2}$$

$$x^2[16 - 8x + x^2 + 16] = (x^2 - 8x + 16)(16 + x^2)$$

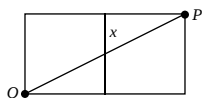
$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube:

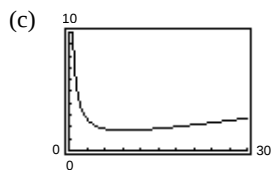


The shortest distance is the line PQ , passing through the midpoint.

16. (a) $s = \frac{v \frac{\text{km}}{\text{hr}} \left(1000 \frac{\text{m}}{\text{km}} \right)}{\left(3600 \frac{\text{sec}}{\text{hr}} \right)} = \frac{5}{18} v$

v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$$d(t) = 0.071s^2 + 0.389s + 0.727$$



$$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ m/sec.

- (b) The distance between the back of the first vehicle and the front of the second vehicle is $d(t)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ more seconds. Hence,

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(d) $T(s) = 0.071s + 0.389 + \frac{6.227}{s}$

$$T'(s) = 0.071 - \frac{6.227}{s^2} \Rightarrow s^2 = \frac{6.227}{0.071}$$

$$\Rightarrow s \approx 9.365 \text{ m/sec}$$

$$T(9.365) \approx 1.719 \text{ seconds}$$

$$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 3.37 \text{ km/hr}$$

(e) $d(9.365) = 10.597 \text{ m}$

18. (a)

x	0	0.5	1	2
$\sqrt{1+x}$	1	1.2247	1.4142	1.7321
$\frac{1}{2}x + 1$	1	1.25	1.5	2

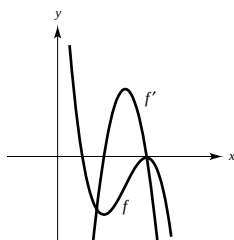
- (b) Let $f(x) = \sqrt{1+x}$. Using the Mean Value Theorem on the interval $[0, x]$, there exists c , $0 < c < x$, satisfying

$$f'(c) = \frac{1}{2\sqrt{1+c}} = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1+x} - 1}{x}.$$

$$\text{Thus } \sqrt{1+x} = \frac{x}{2\sqrt{1+c}} + 1 < \frac{x}{2} + 1 \text{ (because } \sqrt{1+c} > 1).$$

Review Exercises for Chapter 4

2.



$$\begin{aligned} 6. \int \frac{x^3 - 2x^2 + 1}{x^2} dx &= \int (x - 2 + x^{-2}) dx \\ &= \frac{1}{2}x^2 - 2x - \frac{1}{x} + C \end{aligned}$$

$$10. f''(x) = 6(x - 1)$$

$$f'(x) = \int 6(x - 1) dx = 3(x - 1)^2 + C_1$$

Since the slope of the tangent line at (2, 1) is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

$$f'(x) = 3(x - 1)^2$$

$$f(x) = \int 3(x - 1)^2 dx = (x - 1)^3 + C_2$$

$$f(2) = 1 + C_2 = 1 \text{ when } C_2 = 0.$$

$$f(x) = (x - 1)^3$$

$$4. u = 3x$$

$$du = 3 dx$$

$$\int \frac{2}{\sqrt[3]{3x}} dx = \frac{2}{3} \int (3x)^{-1/3} (3) dx = (3x)^{2/3} + C$$

$$8. \int (5 \cos x - 2 \sec^2 x) dx = 5 \sin x - 2 \tan x + C$$

$$12. 45 \text{ mph} = 66 \text{ ft/sec}$$

$$30 \text{ mph} = 44 \text{ ft/sec}$$

$$a(t) = -a$$

$$v(t) = -at + 66 \text{ since } v(0) = 66 \text{ ft/sec.}$$

$$s(t) = -\frac{a}{2}t^2 + 66t \text{ since } s(0) = 0.$$

Solving the system

$$v(t) = -at + 66 = 44$$

$$s(t) = -\frac{a}{2}t^2 + 66t = 264$$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$$475.2 - 264 = 211.2 \text{ ft.}$$

$$14. a(t) = -9.8 \text{ m/sec}^2$$

$$v(t) = -9.8t + v_0 = -9.8t + 40$$

$$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$$

$$(a) v(t) = -9.8t + 40 = 0 \text{ when } t = \frac{40}{9.8} \approx 4.08 \text{ sec.}$$

$$(b) s(4.08) \approx 81.63 \text{ m}$$

$$(c) v(t) = -9.8t + 40 = 20 \text{ when } t = \frac{20}{9.8} \approx 2.04 \text{ sec.}$$

$$(d) s(2.04) \approx 61.2 \text{ m}$$

16. $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5} \sum_{i=1}^5 x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

(b) $\sum_{i=1}^5 \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

(c) $\sum_{i=1}^5 (2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^5 (x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

18. $y = 9 - \frac{1}{4}x^2, \Delta x = 1, n = 4$

$$S(4) = 1 \left[\left(9 - \frac{1}{4}(4) \right) + \left(9 - \frac{1}{4}(9) \right) + \left(9 - \frac{1}{4}(16) \right) + 9 - \frac{1}{4}(25) \right]$$

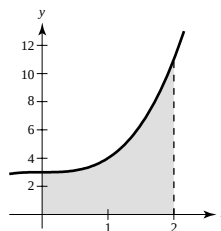
$$\approx 22.5$$

$$s(4) = 1 \left[\left(9 - \frac{1}{4}(9) \right) + \left(9 - \frac{1}{4}(16) \right) + \left(9 - \frac{1}{4}(25) \right) + (9 - 9) \right]$$

$$\approx 14.5$$

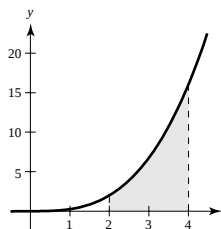
20. $y = x^2 + 3, \Delta x = \frac{2}{n}$ right endpoints

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2i}{n} \right)^2 + 3 \right] \left(\frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left[\frac{4i^2}{n^2} + 3 \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[\frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} + 3n \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \frac{(n+1)(2n+1)}{n^2} + 6 \right] = \frac{8}{3} + 6 = \frac{26}{3} \end{aligned}$$



22. $y = \frac{1}{4}x^3, \Delta x = \frac{2}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{4} \left(2 + \frac{2i}{n} \right)^3 \left(\frac{2}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=1}^n \left[8 + \frac{24i}{n} + \frac{24i^2}{n^2} + \frac{8i^3}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left[1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[n + \frac{3}{n} \frac{n(n+1)}{2} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3} \frac{n^2(n+1)^2}{4} \right] \\ &= 4 + 6 + 4 + 1 = 15 \end{aligned}$$



$$24. (a) S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$$

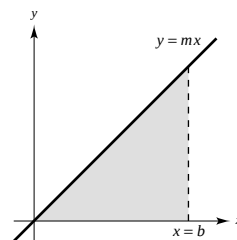
$$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$$

$$(b) S(n) = \sum_{i=1}^n f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^n \left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=1}^n i = \frac{mb^2}{n^2} \left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$$

$$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i = \frac{mb^2}{n^2} \left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$$

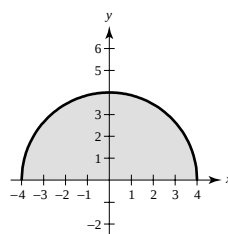
$$(c) \text{Area} = \lim_{n \rightarrow \infty} \frac{mb^2(n+1)}{2n} = \lim_{n \rightarrow \infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}(\text{base})(\text{height})$$

$$(d) \int_0^b mx \, dx = \left[\frac{1}{2}mx^2 \right]_0^b = \frac{1}{2}mb^2$$



$$26. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 3ci(9 - ci^2) \Delta xi = \int_1^3 3x(9 - x^2) \, dx$$

28.



$$\int_{-4}^4 \sqrt{16 - x^2} \, dx = \frac{1}{2} \pi (4)^2 = 8\pi \quad (\text{semicircle})$$

$$30. (a) \int_0^6 f(x) \, dx = \int_0^3 f(x) \, dx + \int_3^6 f(x) \, dx = 4 + (-1) = 3$$

$$(b) \int_6^3 f(x) \, dx = -\int_3^6 f(x) \, dx = -(-1) = 1$$

$$(c) \int_4^4 f(x) \, dx = 0$$

$$(d) \int_3^6 -10 f(x) \, dx = -10 \int_3^6 f(x) \, dx = -10(-1) = 10$$

$$32. \int_1^3 \frac{12}{x^3} \, dx = \left[\frac{12x^{-2}}{-2} \right]_1^3 = \left[\frac{-6}{x^2} \right]_1^3 = \frac{-6}{9} + 6 = \frac{16}{3}, (d)$$

$$34. \int_{-1}^1 (t^2 + 2) \, dt = \left[\frac{t^3}{3} + 2t \right]_{-1}^1 = \frac{14}{3}$$

$$36. \int_{-2}^2 (x^4 + 2x^2 - 5) \, dx = \left[\frac{x^5}{5} + \frac{2x^3}{3} - 5x \right]_{-2}^2$$

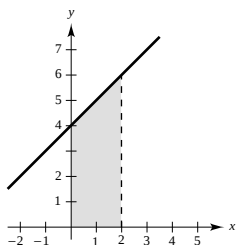
$$= \left(\frac{32}{5} + \frac{16}{3} - 10 \right) - \left(-\frac{32}{5} - \frac{16}{3} + 10 \right)$$

$$= \frac{52}{15}$$

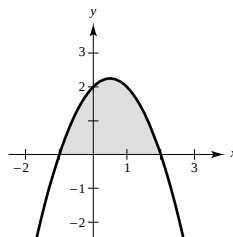
$$38. \int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3} \right) \, dx = \int_1^2 (x^{-2} - x^{-3}) \, dx = \left[-\frac{1}{x} + \frac{1}{2x^2} \right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8} \right) - \left(-1 + \frac{1}{2} \right) = \frac{1}{8}$$

$$40. \int_{-\pi/4}^{\pi/4} \sec^2 t \, dt = \left[\tan t \right]_{-\pi/4}^{\pi/4} = 1 - (-1) = 2$$

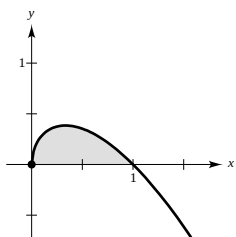
$$42. \int_0^2 (x + 4) dx = \left[\frac{x^2}{2} + 4x \right]_0^2 = 10$$



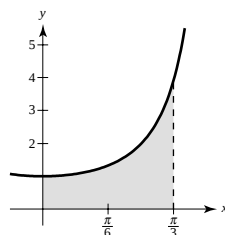
$$\begin{aligned} 44. \int_{-1}^2 (-x^2 + x + 2) dx &= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^2 \\ &= \left(-\frac{8}{3} + 2 + 4 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) \\ &= \frac{10}{3} + \frac{7}{6} = \frac{9}{2} \end{aligned}$$



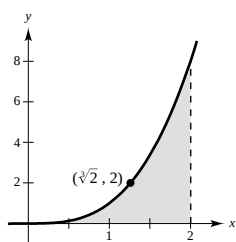
$$\begin{aligned} 46. \int_0^1 \sqrt{x}(1-x) dx &= \int_0^1 (x^{1/2} - x^{3/2}) dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^1 \\ &= \frac{2}{3} - \frac{2}{5} = \frac{4}{15} \end{aligned}$$



$$\begin{aligned} 48. \text{Area} &= \int_0^{\pi/3} \sec^2 x dx \\ &= \tan x \Big|_0^{\pi/3} = \sqrt{3} \end{aligned}$$



$$\begin{aligned} 50. \frac{1}{2-0} \int_0^2 x^3 dx &= \left[\frac{x^4}{8} \right]_0^2 = 2 \\ x^3 &= 2 \\ x &= \sqrt[3]{2} \end{aligned}$$



$$52. F'(x) = \frac{1}{x^2}$$

$$54. F'(x) = \csc^2 x$$

$$\begin{aligned} 56. \int \left(x + \frac{1}{x} \right)^2 dx &= \int (x^2 + 2 + x^{-2}) dx \\ &= \frac{x^3}{3} + 2x - \frac{1}{x} + C \end{aligned}$$

58. $u = x^3 + 3, du = 3x^2 dx$

$$\int x^2 \sqrt{x^3 + 3} dx = \frac{1}{3} \int (x^3 + 3)^{1/2} 3x^2 dx = \frac{2}{9} (x^3 + 3)^{3/2} + C$$

60. $u = x^2 + 6x - 5, du = (2x + 6) dx$

$$\int \frac{x + 3}{(x^2 + 6x - 5)^2} dx = \frac{1}{2} \int \frac{2x + 6}{(x^2 + 6x - 5)^2} dx = \frac{-1}{2} (x^2 + 6x - 5)^{-1} + C = \frac{-1}{2(x^2 + 6x - 5)} + C$$

62. $\int x \sin 3x^2 dx = \frac{1}{6} \int (\sin 3x^2)(6x) dx = -\frac{1}{6} \cos 3x^2 + C$

64. $\int \frac{\cos x}{\sqrt{\sin x}} dx = \int (\sin x)^{-1/2} \cos x dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$

66. $\int \sec 2x \tan 2x dx = \frac{1}{2} \int (\sec 2x \tan 2x)(2) dx = \frac{1}{2} \sec 2x + C$

68. $\int \cot^4 \alpha \csc^2 \alpha d\alpha = - \int (\cot \alpha)^4 (-\csc^2 \alpha) d\alpha = -\frac{1}{5} \cot^5 \alpha + C$

70. $\int_0^1 x^2 (x^3 + 1)^3 dx = \frac{1}{3} \int_0^1 (x^3 + 1)^3 (3x^2) dx = \frac{1}{12} [(x^3 + 1)^4]_0^1 = \frac{1}{12} (16 - 1) = \frac{5}{4}$

72. $\int_3^6 \frac{x}{3\sqrt{x^2 - 8}} dx = \frac{1}{6} \int_3^6 (x^2 - 8)^{-1/2} (2x) dx = \left[\frac{1}{3} (x^2 - 8)^{1/2} \right]_3^6 = \frac{1}{3} (2\sqrt{7} - 1)$

74. $u = x + 1, x = u - 1, dx = du$

When $x = -1, u = 0$. When $x = 0, u = 1$.

$$\begin{aligned} 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx &= 2\pi \int_0^1 (u-1)^2 \sqrt{u} du \\ &= 2\pi \int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2}) du = 2\pi \left[\frac{2}{7} u^{7/2} - \frac{4}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_0^1 = \frac{32\pi}{105} \end{aligned}$$

76. $\int_{-\pi/4}^{\pi/4} \sin 2x dx = 0$ since $\sin 2x$ is an odd function.

78. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a, u = 1 - a$. When $x = b, u = 1 - b$.

$$\begin{aligned} P_{a,b} &= \int_a^b \frac{1155}{32} x^3 (1-x)^{3/2} dx = \frac{1155}{32} \int_{1-a}^{1-b} -(1-u)^3 u^{3/2} du \\ &= \frac{1155}{32} \int_{1-a}^{1-b} (u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2}) du = \frac{1155}{32} \left[\frac{2}{11} u^{11/2} - \frac{2}{3} u^{9/2} + \frac{6}{7} u^{7/2} - \frac{2}{5} u^{5/2} \right]_{1-a}^{1-b} \\ &= \frac{1155}{32} \left[\frac{2u^{5/2}}{1155} (105u^3 - 385u^2 + 495u - 231) \right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_{1-a}^{1-b} \\ \text{(a) } P_{0,0.25} &= \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_1^{0.75} \approx 0.025 = 2.5\% \\ \text{(b) } P_{0.5,1} &= \left[\frac{u^{5/2}}{16} (105u^3 - 385u^2 + 495u - 231) \right]_{0.5}^0 \approx 0.736 = 73.6\% \end{aligned}$$

$$80. \int_0^2 1.75 \sin \frac{\pi t}{2} dt = -\frac{2}{\pi} \left[1.75 \cos \frac{\pi t}{2} \right] = -\frac{2}{\pi} (1.75)(-1 - 1) = \frac{7}{\pi} \approx 2.2282 \text{ liters}$$

Increase is

$$\frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048 \text{ liters.}$$

$$82. \text{ Trapezoidal Rule } (n = 4): \int_0^1 \frac{x^{3/2}}{3 - x^2} dx \approx \frac{1}{8} \left[0 + \frac{2(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{2(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2} \right] \approx 0.172$$

$$\text{Simpson's Rule } (n = 4): \int_0^1 \frac{x^{3/2}}{3 - x^2} dx \approx \frac{1}{12} \left[0 + \frac{4(1/4)^{3/2}}{3 - (1/4)^2} + \frac{2(1/2)^{3/2}}{3 - (1/2)^2} + \frac{4(3/4)^{3/2}}{3 - (3/4)^2} + \frac{1}{2} \right] \approx 0.166$$

Graphing utility: 0.166

$$84. \text{ Trapezoidal Rule } (n = 4): \int_0^{\pi} \sqrt{1 + \sin^2 x} dx \approx 3.820$$

Simpson's Rule ($n = 4$): 3.820

Graphing utility: 3.820

Problem Solving for Chapter 4

$$2. (a) F(x) = \int_2^x \sin t^2 dt$$

x	0	1.0	1.5	1.9	2.0	2.1	2.5	3.0	4.0	5.0
$F(x)$	-0.8048	-0.4945	-0.0265	0.0611	0	-0.0867	-0.3743	-0.0312	-0.0576	-0.2769

$$(b) G(x) = \frac{1}{x-2} \int_2^x \sin t^2 dt$$

x	1.9	1.95	1.99	2.01	2.05	2.1
$G(x)$	-0.6106	-0.6873	-0.7436	-0.7697	-0.8174	-0.8671

$$\lim_{x \rightarrow 2} G(x) \approx -0.75$$

$$(c) F'(2) = \lim_{x \rightarrow 2} \frac{F(x) - F(2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{1}{x-2} \int_2^x \sin t^2 dt$$

$$= \lim_{x \rightarrow 2} G(x)$$

$$\text{Since } F'(x) = \sin x^2, F'(2) = \sin 4 = \lim_{x \rightarrow 2} G(x).$$

(Note: $\sin 4 \approx -0.7568$)

4. Let d be the distance traversed and a be the uniform acceleration.

We can assume that $v(0) = 0$ and $s(0) = 0$. Then

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{1}{2}at^2.$$

$$s(t) = d \text{ when } t = \sqrt{\frac{2d}{a}}.$$

$$\text{The highest speed is } v = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}.$$

The lowest speed is $v = 0$.

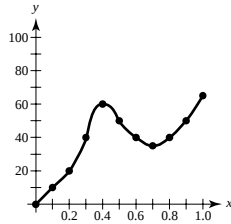
$$\text{The mean speed is } \frac{1}{2}(\sqrt{2ad} + 0) = \sqrt{\frac{ad}{2}}.$$

The time necessary to traverse the distance d at the mean speed is

$$t = \frac{d}{\sqrt{ad/2}} = \sqrt{\frac{2d}{a}}$$

which is the same as the time calculated above.

6. (a)



- (b) v is increasing (positive acceleration) on $(0, 0.4)$ and $(0.7, 1.0)$.

$$(c) \text{ Average acceleration} = \frac{v(0.4) - v(0)}{0.4 - 0} = \frac{60 - 0}{0.4} = 150 \text{ mi/hr}^2$$

- (d) This integral is the total distance traveled in miles.

$$\int_0^1 v(t) dt \approx \frac{1}{10}[0 + 2(20) + 2(60) + 2(40) + 2(40) + 65] = \frac{385}{10} = 38.5 \text{ miles}$$

- (e) One approximation is

$$a(0.8) \approx \frac{v(0.9) - v(0.8)}{0.9 - 0.8} = \frac{50 - 40}{0.1} = 100 \text{ mi/hr}^2$$

(other answers possible)

$$8. \int_0^x f(t)(x-t) dt = \int_0^x x f(t) dt - \int_0^x t f(t) dt = x \int_0^x f(t) dt - \int_0^x t f(t) dt$$

$$\text{Thus, } \frac{d}{dx} \int_0^x f(t)(x-t) dt = x f(x) + \int_0^x f(t) dt - x f(x) = \int_0^x f(t) dt$$

Differentiating the other integral,

$$\frac{d}{dx} \int_0^x \left(\int_0^x f(v) dv \right) dt = \int_0^x f(v) dv.$$

Thus, the two original integrals have equal derivatives,

$$\int_0^x f(t)(x-t) dt = \int_0^x \left(\int_0^t f(v) dv \right) dt + C$$

Letting $x = 0$, we see that $C = 0$.

10. Consider $\int_0^1 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3}$. The corresponding

Riemann Sum using right-hand endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}} \right] \\ &= \frac{1}{n^{3/2}} [\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}] \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}} = \frac{2}{3}.$$

$$\begin{aligned} 12. (a) \text{ Area} &= \int_{-3}^3 (9 - x^2) dx = 2 \int_0^3 (9 - x^2) dx \\ &= 2 \left[9x - \frac{x^3}{3} \right]_0^3 \\ &= 2[27 - 9] = 36 \end{aligned}$$

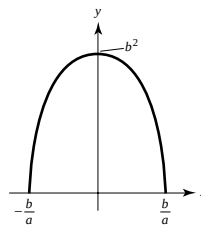
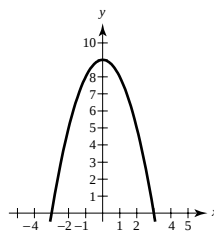
$$(b) \text{ Base} = 6, \text{ height} = 9. \text{ Area} = \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36.$$

(c) Let the parabola be given by $y = b^2 - a^2x^2$, $a, b > 0$.

$$\begin{aligned} \text{Area} &= 2 \int_0^{b/a} (b^2 - a^2x^2) dx \\ &= 2 \left[b^2x - a^2 \frac{x^3}{3} \right]_0^{b/a} \\ &= 2 \left[b^2 \left(\frac{b}{a} \right) - \frac{a^2}{3} \left(\frac{b}{a} \right)^3 \right] \\ &= 2 \left[\frac{b^3}{a} - \frac{1}{3} \frac{b^3}{a} \right] = \frac{4}{3} \frac{b^3}{a} \end{aligned}$$

$$\text{Base} = \frac{2b}{a}, \text{ height} = b^2$$

$$\text{Archimedes' Formula: Area} = \frac{2}{3} \left(\frac{2b}{a} \right) (b^2) = \frac{4}{3} \frac{b^3}{a}$$



14. (a) $(1 + i)^3 = 1 + 3i + 3i^2 + i^3 \Rightarrow (1 + i)^3 - i^3 = 3i^2 + 3i + 1$

(b) $3i^2 + 3i + 1 = (i + 1)^3 - i^3$

$$\begin{aligned}\sum_{i=1}^n (3i^2 + 3i + 1) &= \sum_{i=1}^n [(i + 1)^3 - i^3] \\ &= (2^3 - 1^3) + (3^3 - 2^3) + \cdots + [(n + 1)^3 - n^3] \\ &= (n + 1)^3 - 1\end{aligned}$$

Hence, $(n + 1)^3 = \sum_{i=1}^n (3i^2 + 3i + 1) + 1$

(c) $(n + 1)^3 - 1 = \sum_{i=1}^n (3i^2 + 3i + 1) = \sum_{i=1}^n 3i^2 + \frac{3(n)(n + 1)}{2} + n$

$$\begin{aligned}\Rightarrow \sum_{i=1}^n 3i^2 &= n^3 + 3n^2 + 3n - \frac{3n(n + 1)}{2} - n \\ &= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{2} \\ &= \frac{2n^3 + 3n^2 + n}{2} \\ &= \frac{n(n + 1)(2n + 1)}{2} \\ \Rightarrow \sum_{i=1}^n i^2 &= \frac{n(n + 1)(2n + 1)}{6}\end{aligned}$$

16. (a) $C = 0.1 \int_8^{20} \left[12 \sin \frac{\pi(t - 8)}{12} \right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t - 8)}{12} \right]_8^{20} = \frac{-14.4}{\pi}(-1 - 1) \approx \9.17

(b) $C = 0.1 \int_{10}^{18} \left[12 \sin \frac{\pi(t - 8)}{12} - 6 \right] dt = \left[-\frac{14.4}{\pi} \cos \frac{\pi(t - 8)}{12} - 0.6t \right]_{10}^{18}$
 $= \left[\frac{-14.4}{\pi} \left(\frac{-\sqrt{3}}{2} \right) - 10.8 \right] - \left[\frac{-14.4}{\pi} \left(\frac{\sqrt{3}}{2} \right) - 6 \right] \approx \3.14

Savings $\approx 9.17 - 3.14 = \$6.03$.

18. (a) Let $A = \int_0^b \frac{f(x)}{f(x) + f(b - x)} dx$.

Let $u = b - x$, $du = -dx$.

$$\begin{aligned}A &= \int_b^0 \frac{f(b - u)}{f(b - u) + f(u)} (-du) \\ &= \int_0^b \frac{f(b - u)}{f(b - u) + f(u)} du \\ &= \int_0^b \frac{f(b - x)}{f(b - x) + f(x)} dx\end{aligned}$$

Then,

$$\begin{aligned}2A &= \int_0^b \frac{f(x)}{f(x) + f(b - x)} dx + \int_0^b \frac{f(b - x)}{f(b - x) + f(x)} dx \\ &= \int_0^b 1 dx = b.\end{aligned}$$

Thus, $A = \frac{b}{2}$.

(b) $b = 1 \Rightarrow \int_0^1 \frac{\sin x}{\sin(1 - x) + \sin x} dx = \frac{1}{2}$

CHAPTER 4

Integration

Section 4.1	Antiderivatives and Indefinite Integration	177
Section 4.2	Area	182
Section 4.3	Riemann Sums and Definite Integrals	188
Section 4.4	The Fundamental Theorem of Calculus	192
Section 4.5	Integration by Substitution	197
Section 4.6	Numerical Integration	204
Review Exercises		209
Problem Solving		214

CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

$$1. \frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$$

$$3. \frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$$

$$5. \frac{dy}{dt} = 3t^2$$

$$y = t^3 + C$$

$$\text{Check: } \frac{d}{dt}[t^3 + C] = 3t^2$$

$$7. \frac{dy}{dx} = x^{3/2}$$

$$y = \frac{2}{5}x^{5/2} + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$$

Given

Rewrite

Integrate

Simplify

$$9. \int \sqrt[3]{x} \, dx$$

$$\int x^{1/3} \, dx$$

$$\frac{x^{4/3}}{4/3} + C$$

$$\frac{3}{4}x^{4/3} + C$$

$$11. \int \frac{1}{x\sqrt{x}} \, dx$$

$$\int x^{-3/2} \, dx$$

$$\frac{x^{-1/2}}{-1/2} + C$$

$$-\frac{2}{\sqrt{x}} + C$$

$$13. \int \frac{1}{2x^3} \, dx$$

$$\frac{1}{2} \int x^{-3} \, dx$$

$$\frac{1}{2} \left(\frac{x^{-2}}{-2} \right) + C$$

$$-\frac{1}{4x^2} + C$$

$$15. \int (x + 3) \, dx = \frac{x^2}{2} + 3x + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{x^2}{2} + 3x + C\right] = x + 3$$

$$17. \int (2x - 3x^2) \, dx = x^2 - x^3 + C$$

$$\text{Check: } \frac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$$

$$19. \int (x^3 + 2) \, dx = \frac{1}{4}x^4 + 2x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$$

$$21. \int (x^{3/2} + 2x + 1) \, dx = \frac{2}{5}x^{5/2} + x^2 + x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$$

$$23. \int \sqrt[3]{x^2} \, dx = \int x^{2/3} \, dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$$

$$25. \int \frac{1}{x^3} \, dx = \int x^{-3} \, dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\text{Check: } \frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$$

$$27. \int \frac{x^2 + x + 1}{\sqrt{x}} dx = \int (x^{3/2} + x^{1/2} + x^{-1/2}) dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$$

Check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

$$29. \int (x + 1)(3x - 2) dx = \int (3x^2 + x - 2) dx$$

$$= x^3 + \frac{1}{2}x^2 - 2x + C$$

Check: $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$$= (x + 1)(3x - 2)$$

$$31. \int y^2 \sqrt{y} dy = \int y^{5/2} dy = \frac{2}{7}y^{7/2} + C$$

Check: $\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2 \sqrt{y}$

$$33. \int dx = \int 1 dx = x + C$$

Check: $\frac{d}{dx}(x + C) = 1$

$$35. \int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + C$$

Check: $\frac{d}{dx}(-2 \cos x + 3 \sin x + C) = 2 \sin x + 3 \cos x$

$$37. \int (1 - \csc t \cot t) dt = t + \csc t + C$$

Check: $\frac{d}{dt}(t + \csc t + C) = 1 - \csc t \cot t$

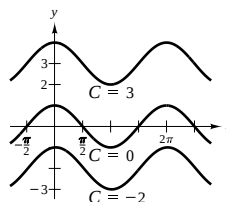
$$39. \int (\sec^2 \theta - \sin \theta) d\theta = \tan \theta + \cos \theta + C$$

Check: $\frac{d}{d\theta}(\tan \theta + \cos \theta + C) = \sec^2 \theta - \sin \theta$

$$41. \int (\tan^2 y + 1) dy = \int \sec^2 y dy = \tan y + C$$

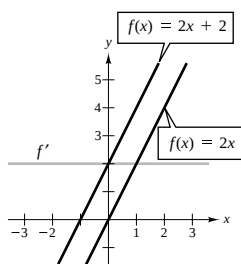
Check: $\frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

$$43. f(x) = \cos x$$



$$45. f'(x) = 2$$

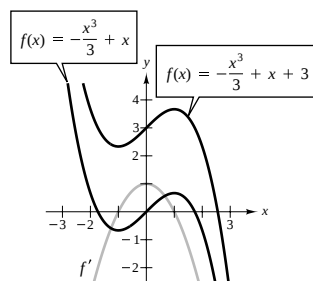
$$f(x) = 2x + C$$



Answers will vary.

$$47. f'(x) = 1 - x^2$$

$$f(x) = x - \frac{x^3}{3} + C$$



Answers will vary.

$$49. \frac{dy}{dx} = 2x - 1, (1, 1)$$

$$y = \int (2x - 1) dx = x^2 - x + C$$

$$1 = (1)^2 - (1) + C \Rightarrow C = 1$$

$$y = x^2 - x + 1$$

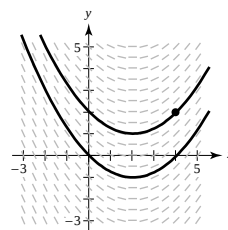
51. $\frac{dy}{dx} = \cos x, (0, 4)$

$$y = \int \cos x \, dx = \sin x + C$$

$$4 = \sin 0 + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

53. (a) Answers will vary.



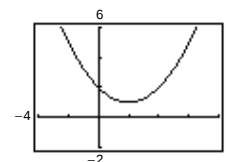
(b) $\frac{dy}{dx} = \frac{1}{2}x - 1, (4, 2)$

$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$



55. $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x \, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5) \, dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

59. $f''(x) = 2$

$$f'(2) = 5$$

$$f(2) = 10$$

$$f'(x) = \int 2 \, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1) \, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$$

$$f(x) = x^2 + x + 4$$

61. $f''(x) = x^{-3/2}$

$$f'(4) = 2$$

$$f(0) = 0$$

$$f'(x) = \int x^{-3/2} \, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int \left(-\frac{2}{\sqrt{x}} + 3\right) \, dx = -4x^{1/2} + 3x + C_2$$

$$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

63. (a) $h(t) = \int (1.5t + 5) \, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$$

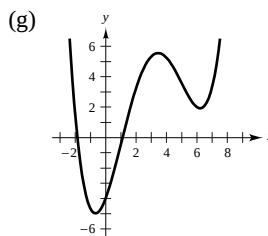
$$h(t) = 0.75t^2 + 5t + 12$$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69 \text{ cm}$

65. $f(0) = -4$. Graph of f' is given.

- (a) $f'(4) \approx -1.0$
 (b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.
 (c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.
 (d) f is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.
 (e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) f'' is a minimum at $x = 3$.



67. $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32 \, dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60) \, dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$$s(t) = -16t^2 + 60t + 6 \text{ Position function}$$

The ball reaches its maximum height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds}$$

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 \approx 62.26 \text{ feet}$$

71. $a(t) = -9.8$

$$v(t) = \int -9.8 \, dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0) \, dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$$

75. $a = -1.6$

$$v(t) = \int -1.6 \, dt = -1.6t + v_0 = -1.6t, \text{ since the stone was dropped, } v_0 = 0.$$

$$s(t) = \int (-1.6t) \, dt = -0.8t^2 + s_0$$

$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

69. From Exercise 68, we have:

$$s(t) = -16t^2 + v_0t$$

$$s'(t) = -32t + v_0 = 0 \text{ when } t = \frac{v_0}{32} = \text{time to reach maximum height.}$$

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35,200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

73. From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$$v(t) = -9.8t + 10 = 0 \text{ (Maximum height when } v = 0.)$$

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

77. $x(t) = t^3 - 6t^2 + 9t - 2 \quad 0 \leq t \leq 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$
 $= 3(t^2 - 4t + 3) = 3(t - 1)(t - 3)$

$a(t) = v'(t) = 6t - 12 = 6(t - 2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t - 2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

79. $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$

$x(t) = \int v(t) dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \Rightarrow C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$ acceleration

81. (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$

$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$

$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$

$\frac{550}{36} = 13a$

$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a\frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$

$s(13) = \frac{275}{234} \frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$

85. $\frac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \frac{22}{15} \text{ ft/sec}$

(a)	t	0	5	10	15	20	25	30
	$V_1(\text{ft/sec})$	0	3.67	10.27	23.47	42.53	66	95.33
	$V_2(\text{ft/sec})$	0	30.8	55.73	74.8	88	93.87	95.33

(c) $S_1(t) = \int V_1(t) dt = \frac{0.1068}{3}t^3 - \frac{0.0416}{2}t^2 + 0.3679t$

$S_2(t) = \int V_2(t) dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$]

$S_1(30) \approx 953.5 \text{ feet}$

$S_2(30) \approx 1970.3 \text{ feet}$

The second car was going faster than the first until the end.

83. Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t - 10)$ when $t = 10 \text{ sec}$.

(a) $s(10) = 3(10)^2 = 300 \text{ ft}$

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mph}$

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

87. $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18,285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

89. True

91. True

93. False. For example, $\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$

95. $f'(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 3x, & 2 \leq x \leq 5 \end{cases}$

$$f(x) = \begin{cases} x + C_1, & 0 \leq x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \leq x \leq 5 \end{cases}$$

$$f(1) = 3 \Rightarrow 1 + C_1 = 3 \Rightarrow C_1 = 2$$

f is continuous: Values must agree at $x = 2$:

$$4 = 6 + C_2 \Rightarrow C_2 = -2$$

$$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ \frac{3x^2}{2} - 2, & 2 \leq x \leq 5 \end{cases}$$

The left and right hand derivatives at $x = 2$ do not agree. Hence f is not differentiable at $x = 2$.

Section 4.2 Area

1. $\sum_{i=1}^5 (2i + 1) = 2 \sum_{i=1}^5 i + \sum_{i=1}^5 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$

3. $\sum_{k=0}^4 \frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

5. $\sum_{k=1}^4 c = c + c + c + c = 4c$

7. $\sum_{i=1}^9 \frac{1}{3i}$

9. $\sum_{j=1}^8 \left[5\left(\frac{j}{8}\right) + 3 \right]$

11. $\frac{2}{n} \sum_{i=1}^n \left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right) \right]$

13. $\frac{3}{n} \sum_{i=1}^n \left[2\left(1 + \frac{3i}{n}\right)^2 \right]$

15. $\sum_{i=1}^{20} 2i = 2 \sum_{i=1}^{20} i = 2 \left[\frac{20(21)}{2} \right] = 420$

$$17. \sum_{i=1}^{20} (i - 1)^2 = \sum_{i=1}^{19} i^2$$

$$= \left[\frac{19(20)(39)}{6} \right] = 2470$$

$$\begin{aligned}
 19. \sum_{i=1}^{15} i(i-1)^2 &= \sum_{i=1}^{15} i^3 - 2 \sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i \\
 &= \frac{15^2(16)^2}{4} - 2 \frac{15(16)(31)}{6} + \frac{15(16)}{2} \\
 &= 14,400 - 2,480 + 120 \\
 &= 12,040
 \end{aligned}$$

$$21. \text{sum seq}(x \sqcap 2 + 3, x, 1, 20, 1) = 2930 \quad (TI-82)$$

$$\begin{aligned}
 \sum_{i=1}^{20} (i^2 + 3) &= \frac{20(20+1)(2(20)+1)}{6} + 3(20) \\
 &= \frac{(20)(21)(41)}{6} + 60 = 2930
 \end{aligned}$$

$$\begin{aligned}
 23. S &= [3 + 4 + \frac{9}{2} + 5](1) = \frac{33}{2} = 16.5 \\
 s &= [1 + 3 + 4 + \frac{9}{2}](1) = \frac{25}{2} = 12.5
 \end{aligned}$$

$$\begin{aligned}
 25. S &= [3 + 3 + 5](1) = 11 \\
 s &= [2 + 2 + 3](1) = 7
 \end{aligned}$$

$$\begin{aligned}
 27. S(4) &= \sqrt{\frac{1}{4}\left(\frac{1}{4}\right)} + \sqrt{\frac{1}{2}\left(\frac{1}{4}\right)} + \sqrt{\frac{3}{4}\left(\frac{1}{4}\right)} + \sqrt{1\left(\frac{1}{4}\right)} = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768 \\
 s(4) &= 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}\left(\frac{1}{4}\right)} + \sqrt{\frac{1}{2}\left(\frac{1}{4}\right)} + \sqrt{\frac{3}{4}\left(\frac{1}{4}\right)} = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518
 \end{aligned}$$

$$\begin{aligned}
 29. S(5) &= 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746 \\
 s(5) &= \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646
 \end{aligned}$$

$$31. \lim_{n \rightarrow \infty} \left[\left(\frac{81}{n^4} \right) \frac{n^2(n+1)^2}{4} \right] = \frac{81}{4} \lim_{n \rightarrow \infty} \left[\frac{n^4 + 2n^3 + n^2}{n^4} \right] = \frac{81}{4}(1) = \frac{81}{4}$$

$$33. \lim_{n \rightarrow \infty} \left[\left(\frac{18}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{18}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{18}{2}(1) = 9$$

$$\begin{aligned}
 35. \sum_{i=1}^n \frac{2i+1}{n^2} &= \frac{1}{n^2} \sum_{i=1}^n (2i+1) = \frac{1}{n^2} \left[2 \frac{n(n+1)}{2} + n \right] = \frac{n+2}{n} = S(n) \\
 S(10) &= \frac{12}{10} = 1.2
 \end{aligned}$$

$$S(100) = 1.02$$

$$S(1000) = 1.002$$

$$S(10,000) = 1.0002$$

$$\begin{aligned}
 37. \sum_{k=1}^n \frac{6k(k-1)}{n^3} &= \frac{6}{n^3} \sum_{k=1}^n (k^2 - k) = \frac{6}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right] \\
 &= \frac{6}{n^3} \left[\frac{2n^2 + 3n + 1 - 3n - 3}{6} \right] = \frac{1}{n^2} [2n^2 - 2] = S(n)
 \end{aligned}$$

$$S(10) = 1.98$$

$$S(100) = 1.9998$$

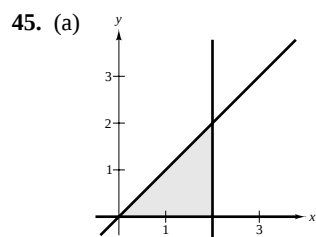
$$S(1000) = 1.999998$$

$$S(10,000) = 1.99999998$$

$$39. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{16i}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{16}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{16}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \left[8 \left(\frac{n^2 + n}{n^2} \right) \right] = 8 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 8$$

$$\begin{aligned}
 41. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i-1)^2 &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{(n-1)(n)(2n-1)}{6} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{6} \left[\frac{2n^3 - 3n^2 + n}{n^3} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(2 - \frac{3}{n} + \frac{1}{n^2} \right) \right] = \frac{1}{3}
 \end{aligned}$$

$$43. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{i}{n} \right) \left(\frac{2}{n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{i=1}^n 1 + \frac{1}{n} \sum_{i=1}^n i \right] = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{1}{n} \left(\frac{n(n+1)}{2} \right) \right] = 2 \lim_{n \rightarrow \infty} \left[1 + \frac{n^2 + n}{2n^2} \right] = 2 \left(1 + \frac{1}{2} \right) = 3$$



(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f(x_{i-1}) \Delta x \\
 &= \sum_{i=1}^n f\left(\frac{2i-2}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[(i-1) \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right)
 \end{aligned}$$

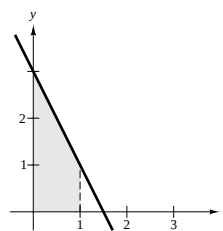
(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \frac{2}{n} = \sum_{i=1}^n \left[i \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right)$$

47. $y = -2x + 3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

$$\begin{aligned}
 s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[-2\left(\frac{i}{n}\right) + 3 \right] \left(\frac{1}{n}\right) \\
 &= 3 - \frac{2}{n^2} \sum_{i=1}^n i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}
 \end{aligned}$$

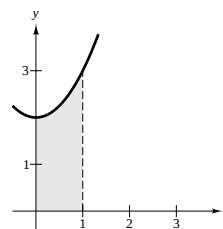
Area = $\lim_{n \rightarrow \infty} s(n) = 2$



49. $y = x^2 + 2$ on $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned}
 S(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(\frac{i}{n}\right)^2 + 2 \right] \left(\frac{1}{n}\right) \\
 &= \left[\frac{1}{n^3} \sum_{i=1}^n i^2 \right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) + 2
 \end{aligned}$$

Area = $\lim_{n \rightarrow \infty} S(n) = \frac{7}{3}$



(e)

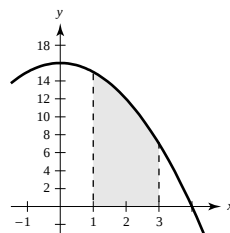
x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

$$\begin{aligned}
 (f) \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[(i-1) \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n (i-1) \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \left[\frac{n(n+1)}{2} - n \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{2(n+1)}{n} - \frac{4}{n} \right] = 2 \\
 \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[i \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i \\
 &= \lim_{n \rightarrow \infty} \left(\frac{4}{n^2} \right) \frac{n(n+1)}{2} \\
 &= \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 2
 \end{aligned}$$

51. $y = 16 - x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{2}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n} \sum_{i=1}^n \left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right] \\ &= \frac{2}{n} \left[15n - \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n} \frac{n(n+1)}{2}\right] \\ &= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1) \end{aligned}$$

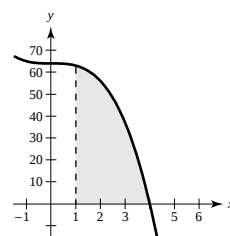
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$



53. $y = 64 - x^3$ on $[1, 4]$. (Note: $\Delta x = \frac{4-1}{n} = \frac{3}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right] \\ &= \frac{3}{n} \left[63n - \frac{27}{n^3} \frac{n^2(n+1)^2}{4} - \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{9}{n} \frac{n(n+1)}{2}\right] \\ &= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2} \frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

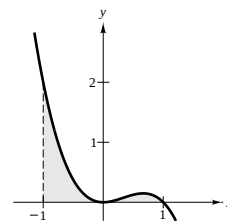


55. $y = x^2 - x^3$ on $[-1, 1]$. (Note: $\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}$)

Again, $T(n)$ is neither an upper nor a lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n} \sum_{i=1}^n 1 - \frac{20}{n^2} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n i^2 - \frac{16}{n^4} \sum_{i=1}^n i^3 \\ &= \frac{4}{n}(n) - \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

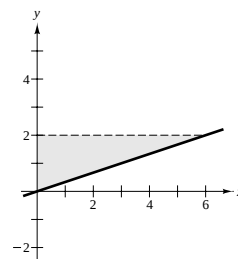


57. $f(y) = 3y, 0 \leq y \leq 2$ (Note: $\Delta y = \frac{2-0}{n} = \frac{2}{n}$)

$$S(n) = \sum_{i=1}^n f(m_i) \Delta y = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n 3\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2} \sum_{i=1}^n i = \left(\frac{12}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(6 + \frac{6}{n}\right) = 6$$

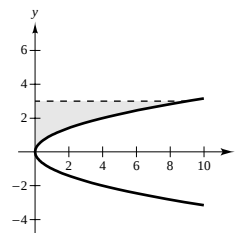


59. $f(y) = y^2, 0 \leq y \leq 3$ (Note: $\Delta y = \frac{3-0}{n} = \frac{3}{n}$)

$$S(n) = \sum_{i=1}^n f\left(\frac{3i}{n}\right) \left(\frac{3}{n}\right) = \sum_{i=1}^n \left(\frac{3i}{n}\right)^2 \left(\frac{3}{n}\right) = \frac{27}{n^3} \sum_{i=1}^n i^2$$

$$= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2} \left(\frac{2n^2 + 3n + 1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$



61. $g(y) = 4y^2 - y^3, 1 \leq y \leq 3$ (Note: $\Delta y = \frac{3-1}{n} = \frac{2}{n}$)

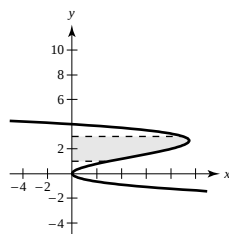
$$S(n) = \sum_{i=1}^n g\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^n \left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3 \right] \frac{2}{n}$$

$$= \frac{2}{n} \sum_{i=1}^n \left[4\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right) \right]$$

$$= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3} \right] = \frac{2}{n} \left[3n + \frac{10}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2} \frac{n^2(n+1)^2}{4} \right]$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



63. $f(x) = x^2 + 3, 0 \leq x \leq 2, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 3] \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \left[\left(\frac{1}{4}\right)^2 + 3 + \left(\frac{3}{4}\right)^2 + 3 + \left(\frac{5}{4}\right)^2 + 3 + \left(\frac{7}{4}\right)^2 + 3 \right]$$

$$= \frac{69}{8}$$

65. $f(x) = \tan x, 0 \leq x \leq \frac{\pi}{4}, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\tan c_i) \left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16} \left(\tan \frac{\pi}{32} + \tan \frac{3\pi}{32} + \tan \frac{5\pi}{32} + \tan \frac{7\pi}{32} \right) \approx 0.345$$

67. $f(x) = \sqrt{x}$ on $[0, 4]$.

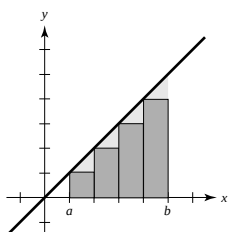
n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is $16/3$)

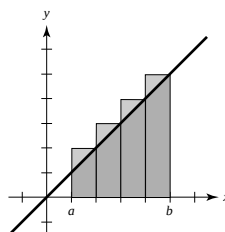
69. $f(x) = \tan\left(\frac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

71. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



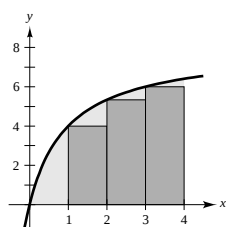
The sum of the areas of these circumscribed rectangles is the upper sum.



We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

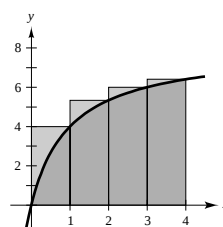
73. (a)



Lower sum:

$$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$$

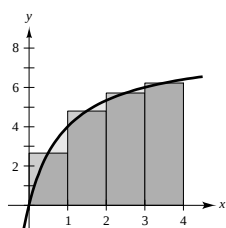
- (b)



Upper sum:

$$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$$

- (c)



Midpoint Rule:

$$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$$

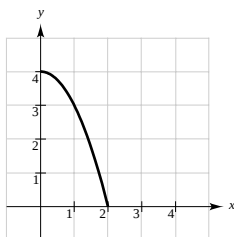
- (d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i-1)(4/n)$. The upper sum uses right endpoints, $i(4/n)$. The Midpoint Rule uses midpoints, $(i-\frac{1}{2})(4/n)$.

- (e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

- (f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

75.

b. $A \approx 6$ square units

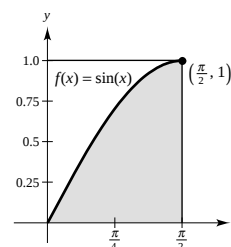
79. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$

Let A_1 = area bounded by $f(x) = \sin x$, the x -axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$.

In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2 . It is keeping track of how many of these points, N_1 , lie in the region whose area is represented by A_1 . Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

The larger N_2 is the better the approximation to A_1 .



81. Suppose there are n rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

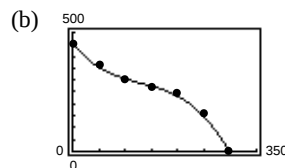
$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1).$$

83. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76,897.5 \text{ ft}^2.$$



Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \frac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i-1)$$

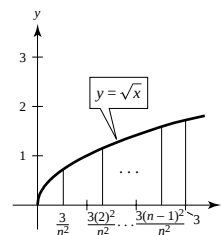
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\frac{3i^2}{n^2}} \frac{3}{n^2} (2i-1)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^n (2i^2 - i)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \left[2 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \rightarrow \infty} 3\sqrt{3} \left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} \right]$$

$$= 3\sqrt{3} \left[\frac{2}{3} - 0 \right] = 2\sqrt{3} \approx 3.464$$



3. $y = 6$ on $[4, 10]$. $\left(\text{Note: } \Delta x = \frac{10 - 4}{n} = \frac{6}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(4 + \frac{6i}{n}\right) \left(\frac{6}{n}\right) = \sum_{i=1}^n 6 \left(\frac{6}{n}\right) = \sum_{i=1}^n \frac{36}{n} = 36$$

$$\int_4^{10} 6 \, dx = \lim_{n \rightarrow \infty} 36 = 36$$

5. $y = x^3$ on $[-1, 1]$. $\left(\text{Note: } \Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2} \sum_{i=1}^n i - \frac{24}{n^3} \sum_{i=1}^n i^2 + \frac{16}{n^4} \sum_{i=1}^n i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^1 x^3 \, dx = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$$

7. $y = x^2 + 1$ on $[1, 2]$. $\left(\text{Note: } \Delta x = \frac{2 - 1}{n} = \frac{1}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty\right)$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(1 + \frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^2 + 1\right] \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right] \left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2} \sum_{i=1}^n i + \frac{1}{n^3} \sum_{i=1}^n i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1) \, dx = \lim_{n \rightarrow \infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

9. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (3c_i + 10) \Delta x_i = \int_{-1}^5 (3x + 10) \, dx$

on the interval $[-1, 5]$.

11. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt{c_i^2 + 4} \Delta x_i = \int_0^3 \sqrt{x^2 + 4} \, dx$

on the interval $[0, 3]$.

13. $\int_0^5 3 \, dx$

15. $\int_{-4}^4 (4 - |x|) \, dx$

17. $\int_{-2}^2 (4 - x^2) \, dx$

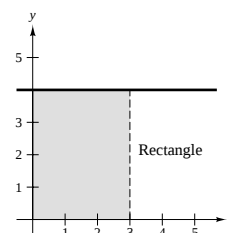
19. $\int_0^\pi \sin x \, dx$

21. $\int_0^2 y^3 \, dy$

23. Rectangle

$$A = bh = 3(4)$$

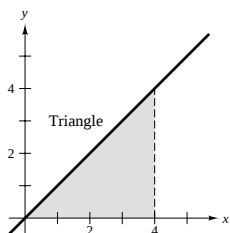
$$A = \int_0^3 4 \, dx = 12$$



25. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

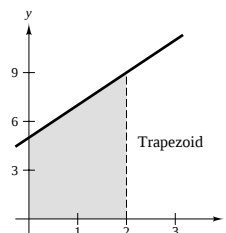
$$A = \int_0^4 x \, dx = 8$$



27. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$$

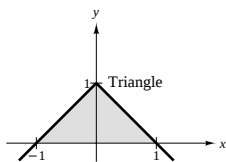
$$A = \int_0^2 (2x + 5) \, dx = 14$$



29. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

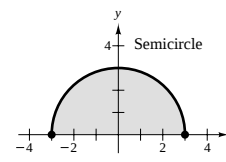
$$A = \int_{-1}^1 (1 - |x|) \, dx = 1$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

$$A = \int_{-3}^3 \sqrt{9 - x^2} \, dx = \frac{9\pi}{2}$$



In Exercises 33–39, $\int_2^4 x^3 \, dx = 60$, $\int_2^4 x \, dx = 6$, $\int_2^4 dx = 2$

$$33. \int_4^2 x \, dx = -\int_2^4 x \, dx = -6$$

$$37. \int_2^4 (x - 8) \, dx = \int_2^4 x \, dx - 8 \int_2^4 dx = 6 - 8(2) = -10$$

$$41. (a) \int_0^7 f(x) \, dx = \int_0^5 f(x) \, dx + \int_5^7 f(x) \, dx = 10 + 3 = 13$$

$$(b) \int_5^0 f(x) \, dx = -\int_0^5 f(x) \, dx = -10$$

$$(c) \int_5^5 f(x) \, dx = 0$$

$$(d) \int_0^5 3f(x) \, dx = 3 \int_0^5 f(x) \, dx = 3(10) = 30$$

$$35. \int_2^4 4x \, dx = 4 \int_2^4 x \, dx = 4(6) = 24$$

$$39. \int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx = \frac{1}{2} \int_2^4 x^3 \, dx - 3 \int_2^4 x \, dx + 2 \int_2^4 dx$$

$$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

$$43. (a) \int_2^6 [f(x) + g(x)] \, dx = \int_2^6 f(x) \, dx + \int_2^6 g(x) \, dx$$

$$= 10 + (-2) = 8$$

$$(b) \int_2^6 [g(x) - f(x)] \, dx = \int_2^6 g(x) \, dx - \int_2^6 f(x) \, dx$$

$$= -2 - 10 = -12$$

$$(c) \int_2^6 2g(x) \, dx = 2 \int_2^6 g(x) \, dx = 2(-2) = -4$$

$$(d) \int_2^6 3f(x) \, dx = 3 \int_2^6 f(x) \, dx = 3(10) = 30$$

$$45. (a) \text{ Quarter circle below } x\text{-axis: } -\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$$

$$(b) \text{ Triangle: } \frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$$

$$(c) \text{ Triangle + Semicircle below } x\text{-axis: } -\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$$

$$(d) \text{ Sum of parts (b) and (c): } 4 - (1 + 2\pi) = 3 - 2\pi$$

$$(e) \text{ Sum of absolute values of (b) and (c): } 4 + (1 + 2\pi) = 5 + 2\pi$$

$$(f) \text{ Answer to (d) plus } 2(10) = 20: (3 - 2\pi) + 20 = 23 - 2\pi$$

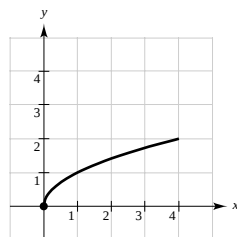
47. The left endpoint approximation will be greater than the actual area: $>$

49. Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

51. $f(x) = \frac{1}{x-4}$

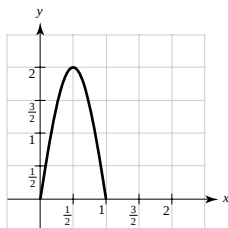
is not integrable on the interval $[3, 5]$ and f has a discontinuity at $x = 4$.

53.



a. $A \approx 5$ square units

55.



d. $\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

57. $\int_0^3 x\sqrt{3-x} \, dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59. $\int_0^{\pi/2} \sin^2 x \, dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True

63. True

65. False

$$\int_0^2 (-x) \, dx = -2$$

67. $f(x) = x^2 + 3x, [0, 8]$

$$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$$

$$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$$

$$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x &= f(1) \Delta x_1 + f(2) \Delta x_2 + f(5) \Delta x_3 + f(8) \Delta x_4 \\ &= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272 \end{aligned}$$

69. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \rightarrow 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

71. Let $f(x) = x^2$, $0 \leq x \leq 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

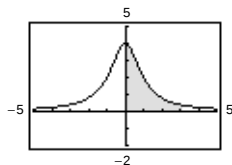
$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^3} [1^2 + 2^2 + 3^2 + \cdots + n^2] &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right) = \frac{1}{3} \end{aligned}$$

Section 4.4 The Fundamental Theorem of Calculus

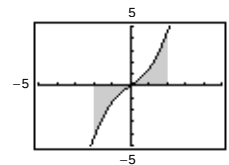
1. $f(x) = \frac{4}{x^2 + 1}$

$\int_0^\pi \frac{4}{x^2 + 1} dx$ is positive.



3. $f(x) = x\sqrt{x^2 + 1}$

$\int_{-2}^2 x\sqrt{x^2 + 1} dx = 0$



5. $\int_0^1 2x dx = [x^2]_0^1 = 1 - 0 = 1$

7. $\int_{-1}^0 (x - 2) dx = \left[\frac{x^2}{2} - 2x \right]_{-1}^0 = 0 - \left(\frac{1}{2} - 2 \right) = -\frac{5}{2}$

9. $\int_{-1}^1 (t^2 - 2) dt = \left[\frac{t^3}{3} - 2t \right]_{-1}^1 = \left(\frac{1}{3} - 2 \right) - \left(-\frac{1}{3} + 2 \right) = -\frac{10}{3}$

11. $\int_0^1 (2t - 1)^2 dt = \int_0^1 (4t^2 - 4t + 1) dt = \left[\frac{4}{3}t^3 - 2t^2 + t \right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

13. $\int_1^2 \left(\frac{3}{x^2} - 1 \right) dx = \left[-\frac{3}{x} - x \right]_1^2 = \left(-\frac{3}{2} - 2 \right) - (-3 - 1) = \frac{1}{2}$

15. $\int_1^4 \frac{u - 2}{\sqrt{u}} du = \int_1^4 (u^{1/2} - 2u^{-1/2}) du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2} \right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4} \right] - \left[\frac{2}{3} - 4 \right] = \frac{2}{3}$

17. $\int_{-1}^1 (\sqrt[3]{t} - 2) dt = \left[\frac{3}{4}t^{4/3} - 2t \right]_{-1}^1 = \left(\frac{3}{4} - 2 \right) - \left(\frac{3}{4} + 2 \right) = -4$

19. $\int_0^1 \frac{x - \sqrt{x}}{3} dx = \frac{1}{3} \int_0^1 (x - x^{1/2}) dx = \frac{1}{3} \left[\frac{x^2}{2} - \frac{2}{3}x^{3/2} \right]_0^1 = \frac{1}{3} \left(\frac{1}{2} - \frac{2}{3} \right) = -\frac{1}{18}$

21. $\int_{-1}^0 (t^{1/3} - t^{2/3}) dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3} \right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5} \right) = -\frac{27}{20}$

23. $\begin{aligned} \int_0^3 |2x - 3| dx &= \int_0^{3/2} (3 - 2x) dx + \int_{3/2}^3 (2x - 3) dx \quad \left(\text{split up the integral at the zero } x = \frac{3}{2} \right) \\ &= \left[3x - x^2 \right]_0^{3/2} + \left[x^2 - 3x \right]_{3/2}^3 = \left(\frac{9}{2} - \frac{9}{4} \right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2} \right) = 2 \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{9}{2} \end{aligned}$

$$\begin{aligned}
 25. \int_0^3 |x^2 - 4| dx &= \int_0^2 (4 - x^2) dx + \int_2^3 (x^2 - 4) dx \\
 &= \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - 4x \right]_2^3 \\
 &= \left(8 - \frac{8}{3} \right) + (9 - 12) - \left(\frac{8}{3} - 8 \right) \\
 &= \frac{23}{3}
 \end{aligned}$$

$$27. \int_0^\pi (1 + \sin x) dx = \left[x - \cos x \right]_0^\pi = (\pi + 1) - (0 - 1) = 2 + \pi$$

$$29. \int_{-\pi/6}^{\pi/6} \sec^2 x dx = \left[\tan x \right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3} \right) = \frac{2\sqrt{3}}{3}$$

$$31. \int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta d\theta = \left[4 \sec \theta \right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$$

$$33. \int_0^3 10,000(t - 6) dt = 10,000 \left[\frac{t^2}{2} - 6t \right]_0^3 = -\$135,000$$

$$35. A = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$37. A = \int_0^3 (3 - x)\sqrt{x} dx = \int_0^3 (3x^{1/2} - x^{3/2}) dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x) \right]_0^3 = \frac{12\sqrt{3}}{5}$$

$$39. A = \int_0^{\pi/2} \cos x dx = \left[\sin x \right]_0^{\pi/2} = 1$$

41. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1) dx = \left[x^3 + x \right]_0^2 = 8 + 2 = 10.$$

43. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x) dx = \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^2 = 4 + 2 = 6.$$

$$45. \int_0^2 (x - 2\sqrt{x}) dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} \right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$(\sqrt{c} - 1)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}} \right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

$$47. \int_{-\pi/4}^{\pi/4} 2 \sec^2 x \, dx = \left[2 \tan x \right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$$

$$f(c) \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 4$$

$$2 \sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm \frac{2}{\sqrt{\pi}}$$

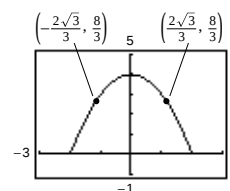
$$c = \pm \operatorname{arcsec} \left(\frac{2}{\sqrt{\pi}} \right)$$

$$= \pm \arccos \frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

$$49. \frac{1}{2 - (-2)} \int_{-2}^2 (4 - x^2) \, dx = \frac{1}{4} \left[4x - \frac{1}{3}x^3 \right]_{-2}^2 = \frac{1}{4} \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{8}{3}$$

$$\text{Average value} = \frac{8}{3}$$

$$4 - x^2 = \frac{8}{3} \text{ when } x^2 = 4 - \frac{8}{3} \text{ or } x = \pm \frac{2\sqrt{3}}{3} \approx \pm 1.155.$$

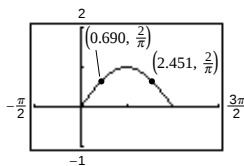


$$51. \frac{1}{\pi - 0} \int_0^{\pi} \sin x \, dx = \left[-\frac{1}{\pi} \cos x \right]_0^{\pi} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$



53. If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$,

$$\text{then } \int_a^b f(x) \, dx = F(b) - F(a).$$

$$55. \int_0^2 f(x) \, dx = -(\text{area of region A}) = -1.5$$

$$57. \int_0^6 |f(x)| \, dx = -\int_0^2 f(x) \, dx + \int_2^6 f(x) \, dx = 1.5 + 5.0 = 6.5$$

$$59. \int_0^6 [2 + f(x)] \, dx = \int_0^6 2 \, dx + \int_0^6 f(x) \, dx \\ = 12 + 3.5 = 15.5$$

$$61. (a) F(x) = k \sec^2 x$$

$$F(0) = k = 500$$

$$F(x) = 500 \sec^2 x$$

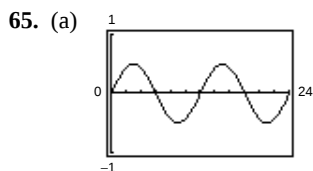
$$(b) \frac{1}{\pi/3 - 0} \int_0^{\pi/3} 500 \sec^2 x \, dx = \frac{1500}{\pi} \left[\tan x \right]_0^{\pi/3}$$

$$= \frac{1500}{\pi} (\sqrt{3} - 0)$$

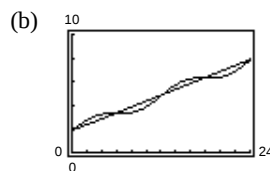
$$\approx 826.99 \text{ newtons}$$

$$\approx 827 \text{ newtons}$$

$$63. \frac{1}{5 - 0} \int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3) \, dt \approx \frac{1}{5} \left[0.08645t^2 + 0.05073t^3 - 0.00935t^4 \right]_0^5 \approx 0.5318 \text{ liter}$$

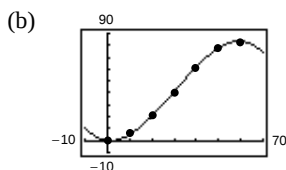


The area above the x -axis equals the area below the x -axis. Thus, the average value is zero.



The average value of S appears to be g .

67. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$



(c) $\int_0^{60} v(t) dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t \right]_0^{60} \approx 2476 \text{ meters}$

69. $F(x) = \int_0^x (t - 5) dt = \left[\frac{t^2}{2} - 5t \right]_0^x = \frac{x^2}{2} - 5x$

$$F(2) = \frac{4}{2} - 5(2) = -8$$

$$F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$$

$$F(8) = \frac{64}{2} - 5(8) = -8$$

71. $F(x) = \int_1^x \frac{10}{v^2} dv = \int_1^x 10v^{-2} dv = \left[\frac{-10}{v} \right]_1^x$

$$= -\frac{10}{x} + 10 = 10 \left(1 - \frac{1}{x} \right)$$

$$F(2) = 10 \left(\frac{1}{2} \right) = 5$$

$$F(5) = 10 \left(\frac{4}{5} \right) = 8$$

$$F(8) = 10 \left(\frac{7}{8} \right) = \frac{35}{4}$$

73. $F(x) = \int_1^x \cos \theta d\theta = \left[\sin \theta \right]_1^x = \sin x - \sin 1$

$$F(2) = \sin 2 - \sin 1 = 0.0678$$

$$F(5) = \sin 5 - \sin 1 \approx -1.8004$$

$$F(8) = \sin 8 - \sin 1 \approx 0.1479$$

75. (a) $\int_0^x (t + 2) dt = \left[\frac{t^2}{2} + 2t \right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\frac{d}{dx} \left[\frac{1}{2}x^2 + 2x \right] = x + 2$

77. (a) $\int_8^x \sqrt[3]{t} dt = \left[\frac{3}{4}t^{4/3} \right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\frac{d}{dx} \left[\frac{3}{4}x^{4/3} - 12 \right] = x^{1/3} = \sqrt[3]{x}$

79. (a) $\int_{x/4}^x \sec^2 t dt = \left[\tan t \right]_{x/4}^x = \tan x - 1$

(b) $\frac{d}{dx} [\tan x - 1] = \sec^2 x$

81. $F(x) = \int_{-2}^x (t^2 - 2t) dt$

$$F'(x) = x^2 - 2x$$

83. $F(x) = \int_{-1}^x \sqrt{t^4 + 1} dt$

$$F'(x) = \sqrt{x^4 + 1}$$

85. $F(x) = \int_0^x t \cos t dt$

$$F'(x) = x \cos x$$

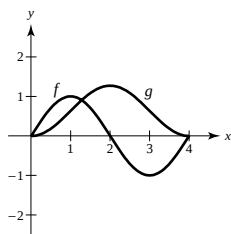
$$\begin{aligned}
 87. \quad F(x) &= \int_x^{x+2} (4t + 1) dt \\
 &= \left[2t^2 + t \right]_x^{x+2} \\
 &= [2(x+2)^2 + (x+2)] - [2x^2 + x] \\
 &= 8x + 10 \\
 F'(x) &= 8
 \end{aligned}$$

$$\begin{aligned}
 89. \quad F(x) &= \int_0^{\sin x} \sqrt{t} dt = \left[\frac{2}{3} t^{3/2} \right]_0^{\sin x} = \frac{2}{3} (\sin x)^{3/2} \\
 F'(x) &= (\sin x)^{1/2} \cos x = \cos x \sqrt{\sin x}
 \end{aligned}$$

Alternate solution

$$\begin{aligned}
 F(x) &= \int_0^{\sin x} \sqrt{t} dt \\
 F'(x) &= \sqrt{\sin x} \frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)
 \end{aligned}$$

$$\begin{aligned}
 93. \quad g(x) &= \int_0^x f(t) dt \\
 g(0) &= 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0
 \end{aligned}$$



g has a relative maximum at $x = 2$.

97. True

$$101. \quad f(x) = \int_0^{1/x} \frac{1}{t^2 + 1} dt + \int_0^x \frac{1}{t^2 + 1} dt$$

By the Second Fundamental Theorem of Calculus, we have

$$\begin{aligned}
 f'(x) &= \frac{1}{(1/x)^2 + 1} \left(-\frac{1}{x^2} \right) + \frac{1}{x^2 + 1} \\
 &= -\frac{1}{1 + x^2} + \frac{1}{x^2 + 1} = 0.
 \end{aligned}$$

Since $f'(x) = 0$, $f(x)$ must be constant.

Alternate solution:

$$\begin{aligned}
 F(x) &= \int_x^{x+2} (4t + 1) dt \\
 &= \int_x^0 (4t + 1) dt + \int_0^{x+2} (4t + 1) dt \\
 &= -\int_0^x (4t + 1) dt + \int_0^{x+2} (4t + 1) dt \\
 F'(x) &= -(4x + 1) + 4(x + 2) + 1 = 8
 \end{aligned}$$

$$\begin{aligned}
 91. \quad F(x) &= \int_0^{x^3} \sin t^2 dt \\
 F'(x) &= \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6
 \end{aligned}$$

$$\begin{aligned}
 95. \quad (a) \quad C(x) &= 5000 \left(25 + 3 \int_0^x t^{1/4} dt \right) \\
 &= 5000 \left(25 + 3 \left[\frac{4}{5} t^{5/4} \right]_0^x \right) \\
 &= 5000 \left(25 + \frac{12}{5} x^{5/4} \right) = 1000(125 + 12x^{5/4})
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad C(1) &= 1000(125 + 12(1)) = \$137,000 \\
 C(5) &= 1000(125 + 12(5)^{5/4}) \approx \$214,721 \\
 C(10) &= 1000(125 + 12(10)^{5/4}) \approx \$338,394
 \end{aligned}$$

$$99. \quad \text{False; } \int_{-1}^1 x^{-2} dx = \int_{-1}^0 x^{-2} dx + \int_0^1 x^{-2} dx$$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

103. $x(t) = t^3 - 6t^2 + 9t - 2$

$$x'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t - 3)(t - 1)$$

$$\begin{aligned} \text{Total distance} &= \int_0^5 |x'(t)| dt \\ &= \int_0^5 3|(t - 3)(t - 1)| dt \\ &= 3 \int_0^1 (t^2 - 4t + 3) dt - 3 \int_1^3 (t^2 - 4t + 3) dt + 3 \int_3^5 (t^2 - 4t + 3) dt \\ &= 4 + 4 + 20 \\ &= 28 \text{ units} \end{aligned}$$

105. Total distance $= \int_1^4 |x'(t)| dt$

$$\begin{aligned} &= \int_1^4 |v(t)| dt \\ &= \int_1^4 \frac{1}{\sqrt{t}} dt \\ &= 2t^{1/2} \Big|_1^4 \\ &= 2(2 - 1) = 2 \text{ units} \end{aligned}$$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x) dx$	$u = g(x)$	$du = g'(x) dx$
1. $\int (5x^2 + 1)^2(10x) dx$	$5x^2 + 1$	$10x dx$
3. $\int \frac{x}{\sqrt{x^2 + 1}} dx$	$x^2 + 1$	$2x dx$
5. $\int \tan^2 x \sec^2 x dx$	$\tan x$	$\sec^2 x dx$
7. $\int (1 + 2x)^4 2 dx = \frac{(1 + 2x)^5}{5} + C$		
Check: $\frac{d}{dx} \left[\frac{(1 + 2x)^5}{5} + C \right] = 2(1 + 2x)^4$		
9. $\int (9 - x^2)^{1/2}(-2x) dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$		
Check: $\frac{d}{dx} \left[\frac{2}{3}(9 - x^2)^{3/2} + C \right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$		

$$11. \int x^3(x^4 + 3)^2 dx = \frac{1}{4} \int (x^4 + 3)^2(4x^3) dx = \frac{1}{4} \frac{(x^4 + 3)^3}{3} + C = \frac{(x^4 + 3)^3}{12} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^4 + 3)^3}{12} + C \right] = \frac{3(x^4 + 3)^2}{12} (4x^3) = (x^4 + 3)^2(x^3)$$

$$13. \int x^2(x^3 - 1)^4 dx = \frac{1}{3} \int (x^3 - 1)^4(3x^2) dx = \frac{1}{3} \left[\frac{(x^3 - 1)^5}{5} \right] + C = \frac{(x^3 - 1)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 - 1)^5}{15} + C \right] = \frac{5(x^3 - 1)^4(3x^2)}{15} = x^2(x^3 - 1)^4$$

$$15. \int t\sqrt{t^2 + 2} dt = \frac{1}{2} \int (t^2 + 2)^{1/2}(2t) dt = \frac{1}{2} \frac{(t^2 + 2)^{3/2}}{3/2} + C = \frac{(t^2 + 2)^{3/2}}{3} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{(t^2 + 2)^{3/2}}{3} + C \right] = \frac{3/2(t^2 + 2)^{1/2}(2t)}{3} = (t^2 + 2)^{1/2}t$$

$$17. \int 5x(1 - x^2)^{1/3} dx = -\frac{5}{2} \int (1 - x^2)^{1/3}(-2x) dx = -\frac{5}{2} \cdot \frac{(1 - x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1 - x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{15}{8}(1 - x^2)^{4/3} + C \right] = -\frac{15}{8} \cdot \frac{4}{3}(1 - x^2)^{1/3}(-2x) = 5x(1 - x^2)^{1/3} = 5x\sqrt[3]{1 - x^2}$$

$$19. \int \frac{x}{(1 - x^2)^3} dx = -\frac{1}{2} \int (1 - x^2)^{-3}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{-2}}{-2} + C = \frac{1}{4(1 - x^2)^2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{4(1 - x^2)^2} + C \right] = \frac{1}{4}(-2)(1 - x^2)^{-3}(-2x) = \frac{x}{(1 - x^2)^3}$$

$$21. \int \frac{x^2}{(1 + x^3)^2} dx = \frac{1}{3} \int (1 + x^3)^{-2}(3x^2) dx = \frac{1}{3} \left[\frac{(1 + x^3)^{-1}}{-1} \right] + C = -\frac{1}{3(1 + x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{3(1 + x^3)} + C \right] = -\frac{1}{3}(-1)(1 + x^3)^{-2}(3x^2) = \frac{x^2}{(1 + x^3)^2}$$

$$23. \int \frac{x}{\sqrt{1 - x^2}} dx = -\frac{1}{2} \int (1 - x^2)^{-1/2}(-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{1/2}}{1/2} + C = -\sqrt{1 - x^2} + C$$

$$\text{Check: } \frac{d}{dx} [-\sqrt{1 - x^2} + C] = -\frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1 - x^2}}$$

$$25. \int \left(1 + \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt = - \int \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) dt = -\frac{[1 + (1/t)]^4}{4} + C$$

$$\text{Check: } \frac{d}{dt} \left[-\frac{[1 + (1/t)]^4}{4} + C \right] = -\frac{1}{4}(4)\left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) = \frac{1}{t^2} \left(1 + \frac{1}{t}\right)^3$$

$$27. \int \frac{1}{\sqrt{2x}} dx = \frac{1}{2} \int (2x)^{-1/2} 2 dx = \frac{1}{2} \left[\frac{(2x)^{1/2}}{1/2} \right] + C = \sqrt{2x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{2x} + C] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$$

$$29. \int \frac{x^2 + 3x + 7}{\sqrt{x}} dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2}) dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$$

Check: $\frac{d}{dx} \left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C \right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$

$$31. \int t^2 \left(t - \frac{2}{t} \right) dt = \int (t^3 - 2t) dt = \frac{1}{4}t^4 - t^2 + C$$

Check: $\frac{d}{dt} \left[\frac{1}{4}t^4 - t^2 + C \right] = t^3 - 2t = t^2 \left(t - \frac{2}{t} \right)$

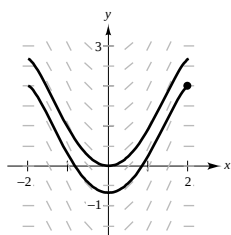
$$33. \int (9 - y)\sqrt{y} dy = \int (9y^{1/2} - y^{3/2}) dy = 9 \left(\frac{2}{3}y^{3/2} \right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$$

Check: $\frac{d}{dy} \left[\frac{2}{5}y^{3/2}(15 - y) + C \right] = \frac{d}{dy} \left[6y^{3/2} - \frac{2}{5}y^{5/2} + C \right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$

$$\begin{aligned} 35. y &= \int \left[4x + \frac{4x}{\sqrt{16 - x^2}} \right] dx \\ &= 4 \int x dx - 2 \int (16 - x^2)^{-1/2} (-2x) dx \\ &= 4 \left(\frac{x^2}{2} \right) - 2 \left[\frac{(16 - x^2)^{1/2}}{1/2} \right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

$$\begin{aligned} 37. y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2} dx \\ &= \frac{1}{2} \int (x^2 + 2x - 3)^{-2} (2x + 2) dx \\ &= \frac{1}{2} \left[\frac{(x^2 + 2x - 3)^{-1}}{-1} \right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

39. (a)

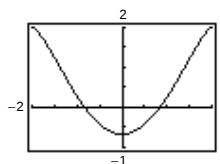


(b) $\frac{dy}{dx} = x\sqrt{4 - x^2}, (2, 2)$

$$\begin{aligned} y &= \int x\sqrt{4 - x^2} dx = -\frac{1}{2} \int (4 - x^2)^{1/2} (-2x dx) \\ &= -\frac{1}{2} \cdot \frac{2}{3} (4 - x^2)^{3/2} + C = -\frac{1}{3} (4 - x^2)^{3/2} + C \end{aligned}$$

$$(2, 2): 2 = -\frac{1}{3} (4 - 2^2)^{3/2} + C \Rightarrow C = 2$$

$$y = -\frac{1}{3} (4 - x^2)^{3/2} + 2$$



$$41. \int \pi \sin \pi x dx = -\cos \pi x + C$$

$$43. \int \sin 2x dx = \frac{1}{2} \int (\sin 2x)(2x) dx = -\frac{1}{2} \cos 2x + C$$

$$45. \int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta = -\int \cos \frac{1}{\theta} \left(-\frac{1}{\theta^2} \right) d\theta = -\sin \frac{1}{\theta} + C$$

$$47. \int \sin 2x \cos 2x \, dx = \frac{1}{2} \int (\sin 2x)(2 \cos 2x) \, dx = \frac{1}{2} \frac{(\sin 2x)^2}{2} + C = \frac{1}{4} \sin^2 2x + C \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = -\frac{1}{2} \int (\cos 2x)(-2 \sin 2x) \, dx = -\frac{1}{2} \frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4} \cos^2 2x + C_1 \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = \frac{1}{2} \int 2 \sin 2x \cos 2x \, dx = \frac{1}{2} \int \sin 4x \, dx = -\frac{1}{8} \cos 4x + C_2$$

$$49. \int \tan^4 x \sec^2 x \, dx = \frac{\tan^5 x}{5} + C = \frac{1}{5} \tan^5 x + C$$

$$\begin{aligned} 51. \int \frac{\csc^2 x}{\cot^3 x} \, dx &= - \int (\cot x)^{-3} (-\csc^2 x) \, dx \\ &= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2 \cot^2 x} + C = \frac{1}{2} \tan^2 x + C = \frac{1}{2} (\sec^2 x - 1) + C = \frac{1}{2} \sec^2 x + C_1 \end{aligned}$$

$$53. \int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx = -\cot x - x + C$$

$$55. f(x) = \int \cos \frac{x}{2} \, dx = 2 \sin \frac{x}{2} + C$$

Since $f(0) = 3 = 2 \sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2 \sin \frac{x}{2} + 3.$$

$$57. u = x + 2, x = u - 2, dx = du$$

$$\begin{aligned} \int x \sqrt{x+2} \, dx &= \int (u-2) \sqrt{u} \, du \\ &= \int (u^{3/2} - 2u^{1/2}) \, du \\ &= \frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} + C \\ &= \frac{2u^{3/2}}{15} (3u - 10) + C \\ &= \frac{2}{15} (x+2)^{3/2} [3(x+2) - 10] + C \\ &= \frac{2}{15} (x+2)^{3/2} (3x-4) + C \end{aligned}$$

$$59. u = 1 - x, x = 1 - u, dx = -du$$

$$\begin{aligned} \int x^2 \sqrt{1-x} \, dx &= - \int (1-u)^2 \sqrt{u} \, du \\ &= - \int (u^{1/2} - 2u^{3/2} + u^{5/2}) \, du \\ &= - \left(\frac{2}{3} u^{3/2} - \frac{4}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right) + C \\ &= -\frac{2u^{3/2}}{105} (35 - 42u + 15u^2) + C \\ &= -\frac{2}{105} (1-x)^{3/2} [35 - 42(1-x) + 15(1-x)^2] + C \\ &= -\frac{2}{105} (1-x)^{3/2} (15x^2 + 12x + 8) + C \end{aligned}$$

61. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2} du$

$$\begin{aligned}
 \int \frac{x^2 - 1}{\sqrt{2x - 1}} dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}} \frac{1}{2} du \\
 &= \frac{1}{8} \int u^{-1/2} [(u^2 + 2u + 1) - 4] du \\
 &= \frac{1}{8} \int (u^{3/2} + 2u^{1/2} - 3u^{-1/2}) du \\
 &= \frac{1}{8} \left(\frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} - 6u^{1/2} \right) + C \\
 &= \frac{u^{1/2}}{60} (3u^2 + 10u - 45) + C \\
 &= \frac{\sqrt{2x - 1}}{60} [3(2x - 1)^2 + 10(2x - 1) - 45] + C \\
 &= \frac{1}{60} \sqrt{2x - 1} (12x^2 + 8x - 52) + C \\
 &= \frac{1}{15} \sqrt{2x - 1} (3x^2 + 2x - 13) + C
 \end{aligned}$$

63. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned}
 \int \frac{-x}{(x + 1) - \sqrt{x + 1}} dx &= \int \frac{-(u - 1)}{u - \sqrt{u}} du \\
 &= - \int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)} du \\
 &= - \int (1 + u^{-1/2}) du \\
 &= -(u + 2u^{1/2}) + C \\
 &= -u - 2\sqrt{u} + C \\
 &= -(x + 1) - 2\sqrt{x + 1} + C \\
 &= -x - 2\sqrt{x + 1} - 1 + C \\
 &= -(x + 2\sqrt{x + 1}) + C_1
 \end{aligned}$$

where $C_1 = -1 + C$.

65. Let $u = x^2 + 1, du = 2x dx$.

$$\int_{-1}^1 x(x^2 + 1)^3 dx = \frac{1}{2} \int_{-1}^1 (x^2 + 1)^3 (2x) dx = \left[\frac{1}{8} (x^2 + 1)^4 \right]_{-1}^1 = 0$$

67. Let $u = x^3 + 1, du = 3x^2 dx$

$$\begin{aligned}
 \int_1^2 2x^2 \sqrt{x^3 + 1} dx &= 2 \cdot \frac{1}{3} \int_1^2 (x^3 + 1)^{1/2} (3x^2) dx \\
 &= \left[\frac{2}{3} \frac{(x^3 + 1)^{3/2}}{3/2} \right]_1^2 \\
 &= \frac{4}{9} \left[(x^3 + 1)^{3/2} \right]_1^2 \\
 &= \frac{4}{9} [27 - 2\sqrt{2}] = 12 - \frac{8}{9}\sqrt{2}
 \end{aligned}$$

69. Let
- $u = 2x + 1$
- ,
- $du = 2 dx$
- .

$$\int_0^4 \frac{1}{\sqrt{2x+1}} dx = \frac{1}{2} \int_0^4 (2x+1)^{-1/2} (2) dx = \left[\sqrt{2x+1} \right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

71. Let
- $u = 1 + \sqrt{x}$
- ,
- $du = \frac{1}{2\sqrt{x}} dx$
- .

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx = 2 \int_1^9 (1+\sqrt{x})^{-2} \left(\frac{1}{2\sqrt{x}} \right) dx = \left[-\frac{2}{1+\sqrt{x}} \right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

- 73.
- $u = 2 - x$
- ,
- $x = 2 - u$
- ,
- $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x-1)\sqrt{2-x} dx = \int_1^0 -[(2-u)-1]\sqrt{u} du = \int_1^0 (u^{3/2} - u^{1/2}) du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3} \right] = \frac{4}{15}$$

- 75.
- $\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right) dx = \left[\frac{3}{2} \sin\left(\frac{2}{3}x\right) \right]_0^{\pi/2} = \frac{3}{2} \left(\frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}}{4}$

- 77.
- $u = x + 1$
- ,
- $x = u - 1$
- ,
- $dx = du$

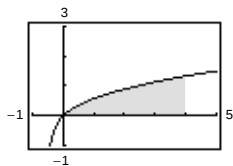
When $x = 0$, $u = 1$. When $x = 7$, $u = 8$.

$$\begin{aligned} \text{Area} &= \int_0^7 x \sqrt[3]{x+1} dx = \int_1^8 (u-1) \sqrt[3]{u} du \\ &= \int_1^8 (u^{4/3} - u^{1/3}) du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} \right]_1^8 = \left(\frac{384}{7} - 12 \right) - \left(\frac{3}{7} - \frac{3}{4} \right) = \frac{1209}{28} \end{aligned}$$

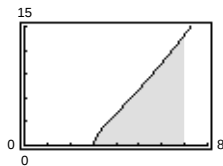
- 79.
- $A = \int_0^\pi (2 \sin x + \sin 2x) dx = -\left[2 \cos x + \frac{1}{2} \cos 2x \right]_0^\pi = 4$

- 81.
- $\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx = 2 \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = \left[2 \tan\left(\frac{x}{2}\right) \right]_{\pi/2}^{2\pi/3} = 2(\sqrt{3} - 1)$

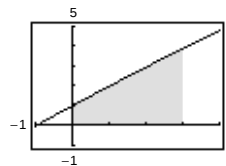
- 83.
- $\int_0^4 \frac{x}{\sqrt{2x+1}} dx \approx 3.333 = \frac{10}{3}$



- 85.
- $\int_3^7 x\sqrt{x-3} dx \approx 28.8 = \frac{144}{5}$



- 87.
- $\int_0^3 \left(\theta + \cos \frac{\theta}{6} \right) d\theta \approx 7.377$



- 89.
- $\int (2x-1)^2 dx = \frac{1}{2} \int (2x-1)^2 (2) dx = \frac{1}{6} (2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$$\int (2x-1)^2 dx = \int (4x^2 - 4x + 1) dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$$

They differ by a constant: $C_2 = C_1 - \frac{1}{6}$.

91. $f(x) = x^2(x^2 + 1)$ is even.

$$\begin{aligned}\int_{-2}^2 x^2(x^2 + 1) dx &= 2 \int_0^2 (x^4 + x^2) dx = 2 \left[\frac{x^5}{5} + \frac{x^3}{3} \right]_0^2 \\ &= 2 \left[\frac{32}{5} + \frac{8}{3} \right] = \frac{272}{15}\end{aligned}$$

93. $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^2 x(x^2 + 1)^3 dx = 0$$

95. $\int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$; the function x^2 is an even function.

$$(a) \int_{-2}^0 x^2 dx = \int_0^2 x^2 dx = \frac{8}{3}$$

$$(c) \int_0^2 (-x^2) dx = - \int_0^2 x^2 dx = -\frac{8}{3}$$

$$(b) \int_{-2}^2 x^2 dx = 2 \int_0^2 x^2 dx = \frac{16}{3}$$

$$(d) \int_{-2}^0 3x^2 dx = 3 \int_0^2 x^2 dx = 8$$

$$97. \int_{-4}^4 (x^3 + 6x^2 - 2x - 3) dx = \int_{-4}^4 (x^3 - 2x) dx + \int_{-4}^4 (6x^2 - 3) dx = 0 + 2 \int_0^4 (6x^2 - 3) dx = 2 \left[2x^3 - 3x \right]_0^4 = 232$$

99. Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

101. $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\int_{-2}^2 x(x^2 + 1)^2 dx = 0$.

$$103. \frac{dV}{dt} = \frac{k}{(t+1)^2}$$

$$V(t) = \int \frac{k}{(t+1)^2} dt = -\frac{k}{t+1} + C$$

$$V(0) = -k + C = 500,000$$

$$V(1) = -\frac{1}{2}k + C = 400,000$$

Solving this system yields $k = -200,000$ and $C = 300,000$. Thus,

$$V(t) = \frac{200,000}{t+1} + 300,000.$$

When $t = 4$, $V(4) = \$340,000$.

$$105. \frac{1}{b-a} \int_a^b \left[74.50 + 43.75 \sin \frac{\pi t}{6} \right] dt = \frac{1}{b-a} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_a^b$$

$$(a) \frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^3 = \frac{1}{3} \left(223.5 + \frac{262.5}{\pi} \right) \approx 102.352 \text{ thousand units}$$

$$(b) \frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_3^6 = \frac{1}{3} \left(447 + \frac{262.5}{\pi} - 223.5 \right) \approx 102.352 \text{ thousand units}$$

$$(c) \frac{1}{12} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^{12} = \frac{1}{12} \left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi} \right) = 74.5 \text{ thousand units}$$

107. $\frac{1}{b-a} \int_a^b [2 \sin(60\pi t) + \cos(120\pi t)] dt = \frac{1}{b-a} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_a^b$

$$(a) \frac{1}{(1/60) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/60} = 60 \left[\left(\frac{1}{30\pi} + 0 \right) - \left(-\frac{1}{30\pi} \right) \right] = \frac{4}{\pi} \approx 1.273 \text{ amps}$$

$$\begin{aligned} \text{(b)} \quad \frac{1}{(1/240) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/240} &= 240 \left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi} \right) - \left(-\frac{1}{30\pi} \right) \right] \\ &= \frac{2}{\pi} (5 - 2\sqrt{2}) \approx 1.382 \text{ amps} \end{aligned}$$

(c) $\frac{1}{(1/30) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/30} = 30 \left[\left(\frac{1}{30\pi} \right) - \left(-\frac{1}{30\pi} \right) \right] = 0 \text{ amps}$

109. False

$$\int (2x + 1)^2 dx = \frac{1}{2} \int (2x + 1)^2 2 dx = \frac{1}{6} (2x + 1)^3 + C$$

111. True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d) dx = \underbrace{\int_{-10}^{10} (ax^3 + cx) dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10} (bx^2 + d) dx}_{\text{Even}} = 0 + 2 \int_0^{10} (bx^2 + d) dx$$

113. True

$$4 \int \sin x \cos x \, dx = 2 \int \sin 2x \, dx = -\cos 2x + C$$

115. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x+h) dx = \int_{a+h}^{b+h} f(u) du = \int_{a+h}^{b+h} f(x) dx.$$

Section 4.6 Numerical Integration

1. Exact: $\int_0^2 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2 dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2 dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2 \right] = \frac{8}{3} \approx 2.6667$

3. Exact: $\int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4.000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{4} \left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{6} \left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3 \right] = \frac{24}{6} = 4.0000$

5. Exact: $\int_0^2 x^3 dx = \left[\frac{1}{4}x^4 \right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{8} \left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0625$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{12} \left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0000$

7. Exact: $\int_4^9 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x} dx \approx \frac{5}{16} \left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3 \right]$
 ≈ 12.6640

Simpson's: $\int_4^9 \sqrt{x} dx \approx \frac{5}{24} \left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3 \right] \approx 12.6667$

9. Exact: $\int_1^2 \frac{1}{(x+1)^2} dx = \left[-\frac{1}{x+1} \right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{8} \left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$
 $= \frac{1}{8} \left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9} \right) \approx 0.1676$

Simpson's: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{12} \left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$
 $= \frac{1}{12} \left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9} \right) \approx 0.1667$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{4} [1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{6} [1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

13. $\int_0^1 \sqrt{x}\sqrt{1-x} dx = \int_0^1 \sqrt{x(1-x)} dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{8} \left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{12} \left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.372$

Graphing utility: 0.393

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{8} \left[\cos 0 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$
 ≈ 0.957

Simpson's: $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{12} \left[\cos 0 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$
 ≈ 0.978

Graphing utility: 0.977

17. Trapezoidal: $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{80} [\sin(1) + 2 \sin(1.025)^2 + 2 \sin(1.05)^2 + 2 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

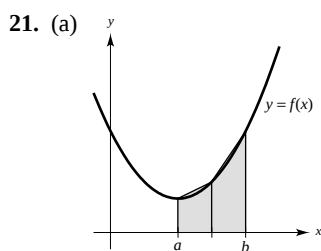
Simpson's: $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{120} [\sin(1) + 4 \sin(1.025)^2 + 2 \sin(1.05)^2 + 4 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Graphing utility: 0.089

19. Trapezoidal: $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{32} \left[0 + 2\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{48} \left[0 + 4\left(\frac{\pi}{16}\right) \tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right) \tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.186$

Graphing utility: 0.186



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

23. $f(x) = x^3$
 $f'(x) = 3x^2$
 $f''(x) = 6x$
 $f'''(x) = 6$
 $f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\leq \frac{(2-0)^3}{12(4^2)}(12) = 0.5$ since

$f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\leq \frac{(2-0)^5}{180(4^4)}(0) = 0$ since

$f^{(4)}(x) = 0$.

25. $f''(x) = \frac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\leq \frac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133,333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \frac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and when $|f^{(4)}(1)| = 24$.

Simpson's: Error $\leq \frac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426,666.67$, $n > 25.56$; let $n = 26$.

27. $f(x) = \sqrt{1+x}$

(a) $f''(x) = -\frac{1}{4(1+x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{1}{4}$.

Trapezoidal: Error $\leq \frac{8}{12n^2} \left(\frac{1}{4} \right) < 0.00001$, $n^2 > 16,666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \frac{-15}{16(1+x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{15}{16}$.

Simpson's: Error $\leq \frac{32}{180n^4} \left(\frac{15}{16} \right) < 0.00001$, $n^4 > 16,666.67$, $n > 11.36$; let $n = 12$.

29. $f(x) = \tan(x^2)$

(a) $f''(x) = 2 \sec^2(x^2)[1 + 4x^2 \tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2} (49.5305) < 0.00001$, $n^2 > 412,754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8 \sec^2(x^2)[12x^2 + (3 + 32x^4) \tan(x^2) + 36x^2 \tan^2(x^2) + 48x^4 \tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4} (9184.4734) < 0.00001$, $n^4 > 5,102,485.22$, $n > 47.53$; let $n = 48$.

31. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \frac{(b-a)^5}{180n^4} (0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3 dx = \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right)^3 + 1 \right] = \frac{1}{6}$

This is the exact value of the integral.

33. $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

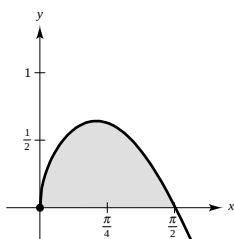
35. $f(x) = \sin \sqrt{x}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

37. $A = \int_0^{\pi/2} \sqrt{x} \cos x \, dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x} \cos x \, dx \approx \frac{\pi}{84} \left[\sqrt{0} \cos 0 + 4 \sqrt{\frac{\pi}{28}} \cos \frac{\pi}{28} + 2 \sqrt{\frac{\pi}{14}} \cos \frac{\pi}{14} + 4 \sqrt{\frac{3\pi}{28}} \cos \frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}} \cos \frac{\pi}{2} \right] \\ \approx 0.701$$



39. $W = \int_0^5 100x \sqrt{125 - x^3} \, dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x \sqrt{125 - x^3} \, dx \approx \frac{5}{3(12)} \left[0 + 400 \left(\frac{5}{12} \right) \sqrt{125 - \left(\frac{5}{12} \right)^3} + 200 \left(\frac{10}{12} \right) \sqrt{125 - \left(\frac{10}{12} \right)^3} \right. \\ \left. + 400 \left(\frac{15}{12} \right) \sqrt{125 - \left(\frac{15}{12} \right)^3} + \cdots + 0 \right] \approx 10,233.58 \text{ ft} \cdot \text{lb}$$

41. $\int_0^{1/2} \frac{6}{\sqrt{1-x^2}} \, dx$ Simpson's Rule, $n = 6$

$$\pi \approx \frac{\left(\frac{1}{2} - 0 \right)}{3(6)} [6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282] \\ \approx \frac{1}{36} [113.098] \approx 3.1416$$

43. Area $\approx \frac{1000}{2(10)} [125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89,250 \text{ sq m}$

45. $\int_0^t \sin \sqrt{x} \, dx = 2, n = 10$

By trial and error, we obtain $t \approx 2.477$.

CHAPTER 4

Integration

Section 4.1	Antiderivatives and Indefinite Integration	450
Section 4.2	Area	456
Section 4.3	Riemann Sums and Definite Integrals	462
Section 4.4	The Fundamental Theorem of Calculus	466
Section 4.5	Integration by Substitution	472
Section 4.6	Numerical Integration	479
Review Exercises		483
Problem Solving		488

CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Even-Numbered Exercises

$$2. \frac{d}{dx} \left(x^4 + \frac{1}{x} + C \right) = 4x^3 - \frac{1}{x^2}$$

$$4. \frac{d}{dx} \left(\frac{2(x^2 + 3)}{3\sqrt{x}} + C \right) = \frac{d}{dx} \left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C \right) \\ = x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$$

$$6. \frac{dr}{d\theta} = \pi$$

$$r = \pi\theta + C$$

$$\text{Check: } \frac{d}{d\theta} [\pi\theta + C] = \pi$$

$$8. \frac{dy}{dx} = 2x^{-3}$$

$$y = \frac{2x^{-2}}{-2} + C = \frac{-1}{x^2} + c$$

$$\text{Check: } \frac{d}{dx} \left[\frac{-1}{x^2} + C \right] = 2x^{-3}$$

Given

Rewrite

Integrate

Simplify

$$10. \int \frac{1}{x^2} dx$$

$$\int x^{-2} dx$$

$$\frac{x^{-1}}{-1} + C$$

$$-\frac{1}{x} + C$$

$$12. \int x(x^2 + 3) dx$$

$$\int (x^3 + 3x) dx$$

$$\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$$

$$\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$$

$$14. \int \frac{1}{(3x^2)} dx$$

$$\frac{1}{9} \int x^{-2} dx$$

$$\frac{1}{9} \left(\frac{x^{-1}}{-1} \right) + C$$

$$\frac{-1}{9x} + C$$

$$16. \int (5 - x) dx = 5x - \frac{x^2}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[5x - \frac{x^2}{2} + C \right] = 5 - x$$

$$18. \int (4x^3 + 6x^2 - 1) dx = x^4 + 2x^3 - x + C$$

$$\text{Check: } \frac{d}{dx} [x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$$

$$20. \int (x^3 - 4x + 2) dx = \frac{x^4}{4} - 2x^2 + 2x + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{x^4}{4} - 2x^2 + 2x + C \right] = x^3 - 4x + 2$$

$$22. \int \left(\sqrt{x} + \frac{1}{2\sqrt{x}} \right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2} \right) dx = \frac{x^{3/2}}{3/2} + \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C = \frac{2}{3}x^{3/2} + x^{1/2} + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{2}{3}x^{3/2} + x^{1/2} + C \right) = x^{1/2} + \frac{1}{2}x^{-1/2} = \sqrt{x} + \frac{1}{2\sqrt{x}}$$

$$24. \int (\sqrt[4]{x^3} + 1) dx = \int (x^{3/4} + 1) dx = \frac{4}{7}x^{7/4} + x + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{4}{7}x^{7/4} + x + C \right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$$

$$26. \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left(-\frac{1}{3x^3} + C \right) = \frac{1}{x^4}$$

$$28. \int \frac{x^2 + 2x - 3}{x^4} dx = \int (x^{-2} + 2x^{-3} - 3x^{-4}) dx$$

$$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C \right] = x^{-2} + 2x^{-3} - 3x^{-4} \\ = \frac{x^2 + 2x - 3}{x^4}$$

$$30. \int (2t^2 - 1)^2 dt = \int (4t^4 - 4t^2 + 1) dt$$

$$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C \right) = 4t^4 - 4t^2 + 1 \\ = (2t^2 - 1)^2$$

$$32. \int (1 + 3t)t^2 dt = \int (t^2 + 3t^3) dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C \right) = t^2 + 3t^3 = (1 + 3t)t^2$$

$$34. \int 3 dt = 3t + C$$

$$\text{Check: } \frac{d}{dt}(3t + C) = 3$$

$$36. \int (t^2 - \sin t) dt = \frac{1}{3}t^3 + \cos t + C$$

$$\text{Check: } \frac{d}{dt} \left(\frac{1}{3}t^3 + \cos t + C \right) = t^2 - \sin t$$

$$38. \int (\theta^2 + \sec^2 \theta) d\theta = \frac{1}{3}\theta^3 + \tan \theta + C$$

$$\text{Check: } \frac{d}{d\theta} \left(\frac{1}{3}\theta^3 + \tan \theta + C \right) = \theta^2 + \sec^2 \theta$$

$$40. \int \sec y (\tan y - \sec y) dy = \int (\sec y \tan y - \sec^2 y) dy$$

$$= \sec y - \tan y + C$$

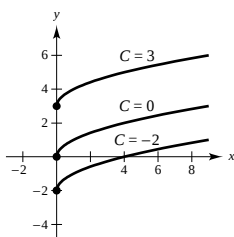
$$\text{Check: } \frac{d}{dy} (\sec y - \tan y + C) = \sec y \tan y - \sec^2 y \\ = \sec y (\tan y - \sec y)$$

$$42. \int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = \int \left(\frac{1}{\sin x} \right) \left(\frac{\cos x}{\sin x} \right) dx$$

$$= \int \csc x \cot x dx = -\csc x + C$$

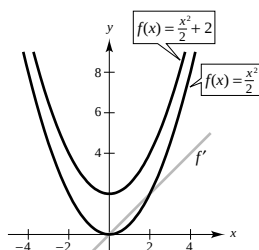
$$\text{Check: } \frac{d}{dx} [-\csc x + C] = \csc x \cot x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ = \frac{\cos x}{1 - \cos^2 x}$$

$$44. f(x) = \sqrt{x}$$



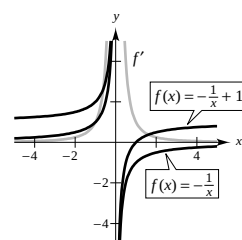
$$46. f'(x) = x$$

$$f(x) = \frac{x^2}{2} + C$$



$$48. f'(x) = \frac{1}{x^2}$$

$$f(x) = -\frac{1}{x} + C$$



50. $\frac{dy}{dx} = 2(x - 1) = 2x - 2, (3, 2)$

$$y = \int 2(x - 1) dx = x^2 - 2x + C$$

$$2 = (3)^2 - 2(3) + C \Rightarrow C = -1$$

$$y = x^2 - 2x - 1$$

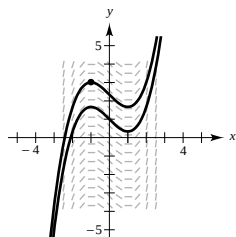
52. $\frac{dy}{dx} = -\frac{1}{x^2} = -x^{-2}, (1, 3)$

$$y = \int -x^{-2} dx = \frac{1}{x} + C$$

$$3 = \frac{1}{1} + C \Rightarrow C = 2$$

$$y = \frac{1}{x} + 2, x > 0$$

54. (a)



(b) $\frac{dy}{dx} = x^2 - 1, (-1, 3)$

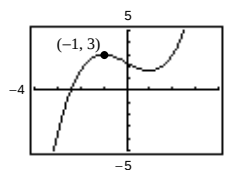
$$y = \frac{x^3}{3} - x + C$$

$$3 = \frac{(-1)^3}{3} - (-1) + C$$

$$3 = -\frac{1}{3} + 1 + C$$

$$C = \frac{7}{3}$$

$$y = \frac{x^3}{3} - x + \frac{7}{3}$$



56. $g'(x) = 6x^2, g(0) = -1$

$$g(x) = \int 6x^2 dx = 2x^3 + C$$

$$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1$$

$$g(x) = 2x^3 - 1$$

58. $f'(s) = 6s - 8s^3, f(2) = 3$

$$f(s) = \int (6s - 8s^3) ds = 3s^2 - 2s^4 + C$$

$$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \Rightarrow C = 23$$

$$f(s) = 3s^2 - 2s^4 + 23$$

60. $f''(x) = x^2$

$$f'(0) = 6$$

$$f(0) = 3$$

$$f'(x) = \int x^2 dx = \frac{1}{3}x^3 + C_1$$

$$f'(0) = 0 + C_1 = 6 \Rightarrow C_1 = 6$$

$$f'(x) = \frac{1}{3}x^3 + 6$$

$$f(x) = \int \left(\frac{1}{3}x^3 + 6 \right) dx = \frac{1}{12}x^4 + 6x + C_2$$

$$f(0) = 0 + 0 + C_2 = 3 \Rightarrow C_2 = 3$$

$$f(x) = \frac{1}{12}x^4 + 6x + 3$$

62. $f''(x) = \sin x$

$$f'(0) = 1$$

$$f(0) = 6$$

$$f'(x) = \int \sin x dx = -\cos x + C_1$$

$$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$$

$$f'(x) = -\cos x + 2$$

$$f(x) = \int (-\cos x + 2) dx = -\sin x + 2x + C_2$$

$$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$$

$$f(x) = -\sin x + 2x + 6$$

$$64. \frac{dP}{dt} = k\sqrt{t}, \quad 0 \leq t \leq 10$$

$$P(t) = \int kt^{1/2} dt = \frac{2}{3}kt^{3/2} + C$$

$$P(0) = 0 + C = 500 \Rightarrow C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

$$68. f''(t) = a(t) = -32 \text{ ft/sec}^2$$

$$f'(0) = v_0$$

$$f(0) = s_0$$

$$f'(t) = v(t) = \int -32 dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \Rightarrow C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0) dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \Rightarrow C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

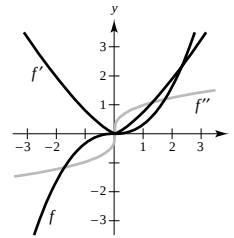
$$72. \text{ From Exercise 71, } f(t) = -4.9t^2 + 1600. \text{ (Using the canyon floor as position 0.)}$$

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \Rightarrow t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



$$70. v_0 = 16 \text{ ft/sec}$$

$$s_0 = 64 \text{ ft}$$

$$(a) \quad s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

$$(b) \quad v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

$$74. \text{ From Exercise 71, } f(t) = -4.9t^2 + v_0t + 2. \text{ If}$$

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9 v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9 v_0^2 + 9.8 v_0^2 = (9.8)^2 198$$

$$4.9 v_0^2 = (9.8)^2 198$$

$$v_0^2 = 3880.8 \Rightarrow v_0 \approx 62.3 \text{ m/sec.}$$

$$76. \int v \, dv = -GM \int \frac{1}{y^2} \, dy$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

$$80. (a) a(t) = \cos t$$

$$v(t) = \int a(t) \, dt = \int \cos t \, dt = \sin t + C_1 = \sin t \quad (\text{since } v_0 = 0)$$

$$f(t) = \int v(t) \, dt = \int \sin t \, dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$$

$$f(t) = -\cos t + 4$$

$$(b) v(t) = 0 = \sin t \text{ for } t = k\pi, k = 0, 1, 2, \dots$$

$$82. \quad v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$$

$$30 \text{ mph} = 44 \text{ ft/sec}$$

$$15 \text{ mph} = 22 \text{ ft/sec}$$

$$a(t) = -a$$

$$v(t) = -at + 66$$

$$s(t) = -\frac{a}{2}t^2 + 66t \quad (\text{Let } s(0) = 0.)$$

$$v(t) = 0 \text{ after car moves 132 ft.}$$

$$-at + 66 = 0 \text{ when } t = \frac{66}{a}.$$

$$\begin{aligned} s\left(\frac{66}{a}\right) &= -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right) \\ &= 132 \text{ when } a = \frac{33}{2} = 16.5. \end{aligned}$$

$$a(t) = -16.5$$

$$v(t) = -16.5t + 66$$

$$s(t) = -8.25t^2 + 66t$$

$$78. x(t) = (t-1)(t-3)^2 \quad 0 \leq t \leq 5$$

$$= t^3 - 7t^2 + 15t - 9$$

$$(a) v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$$

$$a(t) = v'(t) = 6t - 14$$

$$(b) v(t) > 0 \text{ when } 0 < t < \frac{5}{3} \text{ and } 3 < t < 5.$$

$$(c) a(t) = 6t - 14 = 0 \text{ when } t = \frac{7}{3}.$$

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

$$(a) -16.5t + 66 = 44$$

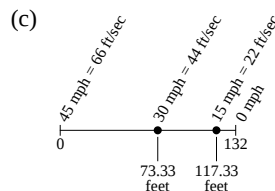
$$t = \frac{22}{16.5} \approx 1.333$$

$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

$$(b) -16.5t + 66 = 22$$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$



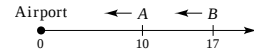
It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

84. No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then $\int_0^{30} |v_2(t)| dt > \int_0^{30} |v_1(t)| dt$.

86. (a) $v = 0.6139t^3 - 5.525t^2 + 0.0492t + 65.9881$ (b) $s(t) = \int v(t)dt = \frac{0.6139t^4}{4} - \frac{5.525t^3}{3} + \frac{0.0492t^2}{2} + 65.9881t$
 (Note: Assume $s(0) = 0$ is initial position)
 $s(6) \approx 196.1$ feet

88. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.

$$\begin{aligned} v_A(t) &= k_A t - 150 & v_B &= k_B t - 250 \\ s_A(t) &= \frac{1}{2}k_A t^2 - 150t + 10 & s_B &= \frac{1}{2}k_B t^2 - 250t + 17 \end{aligned}$$



(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \Rightarrow k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \Rightarrow S_A(t) = S_1(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \Rightarrow k_B = \frac{135}{t_B}$$

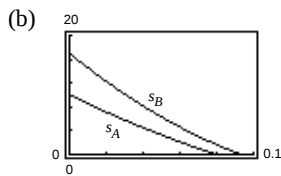
$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

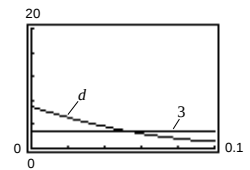
$$t_B = \frac{34}{365}.$$

$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49,275}{34} \Rightarrow S_B(t) = S_2(t) = \frac{49,275}{68}t^2 - 250t + 17$$



(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.



90. True

92. True

94. False. f has an infinite number of antiderivatives, each differing by a constant.

$$\begin{aligned} 96. \quad \frac{d}{dx} [s(x)]^2 + [c(x)]^2 &= 2s(x)s'(x) + 2c(x)c'(x) \\ &= 2s(x)c(x) - 2c(x)s(x) \\ &= 0 \end{aligned}$$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k . Since,

$$s(0) = 0 \text{ and } c(0) = 1, k = 1.$$

Therefore,

$$[s(x)]^2 + [c(x)]^2 = 1.$$

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

Section 4.2 Area

$$2. \quad \sum_{k=3}^6 k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$$

$$4. \quad \sum_{j=3}^5 \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

$$6. \quad \sum_{i=1}^4 [(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$$

$$8. \quad \sum_{i=1}^{15} \frac{5}{1+i}$$

$$10. \quad \sum_{j=1}^4 \left[1 - \left(\frac{j}{4} \right)^2 \right]$$

$$12. \quad \frac{2}{n} \sum_{i=1}^n \left[1 - \left(\frac{2i}{n} - 1 \right)^2 \right]$$

$$14. \quad \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{1 - \left(\frac{i}{n} \right)^2}$$

$$\begin{aligned} 16. \quad \sum_{i=1}^{15} (2i-3) &= 2 \sum_{i=1}^{15} i - 3(15) \\ &= 2 \left[\frac{15(16)}{2} \right] - 45 = 195 \end{aligned}$$

$$\begin{aligned} 18. \quad \sum_{i=1}^{10} (i^2 - 1) &= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 \\ &= \left[\frac{10(11)(21)}{6} \right] - 10 = 375 \end{aligned}$$

$$\begin{aligned} 20. \quad \sum_{i=1}^{10} i(i^2 + 1) &= \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i \\ &= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2} \right] = 3080 \end{aligned}$$

$$22. \quad \text{sum seq}(x \sqcap 3 - 2x, x, 1, 15, 1) = 14,160 \quad (TI-82)$$

$$\begin{aligned} \sum_{i=1}^{15} (i^3 - 2i) &= \frac{(15)^2(15+1)^2}{4} - 2 \frac{15(15+1)}{2} \\ &= \frac{(15)^2(16)^2}{4} - 15(16) = 14,160 \end{aligned}$$

$$\begin{aligned} 24. \quad S &= [5 + 5 + 4 + 2](1) = 16 \\ s &= [4 + 4 + 2 + 0](1) = 10 \end{aligned}$$

$$\begin{aligned} 26. \quad S &= \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2} \right] = \frac{55}{6} \\ s &= \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3} \right] = \frac{9}{2} = 4.5 \end{aligned}$$

$$\begin{aligned} 28. \quad S(8) &= \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2 \right) \frac{1}{4} + (\sqrt{1} + 2) \frac{1}{4} \\ &\quad + \left(\sqrt{\frac{5}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} + (\sqrt{2} + 2) \frac{1}{4} \\ &= \frac{1}{4} \left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2} \right) \approx 6.038 \\ s(8) &= (0+2) \frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2 \right) \frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2 \right) \frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2 \right) \frac{1}{4} \approx 5.685 \end{aligned}$$

$$\begin{aligned}
30. S(5) &= 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) \\
&= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859 \\
s(5) &= \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659
\end{aligned}$$

$$32. \lim_{n \rightarrow \infty} \left[\left(\frac{64}{n^3} \right) \frac{n(n+1)(2n+1)}{6} \right] = \frac{64}{6} \lim_{n \rightarrow \infty} \left[\frac{2n^3 + 3n^2 + n}{n^3} \right] = \frac{64}{6}(2) = \frac{64}{3}$$

$$34. \lim_{n \rightarrow \infty} \left[\left(\frac{1}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{1}{2}(1) = \frac{1}{2}$$

$$\begin{aligned}
36. \sum_{j=1}^n \frac{4j+3}{n^2} &= \frac{1}{n^2} \sum_{j=1}^n (4j+3) = \frac{1}{n^2} \left[\frac{4n(n+1)}{2} + 3n \right] = \frac{2n+5}{n} = S(n) \\
S(10) &= \frac{25}{10} = 2.5
\end{aligned}$$

$$S(100) = 2.05$$

$$S(1000) = 2.005$$

$$S(10,000) = 2.0005$$

$$\begin{aligned}
38. \sum_{i=1}^n \frac{4i^2(i-1)}{n^4} &= \frac{4}{n^4} \sum_{i=1}^n (i^3 - i^2) = \frac{4}{n^4} \left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right] \\
&= \frac{4}{n^3} \left[\frac{n^3 + 2n^2 + n}{4} - \frac{2n^2 + 3n + 1}{6} \right] \\
&= \frac{1}{3n^3} [3n^3 + 6n^2 + 3n - 4n^2 - 6n - 2] \\
&= \frac{1}{3n^3} [3n^3 + 2n^2 - 3n - 2] = S(n)
\end{aligned}$$

$$S(10) = 1.056$$

$$S(100) = 1.006566$$

$$S(1000) = 1.00066567$$

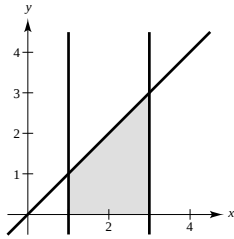
$$S(10,000) = 1.000066657$$

$$40. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n} \right) \left(\frac{2}{n} \right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{4}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \frac{4}{2} \left(1 + \frac{1}{n} \right) = 2$$

$$\begin{aligned}
42. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n} \right)^2 \left(\frac{2}{n} \right) &= \lim_{n \rightarrow \infty} \frac{2}{n^3} \sum_{i=1}^n (n + 2i)^2 \\
&= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[\sum_{i=1}^n n^2 + 4n \sum_{i=1}^n i + 4 \sum_{i=1}^n i^2 \right] \\
&= \lim_{n \rightarrow \infty} \frac{2}{n^3} \left[n^3 + (4n) \left(\frac{n(n+1)}{2} \right) + \frac{4(n)(n+1)(2n+1)}{6} \right] \\
&= 2 \lim_{n \rightarrow \infty} \left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2} \right] \\
&= 2 \left(1 + 2 + \frac{4}{3} \right) = \frac{26}{3}
\end{aligned}$$

$$\begin{aligned}
44. \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) &= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n + 2i)^3 \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (n^3 + 6n^2i + 12ni^2 + 8i^3) \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[n^4 + 6n^2 \left(\frac{n(n+1)}{2}\right) + 12n \left(\frac{n(n+1)(2n+1)}{6}\right) + 8 \left(\frac{n^2(n+1)^2}{4}\right) \right] \\
&= 2 \lim_{n \rightarrow \infty} \left(1 + 3 + \frac{3}{n} + 4 + \frac{6}{n} + \frac{2}{n^2} + 2 + \frac{4}{n} + \frac{2}{n^2}\right) \\
&= 2 \lim_{n \rightarrow \infty} \left(10 + \frac{13}{n} + \frac{4}{n^2}\right) = 20
\end{aligned}$$

46. (a)



$$(b) \quad \Delta x = \frac{3-1}{n} = \frac{2}{n}$$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \dots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \dots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$\begin{aligned}
s(n) &= \sum_{i=1}^n f(x_{i-1}) \Delta x \\
&= \sum_{i=1}^n f\left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right)
\end{aligned}$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right)$$

(e)

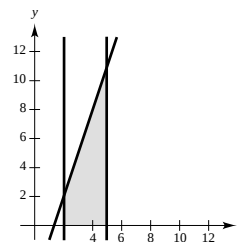
x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

$$\begin{aligned}
(f) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + (i-1)\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \left(\frac{2}{n}\right) \left[n + \frac{2(n(n+1)}{2} - n)\right] \\
&= \lim_{n \rightarrow \infty} \left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n \rightarrow \infty} \left[4 - \frac{2}{n}\right] = 4 \\
\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[1 + i\left(\frac{2}{n}\right)\right] \left(\frac{2}{n}\right) &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[n + \left(\frac{2}{n}\right) \frac{n(n+1)}{2}\right] \\
&= \lim_{n \rightarrow \infty} \left[2 + \frac{2(n+1)}{n}\right] = \lim_{n \rightarrow \infty} \left[4 + \frac{2}{n}\right] = 4
\end{aligned}$$

48. $y = 3x - 4$ on $[2, 5]$. (Note: $\Delta x = \frac{5-2}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \sum_{i=1}^n \left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^n i - 12 \\ &= 6 + \frac{27}{n^2} \left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2} \left(1 + \frac{1}{n}\right) \end{aligned}$$

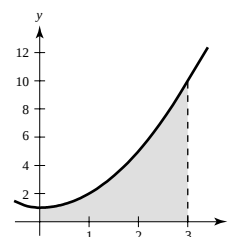
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$



50. $y = x^2 + 1$ on $[0, 3]$. (Note: $\Delta x = \frac{3-0}{n} = \frac{3}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n^3} \sum_{i=1}^n i^2 + \frac{3}{n} \sum_{i=1}^n 1 \\ &= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2} \frac{2n^2 + 3n + 1}{n^2} + 3 \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

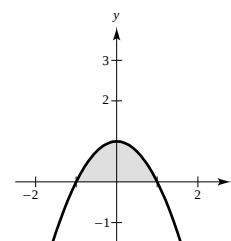


52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\ &= 1 - \frac{1}{n^3} \sum_{i=1}^n i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\frac{1}{2} \text{Area} = \lim_{n \rightarrow \infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

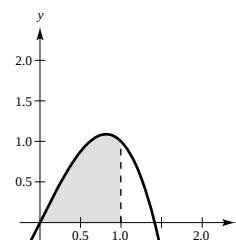


54. $y = 2x - x^3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) \\ &= \frac{2}{n^2} \sum_{i=1}^n i - \frac{1}{n^4} \sum_{i=1}^n i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4}\right] \\ &= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2} \end{aligned}$$

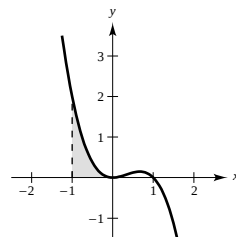
$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$



56. $y = x^2 - x^3$ on $[-1, 0]$. (Note: $\Delta x = \frac{0 - (-1)}{n} = \frac{1}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3 \right] \left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right) \left(\frac{1}{n}\right) \right] = 2 - \frac{5}{n^2} \sum_{i=1}^n i + \frac{4}{n^3} \sum_{i=1}^n i^2 - \frac{1}{n^4} \sum_{i=1}^n i^3 \\ &= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} \end{aligned}$$

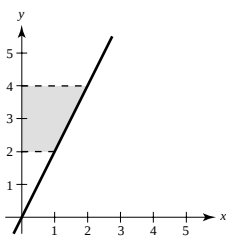
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$



58. $g(y) = \frac{1}{2}y$, $2 \leq y \leq 4$. (Note: $\Delta y = \frac{4 - 2}{n} = \frac{2}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \frac{1}{2} \left(2 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \\ &= \frac{2}{n} \left[n + \frac{1}{n} \frac{n(n+1)}{2} \right] = 2 + \frac{n+1}{n} \end{aligned}$$

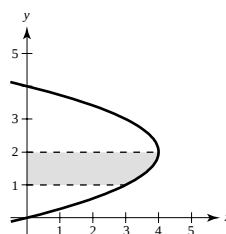
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + 1 = 3$$



60. $f(y) = 4y - y^2$, $1 \leq y \leq 2$. (Note: $\Delta y = \frac{2 - 1}{n} = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n \left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(3 + \frac{2i}{n} - \frac{i^2}{n^2} \right) \\ &= \frac{1}{n} \left[3n + \frac{2}{n} \frac{n(n+1)}{2} - \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6n} \end{aligned}$$

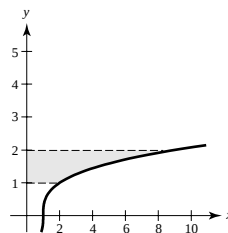
$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



62. $h(y) = y^3 + 1$, $1 \leq y \leq 2$ (Note: $\Delta y = \frac{1}{n}$)

$$\begin{aligned} S(n) &= \sum_{i=1}^n h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^3 + 1 \right] \frac{1}{n} \\ &= \frac{1}{n} \sum_{i=1}^n \left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n} \right) \\ &= \frac{1}{n} \left[2n + \frac{1}{n^3} \frac{n^2(n+1)^2}{4} + \frac{3}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \frac{n(n+1)}{2} \right] \\ &= 2 + \frac{(n+1)^2}{n^2} + \frac{1}{2} \frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



64. $f(x) = x^2 + 4x$, $0 \leq x \leq 4$, $n = 4$

Let $c_i = \frac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1, c_1 = \frac{1}{2}, c_2 = \frac{3}{2}, c_3 = \frac{5}{2}, c_4 = \frac{7}{2}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 4c_i](1) \\ &= \left[\left(\frac{1}{4} + 2 \right) + \left(\frac{9}{4} + 6 \right) + \left(\frac{25}{4} + 10 \right) + \left(\frac{49}{4} + 14 \right) \right] \\ &= 53 \end{aligned}$$

66. $f(x) = \sin x$, $0 \leq x \leq \frac{\pi}{2}$, $n = 4$

Let $c_i = \frac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8}, c_1 = \frac{\pi}{16}, c_2 = \frac{3\pi}{16}, c_3 = \frac{5\pi}{16}, c_4 = \frac{7\pi}{16}$$

$$\begin{aligned} \text{Area} &\approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\sin c_i) \left(\frac{\pi}{8} \right) \\ &= \frac{\pi}{8} \left(\sin \frac{\pi}{16} + \sin \frac{3\pi}{16} + \sin \frac{5\pi}{16} + \sin \frac{7\pi}{16} \right) \approx 1.006 \end{aligned}$$

68. $f(x) = \frac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

70. $f(x) = \cos \sqrt{x}$ on $[0, 2]$.

n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

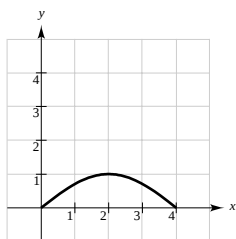
72. See the Definition of Area. Page 259.

74. $f(x) = \sqrt[3]{x}$, $0 \leq x \leq 8$

n	10	20	50	100	200
$s(n)$	10.998	11.519	11.816	11.910	11.956
$S(n)$	12.598	12.319	12.136	12.070	12.036
$M(n)$	12.040	12.016	12.005	12.002	12.001

(Note: exact answer is 12.)

76.

a. $A \approx 3$ square units

78. True. (Theorem 4.3)

80. (a) $\theta = \frac{2\pi}{n}$

(b) $\sin \theta = \frac{h}{r}$

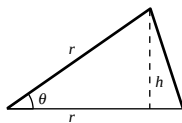
$$h = r \sin \theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r \sin \theta) = \frac{1}{2}r^2 \sin \theta$$

(c) $A_n = n \left(\frac{1}{2}r^2 \sin \frac{2\pi}{n} \right) = \frac{r^2 n}{2} \sin \frac{2\pi}{n} = \pi r^2 \left(\frac{\sin(2\pi/n)}{2\pi/n} \right)$

Let $x = 2\pi/n$. As $n \rightarrow \infty$, $x \rightarrow 0$.

$$\lim_{n \rightarrow \infty} A_n = \lim_{x \rightarrow 0} \pi r^2 \left(\frac{\sin x}{x} \right) = \pi r^2(1) = \pi r^2$$



82. (a) $\sum_{i=1}^n 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k 2i = k(k+1).$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} 2i &= \sum_{i=1}^k 2i + 2(k+1) \\ &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \end{aligned}$$

Which shows that the formula is true for $n = k+1$.

(b) $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$$

$$\begin{aligned} \text{Then we have } \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4} [k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4} (k+2)^2 \end{aligned}$$

which shows that the formula is true for $n = k+1$.

Section 4.3 Riemann Sums and Definite Integrals

2. $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$, $c_i = \frac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt[3]{\frac{i^3}{n^3}} \left[\frac{3i^2 - 3i + 1}{n^3} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n (3i^3 - 3i^2 + i) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[3 \left(\frac{n^2(n+1)^2}{4} \right) - 3 \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4} \right] = \lim_{n \rightarrow \infty} \left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2} \right] = \frac{3}{4} \end{aligned}$$

4. $y = x$ on $[-2, 3]$. (Note: $\Delta x = \frac{3 - (-2)}{n} = \frac{5}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = \sum_{i=1}^n \left(-2 + \frac{5i}{n}\right) \left(\frac{5}{n}\right) = -10 + \frac{25}{n^2} \sum_{i=1}^n i \\ &= -10 + \left(\frac{25}{n^2}\right) \frac{n(n+1)}{2} = -10 + \frac{25}{2} \left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}\end{aligned}$$

$$\int_{-2}^3 x \, dx = \lim_{n \rightarrow \infty} \left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

6. $y = 3x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n 3 \left(1 + \frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) \\ &= \frac{6}{n} \sum_{i=1}^n \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) \\ &= \frac{6}{n} \left[n + \frac{4}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= 6 + 12 \frac{n+1}{n} + 4 \frac{(n+1)(2n+1)}{n^2}\end{aligned}$$

$$\begin{aligned}\int_1^3 3x^2 \, dx &= \lim_{n \rightarrow \infty} \left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2} \right] \\ &= 6 + 12 + 8 = 26\end{aligned}$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. (Note: $\Delta x = \frac{2 - (-1)}{n} = \frac{3}{n}$, $\|\Delta\| \rightarrow 0$ as $n \rightarrow \infty$)

$$\begin{aligned}\sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n f\left(-1 + \frac{3i}{n}\right) \left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(-1 + \frac{3i}{n}\right)^2 + 2 \right] \\ &= \frac{3}{n} \sum_{i=1}^n \left[3 \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2 \right] \\ &= \frac{3}{n} \left[3n - \frac{18}{n} \frac{n(n+1)}{2} + \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} + 2n \right] \\ &= 15 - \frac{27(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2}\end{aligned}$$

$$\begin{aligned}\int_{-1}^2 (3x^2 + 2) \, dx &= \lim_{n \rightarrow \infty} \left[15 - 27 \frac{(n+1)}{n} + \frac{27}{2} \frac{(n+1)(2n+1)}{n^2} \right] \\ &= 15 - 27 + 27 = 15\end{aligned}$$

10. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n 6c_i(4 - c_i)^2 \Delta x_i = \int_0^4 6x(4 - x)^2 \, dx$
on the interval $[0, 4]$.

12. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \left(\frac{3}{c_i^2}\right) \Delta x_i = \int_1^3 \frac{3}{x^2} \, dx$
on the interval $[1, 3]$.

14. $\int_0^2 (4 - 2x) \, dx$

16. $\int_0^2 x^2 \, dx$

18. $\int_{-1}^1 \frac{1}{x^2 + 1} \, dx$

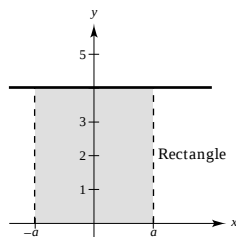
20. $\int_0^{\pi/4} \tan x \, dx$

22. $\int_0^2 (y - 2)^2 dy$

24. Rectangle

$$A = bh = 2(4)(a)$$

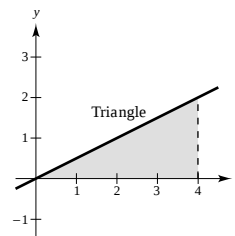
$$A = \int_{-a}^a 4 dx = 8a$$



26. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$$

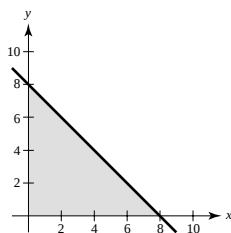
$$A = \int_0^4 \frac{x}{2} dx = 4$$



28. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$$

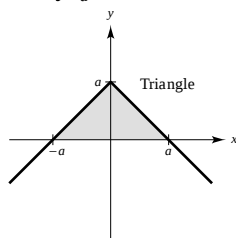
$$A = \int_0^8 (8 - x) dx = 32$$



30. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$$

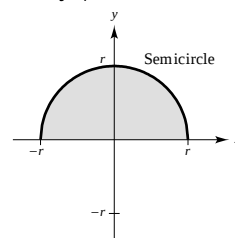
$$A = \int_{-a}^a (a - |x|) dx = a^2$$



32. Semicircle

$$A = \frac{1}{2}\pi r^2$$

$$A = \int_{-r}^r \sqrt{r^2 - x^2} dx = \frac{1}{2}\pi r^2$$



In Exercises 34–40, $\int_2^4 x^3 dx = 60$, $\int_2^4 x dx = 6$, $\int_2^4 dx = 2$.

34. $\int_2^2 x^3 dx = 0$

36. $\int_2^4 15 dx = 15 \int_2^4 dx = 15(2) = 30$

38. $\int_2^4 (x^3 + 4) dx = \int_2^4 x^3 dx + 4 \int_2^4 dx = 60 + 4(2) = 68$

40. $\int_2^4 (6 + 2x - x^3) dx = 6 \int_2^4 dx + 2 \int_2^4 x dx - \int_2^4 x^3 dx$
 $= 6(2) + 2(6) - 60 = -36$

42. (a) $\int_0^6 f(x) dx = \int_0^3 f(x) dx + \int_3^6 f(x) dx = 4 + (-1) = 3$

44. (a) $\int_{-1}^0 f(x) dx = \int_{-1}^1 f(x) dx - \int_0^1 f(x) dx = 0 - 5 = -5$

(b) $\int_6^3 f(x) dx = -\int_3^6 f(x) dx = -(-1) = 1$

(b) $\int_0^1 f(x) dx - \int_1^0 f(x) dx = 5 - (-5) = 10$

(c) $\int_3^3 f(x) dx = 0$

(c) $\int_{-1}^1 3f(x) dx = 3 \int_{-1}^1 f(x) dx = 3(0) = 0$

(d) $\int_3^6 -5f(x) dx = -5 \int_3^6 f(x) dx = -5(-1) = 5$

(d) $\int_0^1 3f(x) dx = 3 \int_0^1 f(x) dx = 3(5) = 15$

46. (a) $\int_0^5 [f(x) + 2] dx = \int_0^5 f(x) dx + \int_0^5 2 dx = 4 + 10 = 14$ (b) $\int_{-2}^3 f(x+2) dx = \int_0^5 f(x) dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x) dx = 2 \int_0^5 f(x) dx = 2(4) = 8$ (f even) (d) $\int_{-5}^5 f(x) dx = 0$ (f odd)

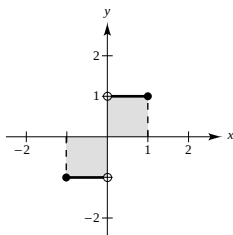
48. The right endpoint approximation will be less than the actual area: $<$

50. The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: $>$

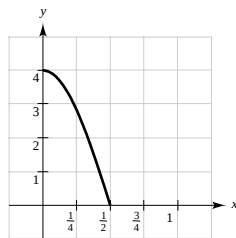
52. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^1 \frac{|x|}{x} dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x -axis for $-1 \leq x \leq 1$. You see that the integral equals 0.

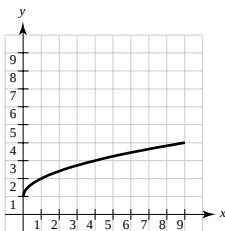


54.



b. $A \approx \frac{4}{3}$ square units

56.



c. Area ≈ 27 .

58. $\int_0^3 \frac{5}{x^2 + 1} dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	7.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

60. $\int_0^3 x \sin x dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

62. False

64. True

66. False

$$\int_0^1 x \sqrt{x} dx \neq \left(\int_0^1 x dx \right) \left(\int_0^1 \sqrt{x} dx \right)$$

$$\int_{-2}^4 x dx = 6$$

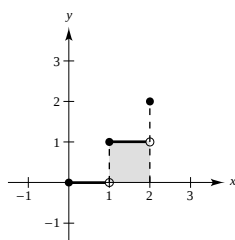
68. $f(x) = \sin x, [0, 2\pi]$

$$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$$

$$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$$

$$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x_i &= f\left(\frac{\pi}{6}\right) \Delta x_1 + f\left(\frac{\pi}{3}\right) \Delta x_2 + f\left(\frac{2\pi}{3}\right) \Delta x_3 + f\left(\frac{3\pi}{2}\right) \Delta x_4 \\ &= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708 \end{aligned}$$

70. To find $\int_0^2 \llbracket x \rrbracket dx$, use a geometric approach.

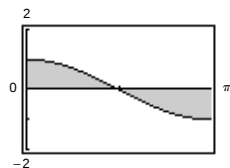
Thus,

$$\int_0^2 \llbracket x \rrbracket dx = 1(2 - 1) = 1.$$

Section 4.4 The Fundamental Theorem of Calculus

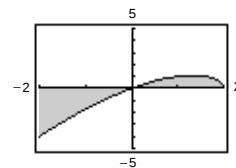
2. $f(x) = \cos x$

$$\int_0^\pi \cos x \, dx = 0$$



4. $f(x) = x\sqrt{2-x}$

$$\int_{-2}^2 x\sqrt{2-x} \, dx \text{ is negative.}$$



6. $\int_2^7 3 \, dv = \left[3v \right]_2^7 = 3(7) - 3(2) = 15$

8. $\int_2^5 (-3v + 4) \, dv = \left[-\frac{3}{2}v^2 + 4v \right]_2^5 = \left(-\frac{75}{2} + 20 \right) - (-6 + 8) = -\frac{39}{2}$

10.
$$\begin{aligned} \int_1^3 (3x^2 + 5x - 4) \, dx &= \left[x^3 + \frac{5x^2}{2} - 4x \right]_1^3 = \left(27 + \frac{45}{2} - 12 \right) - \left(1 + \frac{5}{2} - 4 \right) \\ &= 38 \end{aligned}$$

12.
$$\int_{-1}^1 (t^3 - 9t) \, dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2 \right]_{-1}^1 = \left(\frac{1}{4} - \frac{9}{2} \right) - \left(\frac{1}{4} - \frac{9}{2} \right) = 0$$

14.
$$\int_{-2}^{-1} \left(u - \frac{1}{u^2} \right) \, du = \left[\frac{u^2}{2} + \frac{1}{u} \right]_{-2}^{-1} = \left(\frac{1}{2} - 1 \right) - \left(2 - \frac{1}{2} \right) = -2$$

$$16. \int_{-3}^3 v^{1/3} dv = \left[\frac{3}{4} v^{4/3} \right]_{-3}^3 = \frac{3}{4} [(3\sqrt{-3})^4] - (3\sqrt{-3})^4 = 0$$

$$18. \int_1^8 \sqrt{\frac{2}{x}} dx = \sqrt{2} \int_1^8 x^{-1/2} dx = \left[\sqrt{2}(2)x^{1/2} \right]_1^8 = \left[2\sqrt{2x} \right]_1^8 = 8 - 2\sqrt{2}$$

$$20. \int_0^2 (2-t)\sqrt{t} dt = \int_0^2 (2t^{1/2} - t^{3/2}) dt = \left[\frac{4}{3} t^{3/2} - \frac{2}{5} t^{5/2} \right]_0^2 = \left[\frac{t\sqrt{t}}{15} (20 - 6t) \right]_0^2 = \frac{2\sqrt{2}}{15} (20 - 12) = \frac{16\sqrt{2}}{15}$$

$$22. \int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}} dx = \frac{1}{2} \int_{-8}^{-1} (x^{2/3} - x^{5/3}) dx$$

$$= \frac{1}{2} \left[\frac{3}{5} x^{5/3} - \frac{3}{8} x^{8/3} \right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80} (24 - 15x) \right]_{-8}^{-1} = -\frac{1}{80} (39) + \frac{32}{80} (144) = \frac{4569}{80}$$

$$24. \int_1^4 (3 - 1x - 31) dx = \int_1^3 [3 + (x - 3)] dx + \int_3^4 [3 - (x - 3)] dx$$

$$= \int_1^3 x dx + \int_3^4 (6 - x) dx$$

$$= \left[\frac{x^2}{2} \right]_1^3 + \left[6x - \frac{x^2}{2} \right]_3^4$$

$$= \left(\frac{9}{2} - \frac{1}{2} \right) + \left[(24 - 8) - \left(18 - \frac{9}{2} \right) \right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

$$26. \int_0^4 |x^2 - 4x + 3| dx = \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^4 (x^2 - 4x + 3) dx \quad (\text{split up the integral at the zeros } x = 1, 3)$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x \right]_3^4$$

$$= \left(\frac{1}{3} - 2 + 3 \right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3 \right) + \left(\frac{64}{3} - 32 + 12 \right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

$$28. \int_0^{\pi/4} \frac{1 - \sin^2 \theta}{\cos^2 \theta} d\theta = \int_0^{\pi/4} d\theta = \left[\theta \right]_0^{\pi/4} = \frac{\pi}{4}$$

$$30. \int_{\pi/4}^{\pi/2} (2 - \csc x) dx = \left[2x + \cot x \right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1 \right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$32. \int_{-\pi/2}^{\pi/2} (2t + \cos t) dt = \left[t^2 + \sin t \right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1 \right) - \left(\frac{\pi^2}{4} - 1 \right) = 2$$

$$34. P = \frac{2}{\pi} \int_0^{\pi/2} \sin \theta d\theta = \left[-\frac{2}{\pi} \cos \theta \right]_0^{\pi/2} = -\frac{2}{\pi} (0 - 1) = \frac{2}{\pi} \approx 63.7\%$$

$$36. A = \int_{-1}^1 (1 - x^4) dx = \left[x - \frac{1}{5} x^5 \right]_{-1}^1 = \frac{8}{5}$$

$$38. A = \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$40. A = \int_0^{\pi} (x + \sin x) dx = \left[\frac{x^2}{2} - \cos x \right]_0^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$$

42. Since $y \geq 0$ on $[0, 8]$,

$$\text{Area} = \int_0^8 (1 + x^{1/3}) dx = \left[x + \frac{3}{4}x^{4/3} \right]_0^8 = 8 + \frac{3}{4}(16) = 20$$

44. Since $y \geq 0$ on $[0, 3]$,

$$A = \int_0^3 (3x - x^2) dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3} \right]_0^3 = \frac{9}{2}.$$

$$46. \int_1^3 \frac{9}{x^3} dx = \left[-\frac{9}{2x^2} \right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

$$48. \int_{-\pi/3}^{\pi/3} \cos x dx = \left[\sin x \right]_{-\pi/3}^{\pi/3} = \sqrt{3}$$

$$f(c) \left[\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right] = \sqrt{3}$$

$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

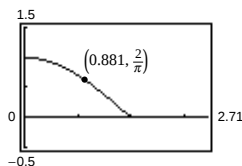
$$c \approx \pm 0.5971$$

$$52. \frac{1}{(\pi/2) - 0} \int_0^{\pi/2} \cos x dx = \left[\frac{2}{\pi} \sin x \right]_0^{\pi/2} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$

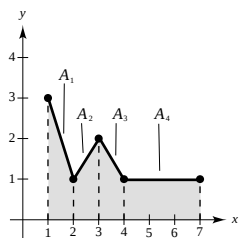


$$54. (a) \int_1^7 f(x) dx = \text{Sum of the areas}$$

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$$

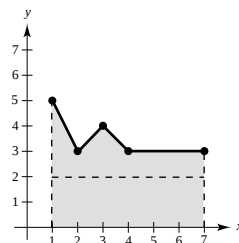
$$= 8$$



$$(b) \text{Average value} = \frac{\int_1^7 f(x) dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$$

$$(c) A = 8 + (6)(2) = 20$$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



$$56. \int_2^6 f(x) dx = (\text{area or region } B) = \int_0^6 f(x) dx - \int_0^2 f(x) dx \\ = 3.5 - (-1.5) = 5.0$$

$$58. \int_0^2 -2f(x) dx = -2 \int_0^2 f(x) dx = -2(-1.5) = 3.0$$

$$60. \text{Average value} = \frac{1}{6} \int_0^6 f(x) dx = \frac{1}{6}(3.5) = 0.5833$$

$$62. \frac{1}{R-0} \int_0^R k(R^2 - r^2) dr = \frac{k}{R} \left[R^2 r - \frac{r^3}{3} \right]_0^R = \frac{2kR^2}{3}$$

$$64. P = 5(\sqrt{t} + 30)$$

(a)

t	1	2	3	4	5	6
P	155	157.071	158.660	160	161.180	162.247

$$\text{Average profit} \approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$$

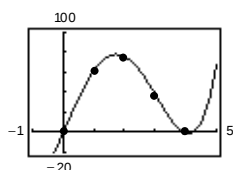
$$(b) \frac{1}{6} \int_{0.5}^{6.5} 5(\sqrt{t} + 30) dt = \frac{1}{6} \left[5 \left(\frac{2}{3} t^{3/2} + 30t \right) \right]_{0.5}^{6.5} \approx \frac{954.061}{6} \approx 159.010$$

(c) The definite integral yields a better approximation.

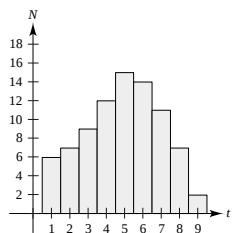
$$66. (a) R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$$

$$(c) \int_0^4 R(t) dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2} \right]_0^4 \\ = 181.957$$

(b)



68. (a) histogram

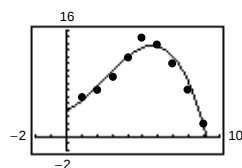


$$(b) [6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980 \text{ customers}$$

(c) Using a graphing utility, you obtain

$$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052t + 4.10317.$$

(d)



$$(e) \int_0^9 N(t) dt \approx 85.162$$

The estimated number of customers is $(85.162)(60) \approx 5110$.

$$(f) \text{Between 3 P.M. and 7 P.M., the number of customers is approximately } \left(\int_3^7 N(t) dt \right) (60) \approx (50.28)(60) \approx 3017.$$

Hence, $3017/240 \approx 12.6$ per minute.

$$\begin{aligned}
 70. F(x) &= \int_2^x (t^3 + 2t - 2) dt = \left[\frac{t^4}{4} + t^2 - 2t \right]_2^x \\
 &= \left(\frac{x^4}{4} + x^2 - 2x \right) - (4 + 4 - 4) \\
 &= \frac{x^4}{4} + x^2 - 2x - 4
 \end{aligned}$$

$$F(2) = 4 + 4 - 4 - 4 = 0 \quad \left[\text{Note: } F(2) = \int_2^2 (t^3 + 2t - 2) dt = 0 \right]$$

$$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$$

$$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$$

$$72. F(x) = \int_2^x \frac{-2}{t^3} dt = - \int_2^x 2t^{-3} dt = \left[\frac{1}{t^2} \right]_2^x = \frac{1}{x^2} - \frac{1}{4}$$

$$F(2) = \frac{1}{4} - \frac{1}{4} = 0$$

$$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$$

$$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$$

$$74. F(x) = \int_0^x \sin \theta d\theta = -\cos \theta \Big|_0^x = -\cos x + \cos 0 = 1 - \cos x$$

$$F(2) = 1 - \cos 2 \approx 1.4161$$

$$F(5) = 1 - \cos 5 \approx 0.7163$$

$$F(8) = 1 - \cos 8 \approx 1.1455$$

$$76. (a) \int_0^x t(t^2 + 1) dt = \int_0^x (t^3 + t) dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2 \right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$$

$$(b) \frac{d}{dx} \left[\frac{1}{4}x^4 + \frac{1}{2}x^2 \right] = x^3 + x = x(x^2 + 1)$$

$$78. (a) \int_4^x \sqrt{t} dt = \left[\frac{2}{3}t^{3/2} \right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$$

$$(b) \frac{d}{dx} \left[\frac{2}{3}x^{3/2} - \frac{16}{3} \right] = x^{1/2} = \sqrt{x}$$

$$80. (a) \int_{\pi/3}^x \sec t \tan t dt = \left[\sec t \right]_{\pi/3}^x = \sec x - 2$$

$$(b) \frac{d}{dx} [\sec x - 2] = \sec x \tan x$$

$$82. F(x) = \int_1^x \frac{t^2}{t^2 + 1} dt$$

$$F'(x) = \frac{x^2}{x^2 + 1}$$

$$84. F(x) = \int_1^x \sqrt[4]{t} dt$$

$$F'(x) = \sqrt[4]{x}$$

$$86. F(x) = \int_0^x \sec^3 t dt$$

$$F'(x) = \sec^3 x$$

$$88. F(x) = \int_{-x}^x t^3 dt = \left[\frac{t^4}{4} \right]_{-x}^x = 0$$

$$F'(x) = 0$$

Alternate solution

$$\begin{aligned} F(x) &= \int_{-x}^x t^3 dt \\ &= \int_{-x}^0 t^3 dt + \int_0^x t^3 dt \\ &= -\int_0^{-x} t^3 dt + \int_0^x t^3 dt \\ F'(x) &= -(-x)^3(-1) + (x^3) = 0 \end{aligned}$$

$$90. F(x) = \int_2^{x^2} t^{-3} dt = \left[\frac{t^{-2}}{-2} \right]_2^{x^2} = \left[-\frac{1}{2t^2} \right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8} \Rightarrow F'(x) = 2x^{-5}$$

Alternate solution: $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

$$92. F(x) = \int_0^{x^2} \sin \theta^2 d\theta$$

$$F'(x) = \sin (x^2)^2 (2x) = 2x \sin x^4$$

94. (a)

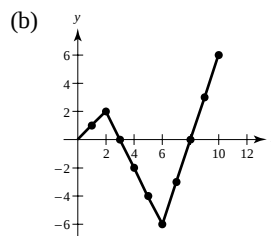
x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$ g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g : $x = 3, x = 8$.



$$96. (a) g(t) = 4 - \frac{4}{t^2}$$

$$\lim_{t \rightarrow \infty} g(t) = 4$$

Horizontal asymptote: $y = 4$

$$\begin{aligned} (b) A(x) &= \int_1^x \left(4 - \frac{4}{t^2} \right) dt \\ &= \left[4t + \frac{4}{t} \right]_1^x = 4x + \frac{4}{x} - 8 \\ &= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x} \end{aligned}$$

$$\lim_{x \rightarrow \infty} A(x) = \lim_{x \rightarrow \infty} \left(4x + \frac{4}{x} - 8 \right) = \infty + 0 - 8 = \infty$$

The graph of $A(x)$ does not have a horizontal asymptote.

98. True

100. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\begin{aligned} \int_{u(x)}^{v(x)} f(t) dt &= \left[F(t) \right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x)) \\ \frac{d}{dx} \left[\int_{u(x)}^{v(x)} f(t) dt \right] &= \frac{d}{dx} [F(v(x)) - F(u(x))] \\ &= F'(v(x))v'(x) - F'(u(x))u'(x) \\ &= f(v(x))v'(x) - f(u(x))u'(x). \end{aligned}$$

$$102. G(x) = \int_0^x \left[s \int_0^s f(t) dt \right] ds$$

$$(a) G(0) = \int_0^0 \left[s \int_0^s f(t) dt \right] ds = 0$$

$$(c) G''(x) = x \cdot f(x) + \int_0^x f(t) dt$$

$$(d) G''(0) = 0 \cdot f(0) + \int_0^0 f(t) dt = 0$$

$$(b) \text{ Let } F(s) = s \int_0^s f(t) dt.$$

$$G(x) = \int_0^x F(s) ds$$

$$G'(x) = F(x) = x \int_0^x f(t) dt$$

$$G'(0) = 0 \int_0^0 f(t) dt = 0$$

$$104. x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$$

$$x'(t) = 3t^2 - 14t + 15$$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)| dt \approx 27.37 \text{ units}$$

Section 4.5 Integration by Substitution

$$\frac{\int f(g(x))g'(x) dx}{\quad} \quad \frac{u = g(x)}{\quad} \quad \frac{du = g'(x) dx}{\quad}$$

$$2. \int x^2 \sqrt{x^3 + 1} dx \quad x^3 + 1 \quad 3x^2 dx$$

$$4. \int \sec 2x \tan 2x dx \quad 2x \quad 2 dx$$

$$6. \int \frac{\cos x}{\sin^2 x} dx \quad \sin x \quad \cos x dx$$

$$8. \int (x^2 - 9)^3 (2x) dx = \frac{(x^2 - 9)^4}{4} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^2 - 9)^4}{4} + C \right] = \frac{4(x^2 - 9)^3}{4} (2x) = (x^2 - 9)^3 (2x)$$

$$10. \int (1 - 2x^2)^{1/3} (-4x) dx = \frac{3}{4} (1 - 2x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{3}{4} (1 - 2x^2)^{4/3} + C \right] = \frac{3}{4} \cdot \frac{4}{3} (1 - 2x^2)^{1/3} (-4x) = (1 - 2x^2)^{1/3} (-4x)$$

$$12. \int x^2 (x^3 + 5)^4 dx = \frac{1}{3} \int (x^3 + 5)^4 (3x^2) dx = \frac{1}{3} \frac{(x^3 + 5)^5}{5} + C = \frac{(x^3 + 5)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 + 5)^5}{15} + C \right] = \frac{5(x^3 + 5)^4 (3x^2)}{15} = (x^3 + 5)^4 x^2$$

$$14. \int x(4x^2 + 3)^3 dx = \frac{1}{8} \int (4x^2 + 3)^3 (8x) dx = \frac{1}{8} \left[\frac{(4x^2 + 3)^4}{4} \right] + C = \frac{(4x^2 + 3)^4}{32} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(4x^2 + 3)^4}{32} + C \right] = \frac{4(4x^2 + 3)^3 (8x)}{32} = x(4x^2 + 3)^3$$

$$16. \int t^3 \sqrt{t^4 + 5} \, dt = \frac{1}{4} \int (t^4 + 5)^{1/2} (4t^3) \, dt = \frac{1}{4} \frac{(t^4 + 5)^{3/2}}{3/2} + C = \frac{1}{6} (t^4 + 5)^{3/2} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{6} (t^4 + 5)^{3/2} + C \right] = \frac{1}{6} \cdot \frac{3}{2} (t^4 + 5)^{1/2} (4t^3) = (t^4 + 5)^{1/2} (t^3)$$

$$18. \int u^2 \sqrt{u^3 + 2} \, du = \frac{1}{3} \int (u^3 + 2)^{1/2} (3u^2) \, du = \frac{1}{3} \frac{(u^3 + 2)^{3/2}}{3/2} + C = \frac{2(u^3 + 2)^{3/2}}{9} + C$$

$$\text{Check: } \frac{d}{du} \left[\frac{2(u^3 + 2)^{3/2}}{9} + C \right] = \frac{2}{9} \cdot \frac{3}{2} (u^3 + 2)^{1/2} (3u^2) = (u^3 + 2)^{1/2} (u^2)$$

$$20. \int \frac{x^3}{(1 + x^4)^2} \, dx = \frac{1}{4} \int (1 + x^4)^{-2} (4x^3) \, dx = -\frac{1}{4} (1 + x^4)^{-1} + C = \frac{-1}{4(1 + x^4)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{-1}{4(1 + x^4)} + C \right] = \frac{1}{4} (1 + x^4)^{-2} (4x^3) = \frac{x^3}{(1 + x^4)^2}$$

$$22. \int \frac{x^2}{(16 - x^3)^2} \, dx = -\frac{1}{3} \int (16 - x^3)^{-2} (-3x^2) \, dx = -\frac{1}{3} \left[\frac{(16 - x^3)^{-1}}{-1} \right] + C = \frac{1}{3(16 - x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3(16 - x^3)} + C \right] = \frac{1}{3} (-1) (16 - x^3)^{-2} (-3x^2) = \frac{x^2}{(16 - x^3)^2}$$

$$24. \int \frac{x^3}{\sqrt{1 + x^4}} \, dx = \frac{1}{4} \int (1 + x^4)^{-1/2} (4x^3) \, dx = \frac{1}{4} \frac{(1 + x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1 + x^4}}{2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{\sqrt{1 + x^4}}{2} + C \right] = \frac{1}{2} \cdot \frac{1}{2} (1 + x^4)^{-1/2} (4x^3) = \frac{x^3}{\sqrt{1 + x^4}}$$

$$26. \int \left[x^2 + \frac{1}{(3x)^2} \right] \, dx = \int \left(x^2 + \frac{1}{9} x^{-2} \right) \, dx = \frac{x^3}{3} + \frac{1}{9} \left(\frac{x^{-1}}{-1} \right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4 - 1}{9x} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{3} x^3 - \frac{1}{9} x^{-1} + C \right] = x^2 + \frac{1}{9} x^{-2} = x^2 + \frac{1}{(3x)^2}$$

$$28. \int \frac{1}{2\sqrt{x}} \, dx = \frac{1}{2} \int x^{-1/2} \, dx = \frac{1}{2} \left(\frac{x^{1/2}}{1/2} \right) + C = \sqrt{x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{x} + C] = \frac{1}{2\sqrt{x}}$$

$$30. \int \frac{t + 2t^2}{\sqrt{t}} \, dt = \int (t^{1/2} + 2t^{3/2}) \, dt = \frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C = \frac{2}{15} t^{3/2} (5 + 6t) + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C \right] = t^{1/2} + 2t^{3/2} = \frac{t + 2t^2}{\sqrt{t}}$$

$$32. \int \left(\frac{t^3}{3} + \frac{1}{4t^2} \right) \, dt = \int \left(\frac{1}{3} t^3 + \frac{1}{4} t^{-2} \right) \, dt = \frac{1}{3} \left(\frac{t^4}{4} \right) + \frac{1}{4} \left(\frac{t^{-1}}{-1} \right) + C = \frac{1}{12} t^4 - \frac{1}{4t} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{12} t^4 - \frac{1}{4t} + C \right] = \frac{1}{3} t^3 + \frac{1}{4t^2}$$

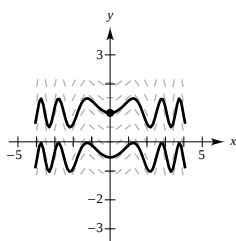
$$34. \int 2\pi y(8 - y^{3/2}) dy = 2\pi \int (8y - y^{5/2}) dy = 2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$$

Check: $\frac{d}{dy} \left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C \right] = \frac{d}{dy} \left[2\pi \left(4y^2 - \frac{2}{7}y^{7/2} \right) + C \right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

$$\begin{aligned} 36. y &= \int \frac{10x^2}{\sqrt{1+x^3}} dx \\ &= \frac{10}{3} \int (1+x^3)^{-1/2} (3x^2) dx \\ &= \frac{10}{3} \left[\frac{(1+x^3)^{1/2}}{1/2} \right] + C \\ &= \frac{20}{3} \sqrt{1+x^3} + C \end{aligned}$$

$$\begin{aligned} 38. y &= \int \frac{x-4}{\sqrt{x^2-8x+1}} dx \\ &= \frac{1}{2} \int (x^2-8x+1)^{-1/2} (2x-8) dx \\ &= \frac{1}{2} \left[\frac{(x^2-8x+1)^{1/2}}{1/2} \right] + C \\ &= \sqrt{x^2-8x+1} + C \end{aligned}$$

40. (a)



(b) $\frac{dy}{dx} = x \cos x^2, (0, 1)$

$$\begin{aligned} y &= \int x \cos x^2 dx = \frac{1}{2} \int \cos(x^2) 2x dx \\ &= \frac{1}{2} \sin(x^2) + C \end{aligned}$$

$$(0, 1): 1 = \frac{1}{2} \sin(0) + C \Rightarrow C = 1$$

$$y = \frac{1}{2} \sin(x^2) + 1$$

$$42. \int 4x^3 \sin x^4 dx = \int \sin x^4 (4x^3) dx = -\cos x^4 + C$$

$$44. \int \cos 6x dx = \frac{1}{6} \int (\cos 6x)(6) dx = \frac{1}{6} \sin 6x + C$$

$$46. \int x \sin x^2 dx = \frac{1}{2} \int (\sin x^2)(2x) dx = -\frac{1}{2} \cos x^2 + C$$

$$48. \int \sec(1-x) \tan(1-x) dx = - \int [\sec(1-x) \tan(1-x)](-1) dx = -\sec(1-x) + C$$

$$50. \int \sqrt{\tan x} \sec^2 x dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3} (\tan x)^{3/2} + C$$

$$52. \int \frac{\sin x}{\cos^3 x} dx = - \int (\cos x)^{-3} (-\sin x) dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2 \cos^2 x} + C = \frac{1}{2} \sec^2 x + C$$

$$54. \int \csc^2\left(\frac{x}{2}\right) dx = 2 \int \csc^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = -2 \cot\left(\frac{x}{2}\right) + C$$

$$56. f(x) = \int \pi \sec \pi x \tan \pi x dx = \sec \pi x + C$$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$$f(x) = \sec \pi x - 1.$$

58. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2} du$

$$\begin{aligned}\int x\sqrt{2x+1} dx &= \int \frac{1}{2}(u-1)\sqrt{u} \frac{1}{2} du \\ &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) du \\ &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{u^{3/2}}{30} (3u - 5) + C \\ &= \frac{1}{30} (2x+1)^{3/2} [3(2x+1) - 5] + C \\ &= \frac{1}{30} (2x+1)^{3/2} (6x-2) + C \\ &= \frac{1}{15} (2x+1)^{3/2} (3x-1) + C\end{aligned}$$

60. $u = 2 - x, x = 2 - u, dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x} dx &= -\int (3-u)\sqrt{u} du \\ &= -\int (3u^{1/2} - u^{3/2}) du \\ &= -\left(2u^{3/2} - \frac{2}{5} u^{5/2} \right) + C \\ &= -\frac{2u^{3/2}}{5} (5-u) + C \\ &= -\frac{2}{5} (2-x)^{3/2} [5 - (2-x)] + C \\ &= -\frac{2}{5} (2-x)^{3/2} (x+3) + C\end{aligned}$$

62. Let $u = x + 4, x = u - 4, du = dx$.

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x+4}} dx &= \int \frac{2(u-4)+1}{\sqrt{u}} du \\ &= \int (2u^{1/2} - 7u^{-1/2}) du \\ &= \frac{4}{3} u^{3/2} - 14u^{1/2} + C \\ &= \frac{2}{3} u^{1/2} (2u - 21) + C \\ &= \frac{2}{3} \sqrt{x+4} [2(x+4) - 21] + C \\ &= \frac{2}{3} \sqrt{x+4} (2x-13) + C\end{aligned}$$

64. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned}\int t \sqrt[3]{t-4} dt &= \int (u+4)u^{1/3} du \\ &= \int (u^{4/3} + 4u^{1/3}) du \\ &= \frac{3}{7} u^{7/3} + 3u^{4/3} + C \\ &= \frac{3u^{4/3}}{7} (u+7) + C \\ &= \frac{3}{7} (t-4)^{4/3} [(t-4) + 7] + C \\ &= \frac{3}{7} (t-4)^{4/3} (t+3) + C\end{aligned}$$

66. Let $u = x^3 + 8, du = 3x^2 dx$.

$$\begin{aligned}\int_{-2}^4 x^2(x^3+8)^2 dx &= \frac{1}{3} \int_{-2}^4 (x^3+8)^2 (3x^2) dx = \left[\frac{1}{3} \frac{(x^3+8)^3}{3} \right]_{-2}^4 \\ &= \frac{1}{9} [(64+8)^3 - (-8+8)^3] = 41,472\end{aligned}$$

68. Let $u = 1 - x^2, du = -2x dx$.

$$\int_0^1 x\sqrt{1-x^2} dx = -\frac{1}{2} \int_0^1 (1-x^2)^{1/2} (-2x) dx = \left[-\frac{1}{3} (1-x^2)^{3/2} \right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

70. Let $u = 1 + 2x^2, du = 4x dx$.

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}} dx = \frac{1}{4} \int_0^2 (1+2x^2)^{-1/2} (4x) dx = \left[\frac{1}{2} \sqrt{1+2x^2} \right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

72. Let $u = 4 + x^2$, $du = 2x \, dx$.

$$\int_0^2 x \sqrt[3]{4 + x^2} \, dx = \frac{1}{2} \int_0^2 (4 + x^2)^{1/3} (2x) \, dx = \left[\frac{3}{8} (4 + x^2)^{4/3} \right]_0^2 = \frac{3}{8} (8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2} \sqrt[3]{4} \approx 3.619$$

74. Let $u = 2x - 1$, $du = 2 \, dx$, $x = \frac{1}{2}(u + 1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\begin{aligned} \int_1^5 \frac{x}{\sqrt{2x-1}} \, dx &= \int_1^9 \frac{1/2(u+1)}{\sqrt{u}} \frac{1}{2} \, du = \frac{1}{4} \int_1^9 (u^{1/2} + u^{-1/2}) \, du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{3/2} + 2u^{1/2} \right]_1^9 \\ &= \frac{1}{4} \left[\left(\frac{2}{3} (27) + 2(3) \right) - \left(\frac{2}{3} + 2 \right) \right] \\ &= \frac{16}{3} \end{aligned}$$

76. $\int_{\pi/3}^{\pi/2} (x + \cos x) \, dx = \left[\frac{x^2}{2} + \sin x \right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1 \right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2} \right) = \frac{5\pi^2}{72} + \frac{2 - \sqrt{3}}{2}$

78. $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

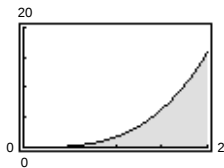
$$\text{Area} = \int_{-2}^6 x^2 \sqrt[3]{x+2} \, dx = \int_0^8 (u-2)^2 \sqrt[3]{u} \, du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3}) \, du = \left[\frac{3}{10} u^{10/3} - \frac{12}{7} u^{7/3} + 3u^{4/3} \right]_0^8 = \frac{4752}{35}$$

80. $A = \int_0^{\pi} (\sin x + \cos 2x) \, dx = \left[-\cos x + \frac{1}{2} \sin 2x \right]_0^{\pi} = 2$

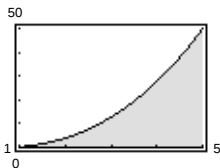
82. Let $u = 2x$, $du = 2 \, dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx = \frac{1}{2} \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x (2) \, dx = \left[-\frac{1}{2} \csc 2x \right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

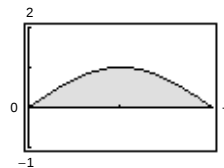
84. $\int_0^2 x^3 \sqrt{x+2} \, dx \approx 7.581$



86. $\int_1^5 x^2 \sqrt{x-1} \, dx \approx 67.505$



88. $\int_0^{\pi/2} \sin 2x \, dx \approx 1.0$



90. $\int \sin x \cos x \, dx = \int (\sin x)^1 (\cos x \, dx) = \frac{\sin^2 x}{2} + C_1$

$$\int \sin x \cos x \, dx = -\int (\cos x)^1 (-\sin x \, dx) = -\frac{\cos^2 x}{2} + C_2$$

$$-\frac{\cos^2 x}{2} + C_2 = -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

92. $f(x) = \sin^2 x \cos x$ is even.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx &= \int_0^{\pi/2} \sin^2 x (\cos x) \, dx \\ &= 2 \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} \\ &= \frac{2}{3}\end{aligned}$$

94. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$$

96. (a) $\int_{-\pi/4}^{\pi/4} \sin x \, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x \, dx = 2 \int_0^{\pi/4} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2 \int_0^{\pi/2} \cos x \, dx = \left[2 \sin x \right]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx = 0$ since $\sin(-x) \cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

98. $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x) \, dx = \int_{-\pi}^{\pi} \sin 3x \, dx + \int_{-\pi}^{\pi} \cos 3x \, dx = 0 + 2 \int_0^{\pi} \cos 3x \, dx = \left[\frac{2}{3} \sin 3x \right]_0^{\pi} = 0$

100. If $u = 5 - x^2$, then $du = -2x \, dx$ and $\int x(5 - x^2)^3 \, dx = -\frac{1}{2} \int (5 - x^2)^3 (-2x) \, dx = -\frac{1}{2} \int u^3 \, du$.

102. $\frac{dQ}{dt} = k(100 - t)^2$

$$Q(t) = \int k(100 - t)^2 \, dt = -\frac{k}{3}(100 - t)^3 + C$$

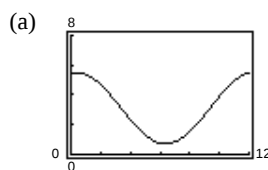
$$Q(100) = C = 0$$

$$Q(t) = -\frac{k}{3}(100 - t)^3$$

$$Q(0) = -\frac{k}{3}(100)^3 = 2,000,000 \Rightarrow k = -6$$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) = \$250,000$.

104. $R = 3.121 + 2.399 \sin(0.524t + 1.377)$

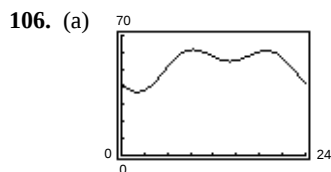


Relative minimum: (6.4, 0.7) or June

Relative maximum: (0.4, 5.5) or January

(b) $\int_0^{12} R(t) \, dt \approx 37.47$ inches

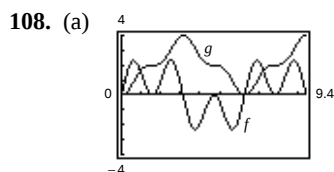
(c) $\frac{1}{3} \int_9^{12} R(t) \, dt \approx \frac{1}{3}(13) = 4.33$ inches



Maximum flow: $R \approx 61.713$ at $t = 9.36$.

[(18.861, 61.178) is a relative maximum.]

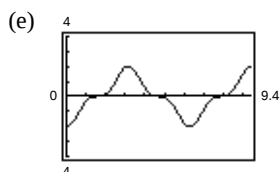
(b) Volume = $\int_0^{24} R(t) \, dt \approx 1272$ (5 thousand of gallons)



(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g .

(d) No, some zeros of f , like $x = \pi/2$, do not correspond to an extrema of g . The graph of g continues to increase after $x = \pi/2$ because f remains above the x -axis.



The graph of h is that of g shifted 2 units downward.

$$\begin{aligned} g(t) &= \int_0^t f(x) dx \\ &= \int_0^{\pi/2} f(x) dx + \int_{\pi/2}^t f(x) dx = 2 + h(t). \end{aligned}$$

110. False

$$\int x(x^2 + 1)^2 dx = \frac{1}{2} \int (x^2 + 1)(2x) dx = \frac{1}{4}(x^2 + 1)^2 + C$$

112. True

$$\int_a^b \sin x dx = [-\cos x]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi} \sin x dx$$

114. False

$$\int \sin^2 2x \cos 2x dx = \frac{1}{2} \int (\sin 2x)^2 (2 \cos 2x) dx = \frac{1}{2} \frac{(\sin 2x)^3}{3} + C = \frac{1}{6} \sin^3 2x + C$$

116. Because f is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= -\int_0^{-a} f(x) dx + \int_0^a f(x) dx. \end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned} \int_{-a}^a f(x) dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x) dx \\ &= -\int_0^a f(u) du + \int_0^a f(x) dx = 0 \end{aligned}$$

Section 4.6 Numerical Integration

2. Exact: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx = \left[\frac{x^3}{6} + x \right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx \approx \frac{1}{8} \left[1 + 2 \left(\frac{(1/4)^2}{2} + 1 \right) + 2 \left(\frac{(1/2)^2}{2} + 1 \right) + 2 \left(\frac{(3/4)^2}{2} + 1 \right) + \left(\frac{1^2}{2} + 1 \right) \right] = \frac{75}{64} \approx 1.1719$

Simpson's: $\int_0^1 \left(\frac{x^2}{2} + 1 \right) dx \approx \frac{1}{12} \left[1 + 4 \left(\frac{(1/4)^2}{2} + 1 \right) + 2 \left(\frac{(1/2)^2}{2} + 1 \right) + 4 \left(\frac{(3/4)^2}{2} + 1 \right) + \left(\frac{1^2}{2} + 1 \right) \right] = \frac{7}{6} \approx 1.1667$

4. Exact: $\int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = 0.5000$

Trapezoidal: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{8} \left[1 + 2 \left(\frac{4}{5} \right)^2 + 2 \left(\frac{4}{6} \right)^2 + 2 \left(\frac{4}{7} \right)^2 + \frac{1}{4} \right] \approx 0.5090$

Simpson's: $\int_1^2 \frac{1}{x^2} dx \approx \frac{1}{12} \left[1 + 4 \left(\frac{4}{5} \right)^2 + 2 \left(\frac{4}{6} \right)^2 + 4 \left(\frac{4}{7} \right)^2 + \frac{1}{4} \right] \approx 0.5004$

6. Exact: $\int_0^8 \sqrt[3]{x} dx = \left[\frac{3}{4} x^{4/3} \right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{2} [0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x} dx \approx \frac{1}{3} [0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

8. Exact: $\int_1^3 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\int_1^3 (4 - x^2) dx \approx \frac{1}{4} \left\{ 3 + 2 \left[4 - \left(\frac{3}{2} \right)^2 \right] + 2(0) + 2 \left[4 - \left(\frac{5}{2} \right)^2 \right] - 5 \right\} = -0.7500$

Simpson's: $\int_1^3 (4 - x^2) dx \approx \frac{1}{6} \left[3 + 4 \left(4 - \frac{9}{4} \right) + 0 + 4 \left(4 - \frac{25}{4} \right) - 5 \right] \approx -0.6667$

10. Exact: $\int_0^2 x \sqrt{x^2 + 1} dx = \frac{1}{3} \left[(x^2 + 1)^{3/2} \right]_0^2 = \frac{1}{3} (5^{3/2} - 1) \approx 3.393$

Trapezoidal: $\int_0^2 x \sqrt{x^2 + 1} dx \approx \frac{1}{4} \left[0 + 2 \left(\frac{1}{2} \right) \sqrt{(1/2)^2 + 1} + 2(1) \sqrt{1^2 + 1} + 2 \left(\frac{3}{2} \right) \sqrt{(3/2)^2 + 1} + 2 \sqrt{2^2 + 1} \right] \approx 3.457$

Simpson's: $\int_0^2 x \sqrt{x^2 + 1} dx \approx \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right) \sqrt{(1/2)^2 + 1} + 2(1) \sqrt{1^2 + 1} + 4 \left(\frac{3}{2} \right) \sqrt{(3/2)^2 + 1} + 2 \sqrt{2^2 + 1} \right] \approx 3.392$

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx \approx \frac{1}{4} \left[1 + 2 \left(\frac{1}{\sqrt{1+(1/2)^3}} \right) + 2 \left(\frac{1}{\sqrt{1+1^3}} \right) + 2 \left(\frac{1}{\sqrt{1+(3/2)^3}} \right) + \frac{1}{3} \right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}} dx \approx \frac{1}{6} \left[1 + 4 \left(\frac{1}{\sqrt{1+(1/2)^3}} \right) + 2 \left(\frac{1}{\sqrt{1+1^3}} \right) + 4 \left(\frac{1}{\sqrt{1+(3/2)^3}} \right) + \frac{1}{3} \right] \approx 1.405$

Graphing utility: 1.402

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x \, dx \approx \frac{\pi}{16} \left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x} \sin x \, dx \approx \frac{\pi}{24} \left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}} \sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}} \sin\left(\frac{7\pi}{8}\right) + 0 \right] \approx 1.458$

Graphing utility: 1.458

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2) \, dx \approx \frac{\sqrt{\pi/4}}{8} \left[\tan 0 + 2 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$
 ≈ 0.271

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2) \, dx \approx \frac{\sqrt{\pi/4}}{12} \left[\tan 0 + 4 \tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2 \tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4 \tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2 \right]$
 ≈ 0.257

Graphing utility: 0.256

18. Trapezoidal: $\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{16} \left[\sqrt{2} + 2\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 2\sqrt{1 + \cos^2(3\pi/8)} + 1 \right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1 + \cos^2 x} \, dx \approx \frac{\pi}{24} \left[\sqrt{2} + 4\sqrt{1 + \cos^2(\pi/8)} + 2\sqrt{1 + \cos^2(\pi/4)} + 4\sqrt{1 + \cos^2(3\pi/8)} + 1 \right] \approx 1.910$

Graphing utility: 1.910

20. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{8} \left[1 + \frac{2 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{2 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x} \, dx \approx \frac{\pi}{12} \left[1 + \frac{4 \sin(\pi/4)}{\pi/4} + \frac{2 \sin(\pi/2)}{\pi/2} + \frac{4 \sin(3\pi/4)}{3\pi/4} + 0 \right] \approx 1.852$

Graphing utility: 1.852

22. Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

24. $f(x) = \frac{1}{x+1}$

$$f'(x) = \frac{-1}{(x+1)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$f'''(x) = \frac{-6}{(x+1)^4}$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5}$$

(a) Trapezoidal: Error $\leq \frac{(1-0)^2}{12(4^2)}(2) = \frac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\leq \frac{(1-0)^5}{180(4^4)}(24) = \frac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

26. $f''(x) = \frac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \frac{1}{12n^2}(2) < 0.00001$, $n^2 > 16,666.67$, $n > 129.10$; let $n = 130$.

$$f^{(4)}(x) = \frac{24}{(1+x)^5} \text{ in } [0, 1]$$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \frac{1}{180n^4}(24) < 0.00001$, $n^4 > 13,333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

28. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\frac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{2}{9}$.

Trapezoidal: Error $\leq \frac{8}{12n^4}\left(\frac{2}{9}\right) < 0.00001$, $n^2 > 14,814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\frac{56}{81(x+1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{56}{81}$.

Simpson's: Error $\leq \frac{32}{180n^4}\left(\frac{56}{81}\right) < 0.00001$, $n^4 > 12,290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

30. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2 \sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19,044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12) \sin(x^2) - 48x^2 \cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15,793.61$, $n > 11.21$; let $n = 12$.

32. The program will vary depending upon the computer or programmable calculator that you use.

34. $f(x) = \sqrt{1-x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

36. $f(x) = \frac{\sin x}{x}$ on $[1, 2]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.7070	0.6597	0.6103	0.6586	0.6593
8	0.6833	0.6594	0.6350	0.6592	0.6593
10	0.6786	0.6594	0.6399	0.6592	0.6593
12	0.6754	0.6594	0.6431	0.6593	0.6593
16	0.6714	0.6594	0.6472	0.6593	0.6593
20	0.6690	0.6593	0.6496	0.6593	0.6593

38. Simpson's Rule: $n = 8$

$$8\sqrt{3} \int_0^{\pi/2} \sqrt{1 - \frac{2}{3} \sin^2 \theta} d\theta \approx \frac{\sqrt{3}\pi}{6} \left[\sqrt{1 - \frac{2}{3} \sin^2 0} + 4\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3} \sin^2 \frac{\pi}{2}} \right] \\ \approx 17.476$$

40. (a) Trapezoidal:

$$\int_0^2 f(x) dx \approx \frac{2}{2(8)} [4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x) dx \approx \frac{2}{3(8)} [4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844$$

$$\text{Integrating, } \int_0^2 y dx \approx 12.53$$

42. Simpson's Rule: $n = 6$

$$\pi = 4 \int_0^1 \frac{1}{1+x^2} dx \approx \frac{4}{3(6)} \left[1 + \frac{4}{1+(1/6)^2} + \frac{2}{1+(2/6)^2} + \frac{4}{1+(3/6)^2} + \frac{2}{1+(4/6)^2} + \frac{4}{1+(5/6)^2} + \frac{1}{2} \right] \\ \approx 3.14159$$

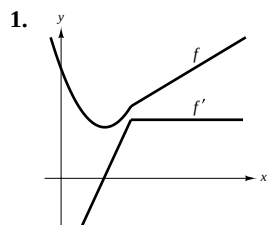
44. Area $\approx \frac{120}{2(12)} [75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$
 $= 7435 \text{ sq m}$

46. The quadratic polynomial

$$p(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3$$

passes through the three points.

Review Exercises for Chapter 4



3. $\int (2x^2 + x - 1) dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

5. $\int \frac{x^3 + 1}{x^2} dx = \int \left(x + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

7. $\int (4x - 3 \sin x) dx = 2x^2 + 3 \cos x + C$

9. $f'(x) = -2x, (-1, 1)$

$$f(x) = \int -2x dx = -x^2 + C$$

When $x = -1$:

$$y = -1 + C = 1$$

$$C = 2$$

$$y = 2 - x^2$$

11. $a(t) = a$

$$v(t) = \int a dt = at + C_1$$

$$v(0) = 0 + C_1 = 0 \text{ when } C_1 = 0.$$

$$v(t) = at$$

$$s(t) = \int at dt = \frac{a}{2}t^2 + C_2$$

$$s(0) = 0 + C_2 = 0 \text{ when } C_2 = 0.$$

$$s(t) = \frac{a}{2}t^2$$

$$s(30) = \frac{a}{2}(30)^2 = 3600 \text{ or}$$

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$$v(30) = 8(30) = 240 \text{ ft/sec}$$

13. $a(t) = -32$

$$v(t) = -32t + 96$$

$$s(t) = -16t^2 + 96t$$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(d) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

15. (a) $\sum_{i=1}^{10} (2i - 1)$

(b) $\sum_{i=1}^n i^3$

(c) $\sum_{i=1}^{10} (4i + 2)$

17. $y = \frac{10}{x^2 + 1}, \Delta x = \frac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2} \left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1} \right]$$

$$\approx 13.0385$$

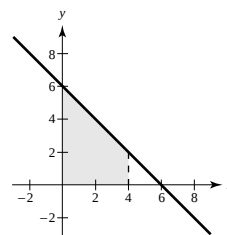
$$s(n) = s(4) = \frac{1}{2} \left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1} \right]$$

$$\approx 9.0385$$

$$9.0385 < \text{Area of Region} < 13.0385$$

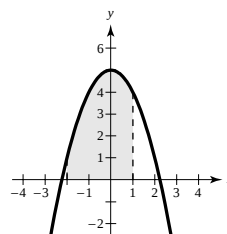
19. $y = 6 - x, \Delta x = \frac{4}{n}, \text{right endpoints}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 - \frac{4i}{n} \right) \frac{4}{n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[6n - \frac{4n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[24 - 8 \frac{n+1}{n} \right] = 24 - 8 = 16 \end{aligned}$$



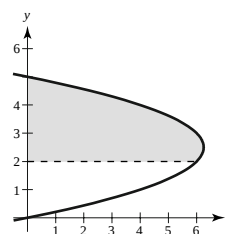
21. $y = 5 - x^2, \Delta x = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(ci) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 - \left(-2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[n + \frac{12n(n+1)}{2} - \frac{9n(n+1)(2n+1)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[3 + 18 \frac{n+1}{n} - \frac{9(n+1)(2n+1)}{n^2} \right] \\ &= 3 + 18 - 9 = 12 \end{aligned}$$



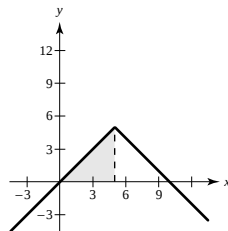
23. $x = 5y - y^2, 2 \leq y \leq 5, \Delta y = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 \left(2 + \frac{3i}{n} \right) - \left(2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[10 + \frac{15i}{n} - 4 - 12 \frac{i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[6n + \frac{3n(n+1)}{2} - \frac{9n(n+1)(2n+1)}{6} \right] \\ &= \left[18 + \frac{9}{2} - 9 \right] = \frac{27}{2} \end{aligned}$$



$$25. \lim_{\|\Delta\| \rightarrow \infty} \sum_{i=1}^n (2ci - 3) \Delta xi = \int_4^6 (2x - 3) dx$$

27.



$$\int_0^5 (5 - |x - 5|) dx = \int_0^5 (5 - (5 - x)) dx = \int_0^5 x dx = \frac{25}{2}$$

(triangle)

$$29. (a) \int_2^6 [f(x) + g(x)] dx = \int_2^6 f(x) dx + \int_2^6 g(x) dx = 10 + 3 = 13$$

$$(b) \int_2^6 [f(x) - g(x)] dx = \int_2^6 f(x) dx - \int_2^6 g(x) dx = 10 - 3 = 7$$

$$(c) \int_2^6 [2f(x) - 3g(x)] dx = 2 \int_2^6 f(x) dx - 3 \int_2^6 g(x) dx = 2(10) - 3(3) = 11$$

$$(d) \int_2^6 5f(x) dx = 5 \int_2^6 f(x) dx = 5(10) = 50$$

$$31. \int_1^8 (\sqrt[3]{x} + 1) dx = \left[\frac{3}{4} x^{4/3} + x \right]_1^8 = \left[\frac{3}{4}(16) + 8 \right] - \left[\frac{3}{4} + 1 \right] = \frac{73}{4} \quad (c)$$

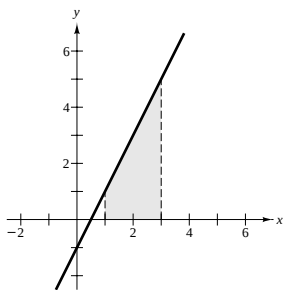
$$33. \int_0^4 (2 + x) dx = \left[2x + \frac{x^2}{2} \right]_0^4 = 8 + \frac{16}{2} = 16$$

$$35. \int_{-1}^1 (4t^3 - 2t) dt = \left[t^4 - t^2 \right]_{-1}^1 = 0$$

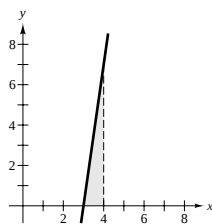
$$37. \int_4^9 x \sqrt{x} dx = \int_4^9 x^{3/2} dx = \left[\frac{2}{5} x^{5/2} \right]_4^9 = \frac{2}{5} [(\sqrt{9})^5 - (\sqrt{4})^5] = \frac{2}{5} (243 - 32) = \frac{422}{5}$$

$$39. \int_0^{3\pi/4} \sin \theta d\theta = \left[-\cos \theta \right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2} \right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$$

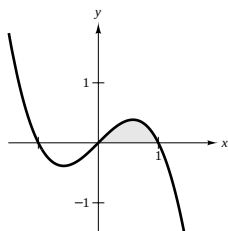
$$41. \int_1^3 (2x - 1) dx = \left[x^2 - x \right]_1^3 = 6$$



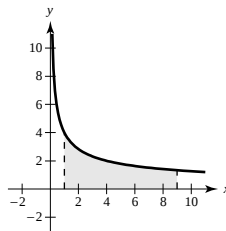
$$\begin{aligned} 43. \int_3^4 (x^2 - 9) dx &= \left[\frac{x^3}{3} - 9x \right]_3^4 \\ &= \left(\frac{64}{3} - 36 \right) - (9 - 27) \\ &= \frac{64}{3} - \frac{54}{3} = \frac{10}{3} \end{aligned}$$



$$45. \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$



$$47. \text{Area} = \int_1^9 \frac{4}{\sqrt{x}} dx = \left[\frac{4x^{1/2}}{(1/2)} \right]_1^9 = 8(3 - 1) = 16$$

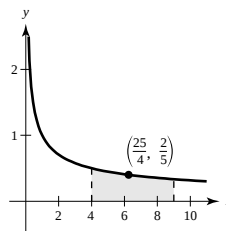


$$49. \frac{1}{9-4} \int_4^9 \frac{1}{\sqrt{x}} dx = \left[\frac{1}{5} 2\sqrt{x} \right]_4^9 = \frac{2}{5}(3 - 2) = \frac{2}{5} \text{ Average value}$$

$$\frac{2}{5} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = \frac{5}{2}$$

$$x = \frac{25}{4}$$



$$51. F'(x) = x^2 \sqrt{1+x^3}$$

$$53. F'(x) = x^2 + 3x + 2$$

$$55. \int (x^2 + 1)^3 dx = \int (x^6 + 3x^4 + 3x^2 + 1) dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$$

$$57. u = x^3 + 3, du = 3x^2 dx$$

$$\int \frac{x^2}{\sqrt{x^3 + 3}} dx = \int (x^3 + 3)^{-1/2} x^2 dx = \frac{1}{3} \int (x^3 + 3)^{-1/2} 3x^2 dx = \frac{2}{3} (x^3 + 3)^{1/2} + C$$

$$59. u = 1 - 3x^2, du = -6x dx$$

$$\int x(1 - 3x^2)^4 dx = -\frac{1}{6} \int (1 - 3x^2)^4 (-6x dx) = -\frac{1}{30} (1 - 3x^2)^5 + C = \frac{1}{30} (3x^2 - 1)^5 + C$$

$$61. \int \sin^3 x \cos x dx = \frac{1}{4} \sin^4 x + C$$

$$63. \int \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta = \int (1 - \cos \theta)^{-1/2} \sin \theta d\theta = 2(1 - \cos \theta)^{1/2} + C = 2\sqrt{1 - \cos \theta} + C$$

$$65. \int \tan^n x \sec^2 x dx = \frac{\tan^{n+1} x}{n+1} + C, n \neq -1$$

$$67. \int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x dx = \frac{1}{\pi} \int (1 + \sec \pi x)^2 (\pi \sec \pi x \tan \pi x) dx = \frac{1}{3\pi} (1 + \sec \pi x)^3 + C$$

$$69. \int_{-1}^2 x(x^2 - 4) dx = \frac{1}{2} \int_{-1}^2 (x^2 - 4)(2x) dx = \frac{1}{2} \left[\frac{(x^2 - 4)^2}{2} \right]_{-1}^2 = \frac{1}{4} [0 - 9] = -\frac{9}{4}$$

$$71. \int_0^3 \frac{1}{\sqrt{1+x}} dx = \int_0^3 (1+x)^{-1/2} dx = \left[2(1+x)^{1/2} \right]_0^3 = 4 - 2 = 2$$

$$73. u = 1 - y, y = 1 - u, dy = -du$$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$\begin{aligned} 2\pi \int_0^1 (y+1)\sqrt{1-y} dy &= 2\pi \int_1^0 -[(1-u)+1]\sqrt{u} du \\ &= 2\pi \int_1^0 (u^{3/2} - 2u^{1/2}) du = 2\pi \left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} \right]_1^0 = \frac{28\pi}{15} \end{aligned}$$

$$75. \int_0^\pi \cos\left(\frac{x}{2}\right) dx = 2 \int_0^\pi \cos\left(\frac{x}{2}\right) \frac{1}{2} dx = \left[2 \sin\left(\frac{x}{2}\right) \right]_0^\pi = 2$$

$$77. u = 1 - x, x = 1 - u, dx = -du$$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$\begin{aligned} P_{a,b} &= \int_a^b \frac{15}{4} x \sqrt{1-x} dx = \frac{15}{4} \int_{1-a}^{1-b} -(1-u)\sqrt{u} du \\ &= \frac{15}{4} \int_{1-a}^{1-b} (u^{3/2} - u^{1/2}) du = \frac{15}{4} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_{1-a}^{1-b} = \frac{15}{4} \left[\frac{2u^{3/2}}{15} (3u - 5) \right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_a^b \end{aligned}$$

$$(a) P_{0.50, 0.75} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_{0.50}^{0.75} = 0.353 = 35.3\%$$

$$(b) P_{0,b} = \left[-\frac{(1-x)^{3/2}}{2} (3x+2) \right]_0^b = -\frac{(1-b)^{3/2}}{2} (3b+2) + 1 = 0.5$$

$$(1-b)^{3/2} (3b+2) = 1$$

$$b \approx 0.586 = 58.6\%$$

$$79. p = 1.20 + 0.04t$$

$$C = \frac{15,000}{M} \int_t^{t+1} p ds$$

(a) 2000 corresponds to $t = 10$.

(b) 2005 corresponds to $t = 15$.

$$\begin{aligned} C &= \frac{15,000}{M} \int_{10}^{11} [1.20 + 0.04t] dt \\ &= \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{10}^{11} = \frac{24,300}{M} \end{aligned}$$

$$C = \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{15}^{16} = \frac{27,300}{M}$$

$$81. \text{Trapezoidal Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{8} \left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.257$$

$$\text{Simpson's Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{12} \left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.254$$

Graphing utility: 0.254

$$83. \text{Trapezoidal Rule } (n = 4): \int_0^{\pi/2} \sqrt{x} \cos x dx \approx 0.637$$

Simpson's Rule ($n = 4$): 0.685

Graphing Utility: 0.704

$$85. (a) R < I < T < L$$

$$(b) S(4) = \frac{4-0}{3(4)} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$$

$$\approx \frac{1}{3} \left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4} \right] \approx 5.417$$

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t} dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$$L'(1) = 1$$

(c) $L(x) = 1 = \int_1^x \frac{1}{t} dt$ for $x \approx 2.718$

$$\int_1^{2.718} \frac{1}{t} dt = 0.999896 \quad (\text{Note: The exact value of } x \text{ is } e, \text{ the base of the natural logarithm function.})$$

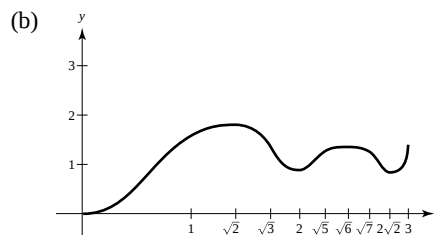
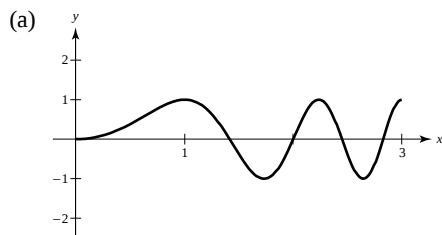
(d) We first show that $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{t} dt$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1} dt$.

Then $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{ux_1} (x_1 du) = \int_{1/x_1}^1 \frac{1}{u} du = \int_{1/x_1}^1 \frac{1}{t} dt$.

Now,
$$\begin{aligned} L(x_1 x_2) &= \int_1^{x_1 x_2} \frac{1}{t} dt = \int_{1/x_1}^{x_2} \frac{1}{u} du \quad (\text{using } u = \frac{t}{x_1}) \\ &= \int_{1/x_1}^1 \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du \\ &= \int_1^{x_1} \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du \\ &= L(x_1) + L(x_2). \end{aligned}$$

3. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$



The zeros of $y = \sin \frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin \frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}, n \text{ integer.}$

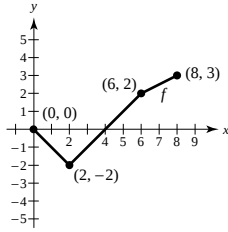
Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n}, n \text{ integer}$

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \text{ and } \sqrt{7}$.

5. (a)



$$(c) f(x) = \begin{cases} -x, & 0 \leq x < 2 \\ x - 4, & 2 \leq x < 6 \\ \frac{1}{2}x - 1, & 6 \leq x \leq 8 \end{cases}$$

$$F(x) = \int_0^x f(t) dt = \begin{cases} (-x^2/2), & 0 \leq x < 2 \\ (x^2/2) - 4x + 4, & 2 \leq x < 6 \\ (1/4)x^2 - x - 5, & 6 \leq x \leq 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 4)$ and increasing on $(4, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

$$7. (a) \int_{-1}^1 \cos x \, dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2 \cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$$

$$\int_{-1}^1 \cos x \, dx = \sin x \Big|_{-1}^1 = 2 \sin(1) \approx 1.6829$$

$$\text{Error. } |1.6829 - 1.6758| = 0.0071$$

$$(b) \int_{-1}^1 \frac{1}{1+x^2} \, dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$$

(Note: exact answer is $\pi/2 \approx 1.5708$)

9. Consider $F(x) = [f(x)]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. Thus,

$$\begin{aligned} \int_a^b f(x)f'(x) \, dx &= \int_a^b \frac{1}{2}F'(x) \, dx \\ &= \left[\frac{1}{2}F(x)\right]_a^b \\ &= \frac{1}{2}[F(b) - F(a)] \\ &= \frac{1}{2}[f(b)^2 - f(a)^2] \end{aligned}$$

(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

$$(d) F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$$

$x = 2$ is a point of inflection, whereas $x = 6$ is not. (f is not continuous at $x = 6$.)

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$$\int_{-1}^1 p(x) \, dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^1 = \frac{2b}{3} + 2d$$

$$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$$

$$11. \text{ Consider } \int_0^1 x^5 \, dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}.$$

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5 \right] \\ &= \frac{1}{n^6} [1^5 + 2^5 + \cdots + n^5] \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}.$$

13. By Theorem 4.8, $0 < f(x) \leq M \Rightarrow \int_a^b f(x) \, dx \leq \int_a^b M \, dx = M(b - a)$.

Similarly, $m \leq f(x) \Rightarrow m(b - a) = \int_a^b m \, dx \leq \int_a^b f(x) \, dx$.

Thus, $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$. On the interval $[0, 1]$, $1 \leq \sqrt{1 + x^4} \leq \sqrt{2}$ and $b - a = 1$.

Thus, $1 \leq \int_0^1 \sqrt{1 + x^4} \, dx \leq \sqrt{2}$. (**Note:** $\int_0^1 \sqrt{1 + x^4} \, dx \approx 1.0894$)

15. Since $-|f(x)| \leq f(x) \leq |f(x)|$,

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx \Rightarrow \left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

17. $\frac{1}{365} \int_0^{365} 100,000 \left[1 + \sin \frac{2\pi(t - 60)}{365} \right] dt = \frac{100,000}{365} \left[t - \frac{365}{2\pi} \cos \frac{2\pi(t - 60)}{365} \right]_0^{365} = 100,000 \text{ lbs.}$

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

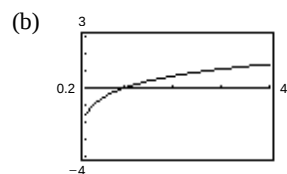
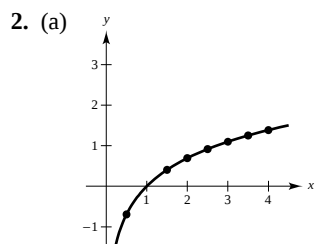
Section 5.1	The Natural Logarithmic Function: Differentiation	493
Section 5.2	The Natural Logarithmic Function: Integration	498
Section 5.3	Inverse Functions	503
Section 5.4	Exponential Functions: Differentiation and Integration . .	509
Section 5.5	Bases Other than e and Applications	516
Section 5.6	Differential Equations: Growth and Decay	522
Section 5.7	Differential Equations: Separation of Variables	527
Section 5.8	Inverse Trigonometric Functions: Differentiation	535
Section 5.9	Inverse Trigonometric Functions: Integration	539
Section 5.10	Hyperbolic Functions	543
Review Exercises		548
Problem Solving		554

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Even-Numbered Exercises



The graphs are identical.

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t} dt \approx 2.1163$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t} dt \approx -0.5108$

8. $f(x) = -\ln x$

Reflection in the x -axis

Matches (d)

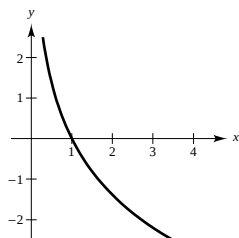
10. $f(x) = -\ln(-x)$

Reflection in the y -axis and the x -axis

Matches (c)

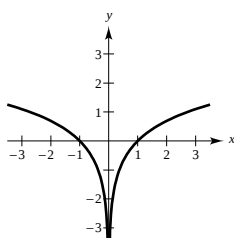
12. $f(x) = -2 \ln x$

Domain: $x > 0$



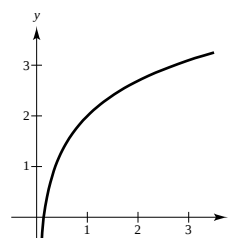
14. $f(x) = \ln|x|$

Domain: $x \neq 0$



16. $g(x) = 2 + \ln x$

Domain: $x > 0$



18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

20. $\ln \sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2} \ln 2$

22. $\ln xyz = \ln x + \ln y + \ln z$

24. $\ln \sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right) \ln(a-1)$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

28. $\ln \frac{1}{e} = \ln 1 - \ln e = -1$

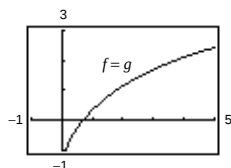
$$30. 3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$$

$$= \ln \frac{x^3 y^2}{z^4}$$

$$32. 2[\ln x - \ln(x+1) - \ln(x-1)] = 2 \ln \frac{x}{(x+1)(x-1)} = \ln \left(\frac{x}{x^2-1} \right)^2$$

$$34. \frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)] = \frac{3}{2} \ln \frac{x^2+1}{(x+1)(x-1)} = \ln \sqrt{\left(\frac{x^2+1}{x^2-1} \right)^3}$$

36.



$$38. \lim_{x \rightarrow 6^-} \ln(6-x) = -\infty$$

$$40. \lim_{x \rightarrow 5^+} \ln \frac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$$

$$42. y = \ln x^{3/2} = \frac{3}{2} \ln x$$

$$y' = \frac{3}{2x}$$

$$\text{At } (1, 0), y' = \frac{3}{2}.$$

$$44. y = \ln x^{1/2} = \frac{1}{2} \ln x$$

$$y' = \frac{1}{2x}$$

$$\text{At } (1, 0), y' = \frac{1}{2}.$$

$$46. h(x) = \ln(2x^2 + 1)$$

$$h'(x) = \frac{1}{2x^2 + 1}(4x) = \frac{4x}{2x^2 + 1}$$

$$48. y = x \ln x$$

$$\frac{dy}{dx} = x \left(\frac{1}{x} \right) + \ln x = 1 + \ln x$$

$$50. y = \ln \sqrt{x^2 - 4} = \frac{1}{2} \ln(x^2 - 4)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{2x}{x^2 - 4} \right) = \frac{x}{x^2 - 4}$$

$$52. f(x) = \ln \left(\frac{2x}{x+3} \right) = \ln 2x - \ln(x+3)$$

$$f'(x) = \frac{1}{x} - \frac{1}{x+3} = \frac{3}{x(x+3)}$$

$$54. h(t) = \frac{\ln t}{t}$$

$$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$$

$$56. y = \ln(\ln x)$$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

$$58. y = \ln \sqrt[3]{\frac{x-1}{x+1}} = \frac{1}{3} [\ln(x-1) - \ln(x+1)]$$

$$y' = \frac{1}{3} \left[\frac{1}{x-1} - \frac{1}{x+1} \right] = \frac{1}{3} \frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$$

$$60. f(x) = \ln(x + \sqrt{4+x^2})$$

$$\begin{aligned} f'(x) &= \frac{1}{x + \sqrt{4+x^2}} \left(1 + \frac{x}{\sqrt{4+x^2}} \right) \\ &= \frac{1}{\sqrt{4+x^2}} \end{aligned}$$

$$62. \quad y = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln\left(\frac{2+\sqrt{x^2+4}}{x}\right) = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln(2+\sqrt{x^2+4}) + \frac{1}{4} \ln x$$

$$\frac{dy}{dx} = \frac{-2x^2(x/\sqrt{x^2+4}) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4} \left(\frac{1}{2+\sqrt{x^2+4}} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}} \cdot \frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

$$\begin{aligned} \text{Hence, } \frac{dy}{dx} &= \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4} \left(\frac{2-\sqrt{x^2+4}}{-x^2} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x} \\ &= \frac{-1 + (1/2)(2-\sqrt{x^2+4})}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} \\ &= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3} \end{aligned}$$

$$64. \quad y = \ln|\csc x|$$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

$$66. \quad y = \ln|\sec x + \tan x|$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x \end{aligned}$$

$$68. \quad y = \ln\sqrt{1+\sin^2 x} = \frac{1}{2} \ln(1+\sin^2 x)$$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right) \frac{2 \sin x \cos x}{1+\sin^2 x} = \frac{\sin x \cos x}{1+\sin^2 x}$$

$$70. \quad g(x) = \int_1^{\ln x} (t^2 + 3) dt$$

$$g'(x) = [(\ln x)^2 + 3] \frac{d}{dx}(\ln x) = \frac{(\ln x)^2 + 3}{x}$$

(Second Fundamental Theorem of Calculus)

$$72. \quad (a) \quad y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$$

$$\begin{aligned} \frac{dy}{dx} &= -2x - \frac{1}{(1/2)x + 1} \left(\frac{1}{2}\right) \\ &= -2x - \frac{1}{x+2} \end{aligned}$$

$$\text{When } x = 0, \frac{dy}{dx} = -\frac{1}{2}.$$

$$\text{Tangent line: } y - 4 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 4$$

$$74. \quad \ln(xy) + 5x = 30$$

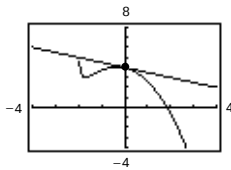
$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

(b)



76. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x \ln x - 4x - x(-3 + \ln x) = 0$$

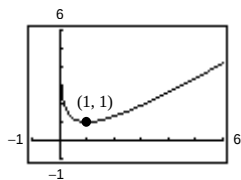
78. $y = x - \ln x$

Domain: $x > 0$

$$y' = 1 - \frac{1}{x} = 0 \text{ when } x = 1.$$

$$y'' = \frac{1}{x^2} > 0$$

Relative minimum: $(1, 1)$



80. $y = \frac{\ln x}{x}$

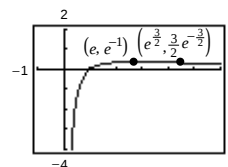
Domain: $x > 0$

$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

Relative maximum: (e, e^{-1})

Point of inflection: $(e^{3/2}, \frac{3}{2}e^{-3/2})$



82. $y = x^2 \ln \frac{x}{4}$. Domain $x > 0$

$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when}$$

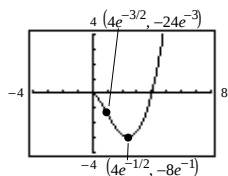
$$-1 = 2 \ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$$y'' = 0 \text{ when } x = 4e^{-3/2}$$

Relative minimum: $(4e^{-1/2}, -8e^{-1})$

Point of inflection: $(4e^{-3/2}, -24e^{-3})$



84. $f(x) = x \ln x$, $f(1) = 0$

$$f'(x) = 1 + \ln x$$
, $f'(1) = 1$

$$f''(x) = \frac{1}{x}$$
, $f''(1) = 1$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1$$
, $P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

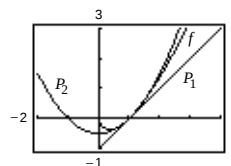
$$= (x - 1) + \frac{1}{2}(x - 1)^2$$
, $P_2(1) = 0$

$$P_1'(x) = 1$$
, $P_1'(1) = 1$

$$P_2'(x) = 1 + (x - 1) = x$$
, $P_2'(1) = 1$

$$P_2''(x) = 1$$
, $P_2''(1) = 1$

The values of f , P_1 , P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



86. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{4 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	-0.3069	-0.0049	0.0000

Approximate root: $x = 2.208$

90. $y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$

$$\ln y = \frac{1}{2} [\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \right]$$

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{\frac{x^2 - 1}{x^2 + 1}} \left[\frac{2x}{x^4 - 1} \right] \\ &= \frac{(x^2 - 1)^{1/2} 2x}{(x^2 + 1)^{1/2} (x^2 - 1)(x^2 + 1)} \\ &= \frac{2x}{(x^2 + 1)^{3/2} (x^2 - 1)^{1/2}} \end{aligned}$$

94. The base of the natural logarithmic function is e .

96. $g(x) = \ln f(x)$, $f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

- (a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

88. $y = \sqrt{(x-1)(x-2)(x-3)}$

$$\ln y = \frac{1}{2} [\ln(x-1) + \ln(x-2) + \ln(x-3)]$$

$$\begin{aligned} \frac{1}{y} \left(\frac{dy}{dx} \right) &= \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} \right] \\ &= \frac{1}{2} \left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} \right] \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2 - 12x + 11}{2y} \\ &= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}} \end{aligned}$$

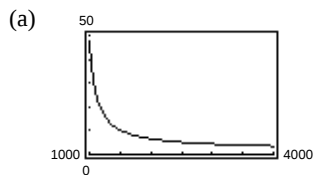
92. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$

$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$

$$\begin{aligned} \frac{dy}{dx} &= y \left[\frac{-2}{x^2 - 1} + \frac{-4}{x^2 - 4} \right] = y \left[\frac{-6x^2 + 12}{(x^2 - 1)(x^2 - 4)} \right] \\ &= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2 - 2)}{(x+1)(x-1)(x+2)(x-2)} \\ &= -\frac{6(x^2 - 2)}{(x-1)^2(x-2)^2} \end{aligned}$$

98. $t = \frac{5.315}{-6.7968 + \ln x}$, $1000 < x$



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280,178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384,642.00$$

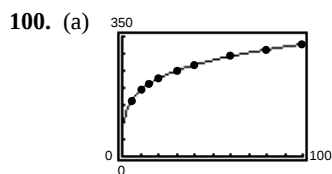
(d) $\frac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2} \left(\frac{1}{x} \right)$

$$= -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

- (e) There are two obvious benefits to paying a higher monthly payment:

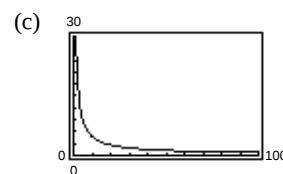
1. The term is lower
2. The total amount paid is lower.



(b) $T'(p) = \frac{34.96}{p} + \frac{3.955}{\sqrt{p}}$

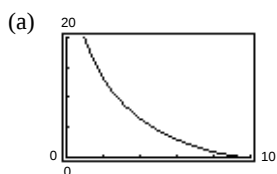
$T'(10) \approx 4.75 \text{ deg/lb/in}^2$

$T'(70) \approx 0.97 \text{ deg/lb/in}^2$



$\lim_{p \rightarrow \infty} T'(p) = 0$

102. $y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10 [\ln(10 + \sqrt{100 - x^2}) - \ln x] - \sqrt{100 - x^2}$



(c) $\lim_{x \rightarrow 10^-} \frac{dy}{dx} = 0$

(b)
$$\begin{aligned} \frac{dy}{dx} &= 10 \left[\frac{-x}{\sqrt{100 - x^2}(10 + \sqrt{100 - x^2})} - \frac{1}{x} \right] + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1 \right] - \frac{10}{x} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} \\ &= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x} \\ &= \frac{x(10 - \sqrt{100 - x^2})}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x} \end{aligned}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

104. $y = \ln x$

$y' = \frac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

106. False

π is a constant.

$\frac{d}{dx}[\ln \pi] = 0$

Section 5.2 The Natural Logarithmic Function: Integration

2. $\int \frac{10}{x} dx = 10 \int \frac{1}{x} dx = 10 \ln|x| + C$

6.
$$\begin{aligned} \int \frac{1}{3x + 2} dx &= \frac{1}{3} \int \frac{1}{3x + 2} (3) dx \\ &= \frac{1}{3} \ln|3x + 2| + C \end{aligned}$$

4. $u = x - 5, du = dx$

$\int \frac{1}{x - 5} dx = \ln|x - 5| + C$

8. $u = 3 - x^3, du = -3x^2 dx$

$$\begin{aligned} \int \frac{x^2}{3 - x^3} dx &= -\frac{1}{3} \int \frac{1}{3 - x^3} (-3x^2) dx \\ &= -\frac{1}{3} \ln|3 - x^3| + C \end{aligned}$$

10. $u = 9 - x^2, du = -2x dx$

$$\int \frac{x}{\sqrt{9-x^2}} dx = -\frac{1}{2} \int (9-x^2)^{-1/2} (-2x) dx = -\sqrt{9-x^2} + C$$

12. $\int \frac{x(x+2)}{x^3+3x^2-4} dx = \frac{1}{3} \int \frac{3x^2+6x}{x^3+3x^2-4} dx \quad (u = x^3+3x^2-4)$

$$= \frac{1}{3} \ln|x^3+3x^2-4| + C$$

14. $\int \frac{2x^2+7x-3}{x-2} dx = \int \left(2x+11+\frac{19}{x-2} \right) dx$

$$= x^2+11x+19 \ln|x-2| + C$$

16. $\int \frac{x^3-6x-20}{x+5} dx = \int \left(x^2-5x+19-\frac{115}{x+5} \right) dx$

$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115 \ln|x+5| + C$$

18. $\int \frac{x^3-3x^2+4x-9}{x^2+3} dx = \int \left(-3+x+\frac{x}{x^2+3} \right) dx$

$$= -3x + \frac{x^2}{2} + \frac{1}{2} \ln(x^2+3) + C$$

20. $\int \frac{1}{x \ln(x^3)} dx = \frac{1}{3} \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$$= \frac{1}{3} \ln|\ln|x|| + C$$

22. $u = 1 + x^{1/3}, du = \frac{1}{3x^{2/3}} dx$

$$\int \frac{1}{x^{2/3}(1+x^{1/3})} dx = 3 \int \frac{1}{1+x^{1/3}} \left(\frac{1}{3x^{2/3}} \right) dx$$

$$= 3 \ln|1+x^{1/3}| + C$$

24. $\int \frac{x(x-2)}{(x-1)^3} dx = \int \frac{x^2-2x+1-1}{(x-1)^3} dx$

$$= \int \frac{(x-1)^2}{(x-1)^3} dx - \int \frac{1}{(x-1)^3} dx$$

$$= \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^3} dx$$

$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

26. $u = 1 + \sqrt{3x}, du = \frac{3}{2\sqrt{3x}} dx \Rightarrow dx = \frac{2}{3}(u-1) du$

$$\int \frac{1}{1+\sqrt{3x}} dx = \int \frac{1}{u} \frac{2}{3}(u-1) du$$

$$= \frac{2}{3} \int \left(1 - \frac{1}{u} \right) du$$

$$= \frac{2}{3} [u - \ln|u|] + C$$

$$= \frac{2}{3} [1 + \sqrt{3x} - \ln(1 + \sqrt{3x})] + C$$

$$= \frac{2}{3} \sqrt{3x} - \frac{2}{3} \ln(1 + \sqrt{3x}) + C_1$$

$$28. u = x^{1/3} - 1, du = \frac{1}{3x^{2/3}} dx \Rightarrow dx = 3(u + 1)^2 du$$

$$\begin{aligned} \int \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1} dx &= \int \frac{u + 1}{u} 3(u + 1)^2 du \\ &= 3 \int \frac{u + 1}{u} (u^2 + 2u + 1) du \\ &= 3 \int \left(u^2 + 3u + 3 + \frac{1}{u} \right) du \\ &= 3 \left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u| \right] + C \\ &= 3 \left[\frac{(x^{1/3} - 1)^3}{3} + \frac{3(x^{1/3} - 1)^2}{2} + 3(x^{1/3} - 1) + \ln|x^{1/3} - 1| \right] + C \\ &= 3 \ln|x^{1/3} - 1| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1 \end{aligned}$$

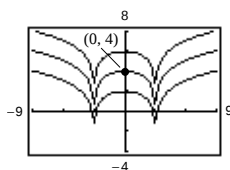
$$\begin{aligned} 30. \int \tan 5\theta d\theta &= \frac{1}{5} \int \frac{5 \sin 5\theta}{\cos 5\theta} d\theta \\ &= -\frac{1}{5} \ln|\cos 5\theta| + C \end{aligned}$$

$$\begin{aligned} 32. \int \sec \frac{x}{2} dx &= 2 \int \sec \frac{x}{2} \left(\frac{1}{2} \right) dx \\ &= 2 \ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + C \end{aligned}$$

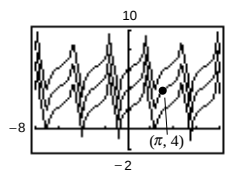
$$\begin{aligned} 34. u = \cot t, du &= -\csc^2 t dt \\ \int \frac{\csc^2 t}{\cot t} dt &= -\ln|\cot t| + C \end{aligned}$$

$$\begin{aligned} 36. \int (\sec t + \tan t) dt &= \ln|\sec t + \tan t| - \ln|\cos t| + C \\ &= \ln \left| \frac{\sec t + \tan t}{\cos t} \right| + C \\ &= \ln|\sec t(\sec t + \tan t)| + C \end{aligned}$$

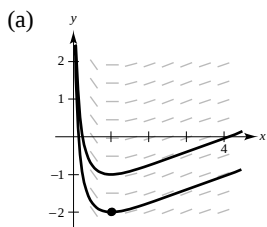
$$\begin{aligned} 38. y &= \int \frac{2x}{x^2 - 9} dx \\ &= \ln|x^2 - 9| + C \\ (0, 4): 4 &= \ln|0 - 9| + C \Rightarrow C = 4 - \ln 9 \\ y &= \ln|x^2 - 9| + 4 - \ln 9 \end{aligned}$$



$$\begin{aligned} 40. r &= \int \frac{\sec^2 t}{\tan t + 1} dt \\ &= \ln|\tan t + 1| + C \\ (\pi, 4): 4 &= \ln|0 + 1| + C \Rightarrow C = 4 \\ r &= \ln|\tan t + 1| + 4 \end{aligned}$$



$$42. \frac{dy}{dx} = \frac{\ln x}{x}, (1, -2)$$



$$\begin{aligned} (b) \quad y &= \int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + C \\ y(1) &= -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \Rightarrow C = -2 \\ \text{Hence, } y &= \frac{(\ln x)^2}{2} - 2. \end{aligned}$$

$$44. \int_{-1}^1 \frac{1}{x+2} dx = \left[\ln|x+2| \right]_{-1}^1 = \ln 3 - \ln 1 = \ln 3$$

$$46. u = \ln x, du = \frac{1}{x} dx$$

$$\int_e^{e^2} \frac{1}{x \ln x} dx = \int_e^{e^2} \left(\frac{1}{\ln x} \right) \frac{1}{x} dx = \left[\ln|\ln|x|| \right]_e^{e^2} = \ln 2$$

$$48. \int_0^1 \frac{x-1}{x+1} dx = \int_0^1 1 dx + \int_0^1 \frac{-2}{x+1} dx$$

$$= \left[x - 2 \ln|x+1| \right]_0^1 = 1 - 2 \ln 2$$

$$50. \int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2 d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2 \csc 2\theta \cot 2\theta + \cot^2 2\theta) d\theta$$

$$= \int_{0.1}^{0.2} (2 \csc^2 2\theta - 2 \csc 2\theta \cot 2\theta - 1) d\theta$$

$$= \left[-\cot 2\theta + \csc 2\theta - \theta \right]_{0.1}^{0.2} \approx 0.0024$$

$$52. \ln|\sin x| + C = \ln \left| \frac{1}{\csc x} \right| + C = -\ln|\csc x| + C$$

$$54. -\ln|\csc x + \cot x| + C = -\ln \left| \frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)} \right| + C = -\ln \left| \frac{\csc^2 x - \cot^2 x}{\csc x - \cot x} \right| + C$$

$$= -\ln \left| \frac{1}{\csc x - \cot x} \right| + C = \ln|\csc x - \cot x| + C$$

$$56. \int \frac{1-\sqrt{x}}{1+\sqrt{x}} dx = -(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 4 \ln(1+\sqrt{x}) + C_1$$

$$= 4\sqrt{x} - x - 4 \ln(1+\sqrt{x}) + C \text{ where } C = C_1 + 5.$$

$$58. \int \frac{\tan^2 2x}{\sec 2x} dx = \frac{1}{2} [\ln|\sec 2x + \tan 2x| - \sin 2x] + C$$

$$60. \int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x} dx = \left[\ln|\sec x + \tan x| - 2 \sin x \right]_{-\pi/4}^{\pi/4}$$

$$= \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) - 2\sqrt{2} \approx -1.066$$

Note: In Exercises 62 and 64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

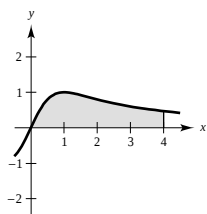
$$62. F(x) = \int_0^x \tan t dt$$

$$F'(x) = \tan x$$

$$64. F(x) = \int_1^{x^2} \frac{1}{t} dt$$

$$F'(x) = \frac{2x}{x^2} = \frac{2}{x}$$

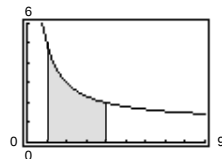
66.



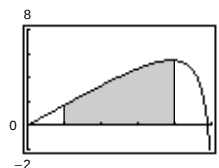
$$A \approx 3$$

Matches (a)

$$\begin{aligned} 68. A &= \int_1^4 \frac{x+4}{x} dx = \int_1^4 \left(1 + \frac{4}{x}\right) dx \\ &= \left[x + 4 \ln x \right]_1^4 \\ &= 4 + 4 \ln 4 - 1 \\ &= 3 + 4 \ln 4 \approx 8.5452 \end{aligned}$$



$$\begin{aligned} 70. \int_1^4 (2x - \tan(0.3x)) dx &= \left[x^2 + \frac{10}{3} \ln |\cos(0.3x)| \right]_1^4 \\ &= \left[16 + \frac{10}{3} \ln \cos(1.2) \right] - \left[1 + \frac{10}{3} \ln \cos(0.3) \right] \approx 11.7686 \end{aligned}$$

72. Substitution: ($u = x^2 + 4$) and Power Rule74. Substitution: ($u = \tan x$) and Log Rule

76. Answers will vary.

$$\begin{aligned} 78. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{4(x+1)}{x^2} dx \\ &= 2 \int_2^4 \left(\frac{1}{x} + \frac{1}{x^2} \right) dx \\ &= 2 \left[\ln x - \frac{1}{x} \right]_2^4 \\ &= 2 \left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2} \right] \\ &= 2 \left[\ln 2 + \frac{1}{4} \right] = \ln 4 + \frac{1}{2} \approx 1.8863 \end{aligned}$$

$$\begin{aligned} 80. \text{Average value} &= \frac{1}{2-0} \int_0^2 \sec \frac{\pi x}{6} dx \\ &= \left[\frac{1}{2} \left(\frac{6}{\pi} \right) \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\ &= \frac{3}{\pi} [\ln(2 + \sqrt{3}) - \ln(1 + 0)] \\ &= \frac{3}{\pi} \ln(2 + \sqrt{3}) \end{aligned}$$

$$\begin{aligned} 82. t &= \frac{10}{\ln 2} \int_{250}^{300} \frac{1}{T-100} dT \\ &= \frac{10}{\ln 2} [\ln(T-100)]_{250}^{300} = \frac{10}{\ln 2} [\ln 200 - \ln 150] = \frac{10}{\ln 2} \left[\ln \left(\frac{4}{3} \right) \right] \approx 4.1504 \text{ units of time} \end{aligned}$$

$$84. \frac{dS}{dt} = \frac{k}{t}$$

$$S(t) = \int \frac{k}{t} dt = k \ln |t| + C = k \ln t + C \text{ since } t > 1.$$

$$S(2) = k \ln 2 + C = 200$$

$$S(4) = k \ln 4 + C = 300$$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

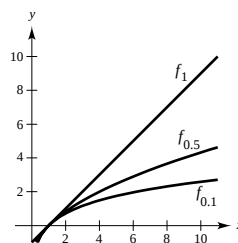
$$S(t) = \frac{100 \ln t}{\ln 2} + 100 = 100 \left[\frac{\ln t}{\ln 2} + 1 \right].$$

86. $k = 1$: $f_1(x) = x - 1$

$$k = 0.5: f_{0.5}(x) = \frac{\sqrt{x} - 1}{0.5} = 2(\sqrt{x} - 1)$$

$$k = 0.1: f_{0.1}(x) = \frac{\sqrt[10]{x} - 1}{0.1} = 10(\sqrt[10]{x} - 1)$$

$$\lim_{k \rightarrow 0^+} f_k(x) = \ln x$$



88. False

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

90. False; the integrand has a nonremovable discontinuity at $x = 0$.

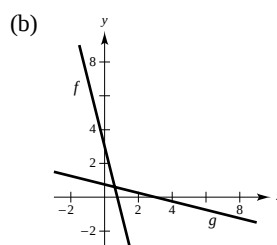
Section 5.3 Inverse Functions

2. (a) $f(x) = 3 - 4x$

$$g(x) = \frac{3 - x}{4}$$

$$f(g(x)) = f\left(\frac{3 - x}{4}\right) = 3 - 4\left(\frac{3 - x}{4}\right) = x$$

$$g(f(x)) = g(3 - 4x) = \frac{3 - (3 - 4x)}{4} = x$$

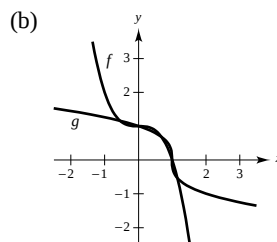


4. (a) $f(x) = 1 - x^3$

$$g(x) = \sqrt[3]{1 - x}$$

$$\begin{aligned} f(g(x)) &= f(\sqrt[3]{1 - x}) = 1 - (\sqrt[3]{1 - x})^3 \\ &= 1 - (1 - x) = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(1 - x^3) \\ &= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x \end{aligned}$$

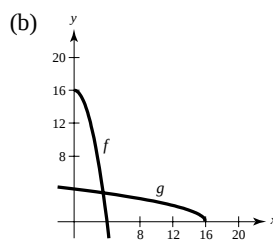


6. (a) $f(x) = 16 - x^2$, $x \geq 0$

$$g(x) = \sqrt{16 - x}$$

$$\begin{aligned} f(g(x)) &= f(\sqrt{16 - x}) = 16 - (\sqrt{16 - x})^2 \\ &= 16 - (16 - x) = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(16 - x^2) = \sqrt{16 - (16 - x^2)} \\ &= \sqrt{x^2} = x \end{aligned}$$

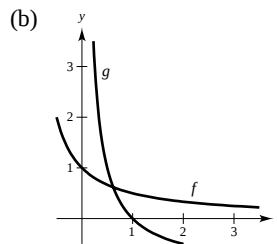


8. (a) $f(x) = \frac{1}{1 + x}$, $x \geq 0$

$$g(x) = \frac{1 - x}{x}$$
, $0 < x \leq 1$

$$f(g(x)) = f\left(\frac{1 - x}{x}\right) = \frac{1}{1 + \frac{1 - x}{x}} = \frac{1}{\frac{1 + x}{x}} = \frac{x}{1 + x}$$

$$g(f(x)) = g\left(\frac{1}{1 + x}\right) = \frac{1 - \frac{1}{1 + x}}{\frac{1}{1 + x}} = \frac{\frac{1 + x - 1}{1 + x}}{\frac{1}{1 + x}} = \frac{x}{1 + x} \cdot \frac{1 + x}{1} = x$$

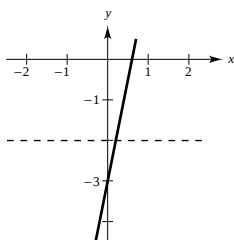


10. Matches (b)

12. Matches (d)

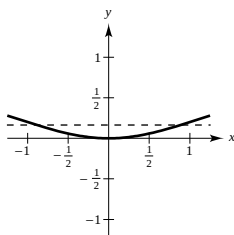
14. $f(x) = 5x - 3$

One-to-one; has an inverse



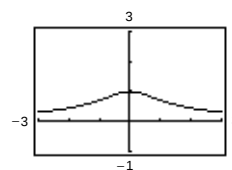
16. $F(x) = \frac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



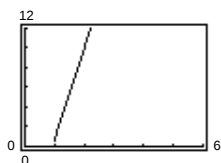
18. $g(t) = \frac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



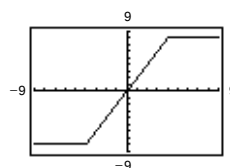
20. $f(x) = 5x\sqrt{x-1}$

One-to-one; has an inverse



22. $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse



24. $f(x) = \cos \frac{3x}{2}$

$$f'(x) = -\frac{3}{2} \sin \frac{3x}{2} = 0 \text{ when } x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$$

 f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \geq 0 \text{ for all } x.$$

 f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = \ln(x - 3), x > 3$

$$f'(x) = \frac{1}{x - 3} > 0 \text{ for } x > 3.$$

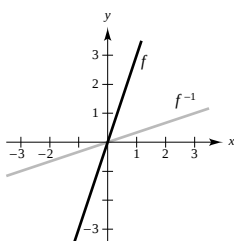
 f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

30. $f(x) = 3x = y$

$$x = \frac{y}{3}$$

$$y = \frac{x}{3}$$

$$f^{-1}(x) = \frac{x}{3}$$

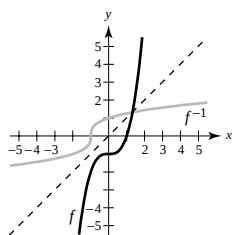


32. $f(x) = x^3 - 1 = y$

$$x = \sqrt[3]{y + 1}$$

$$y = \sqrt[3]{x + 1}$$

$$f^{-1}(x) = \sqrt[3]{x + 1}$$

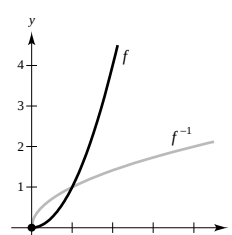


34. $f(x) = x^2 = y, 0 \leq x$

$$x = \sqrt{y}$$

$$y = \sqrt{x}$$

$$f^{-1}(x) = \sqrt{x}$$

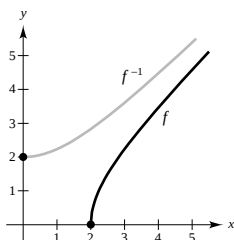


36. $f(x) = \sqrt{x^2 - 4} = y, x \geq 2$

$$x = \sqrt{y^2 + 4}$$

$$y = \sqrt{x^2 - 4}$$

$$f^{-1}(x) = \sqrt{x^2 + 4}, x \geq 0$$

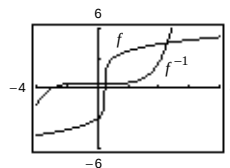


38. $f(x) = 3\sqrt[5]{2x - 1} = y$

$$x = \frac{y^5 + 243}{486}$$

$$y = \frac{x^5 + 243}{486}$$

$$f^{-1}(x) = \frac{x^5 + 243}{486}$$



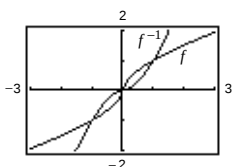
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

40. $f(x) = x^{3/5} = y$

$$x = y^{5/3}$$

$$y = x^{3/5}$$

$$f^{-1}(x) = x^{3/5}$$



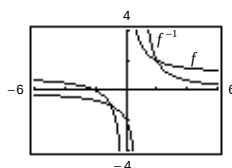
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

42. $f(x) = \frac{x + 2}{x} = y$

$$x = \frac{2}{y - 1}$$

$$y = \frac{2}{x - 1}$$

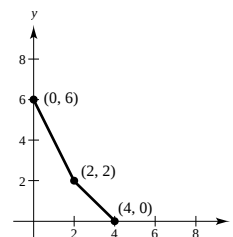
$$f^{-1}(x) = \frac{2}{x - 1}$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

44.

x	0	2	4
$f^{-1}(x)$	6	2	0



46. $f(x) = k(2 - x - x^3)$ is one-to-one for all $k \neq 0$. Since $f^{-1}(3) = -2$, $f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \Rightarrow k = \frac{1}{4}$.

48. $f(x) = |x + 2|$ on $[-2, \infty)$

$$f'(x) = \frac{|x + 2|}{x + 2}(1) = 1 > 0 \text{ on } (-2, \infty)$$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

50. $f(x) = \cot x$ on $(0, \pi)$

$$f'(x) = -\csc^2 x < 0 \text{ on } (0, \pi)$$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \sec x$ on $\left[0, \frac{\pi}{2}\right)$

$$f'(x) = \sec x \tan x > 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

54. $f(x) = 2 - \frac{3}{x^2} = y$ on $(0, 10)$

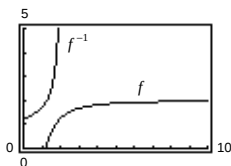
$$2x^2 - 3 = x^2y$$

$$x^2(2 - y) = 3$$

$$x = \pm \sqrt{\frac{3}{2 - y}}$$

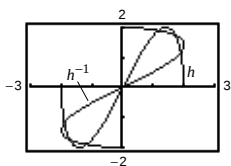
$$y = \pm \sqrt{\frac{3}{2 - x}}$$

$$f^{-1}(x) = \sqrt{\frac{3}{2 - x}}, x < 2$$



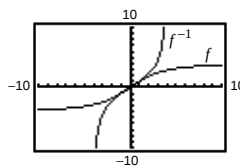
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

56. (a), (b)



(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

60. $f(x) = -3$

Not one-to-one; does not have an inverse

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$$ax + b = y$$

$$x = \frac{y - b}{a}$$

$$y = \frac{x - b}{a}$$

$$f^{-1}(x) = \frac{x - b}{a}, a \neq 0$$

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$$16 - x^4 = y$$

$$16 - y = x^4$$

$$\sqrt[4]{16 - y} = x$$

$$\sqrt[4]{16 - x} = y$$

$$f^{-1}(x) = \sqrt[4]{16 - x}, x \leq 16$$

66. $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$$x - 3 = y$$

$$x = y + 3$$

$$y = x + 3$$

$$f^{-1}(x) = x + 3, x \geq 0$$

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A .

72. $f(x) = \frac{1}{27}(x^5 + 2x^3); f(-3) = \frac{1}{27}(-243 - 54) = -11 = a$.

$$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$$

$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))} = \frac{1}{f'(-3)} = \frac{27}{5(-3)^4 + 6(-3)^2} = \frac{1}{17}$$

74. $f(x) = \cos 2x, f(0) = 1 = a$

$$f'(x) = -2 \sin 2x$$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-2 \sin 0} = \frac{1}{0} \text{ which is undefined.}$$

76. $f(x) = \sqrt{x-4}, f(8) = 2 = a$

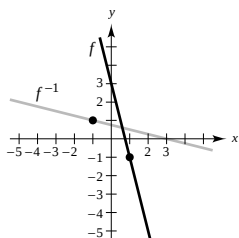
$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)} = \frac{1}{1/(2\sqrt{8-4})} = \frac{1}{1/4} = 4$$

78. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, (1, -1)$

$$f'(x) = -4$$

$$f'(1) = -4$$

$$f^{-1}(x) = \frac{3-x}{4}, (-1, 1)$$

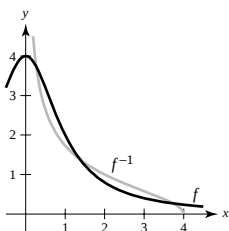
$$(f^{-1})'(x) = -\frac{1}{4}$$

$$(f^{-1})'(-1) = -\frac{1}{4}$$

80. (a) Domain $f = [0, \infty)$, Domain $f^{-1} = (0, 4]$

(b) Range $f = (0, 4]$, Range $f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \frac{4}{1+x^2}$

$$f'(x) = \frac{-8x}{(x^2+1)^2}, f'(1) = -2$$

$$f^{-1}(x) = \sqrt{\frac{4-x}{x}}$$

$$(f^{-1})'(x) = \frac{-2}{x^2 \sqrt{\frac{4-x}{x}}}, (f^{-1})'(2) = -\frac{1}{2}$$

82. $x = 2 \ln(y^2 - 3)$

$$1 = 2 \frac{1}{y^2 - 3} 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2 - 3}{4y}. \text{ At } (0, 4), \frac{dy}{dx} = \frac{16 - 3}{16} = \frac{13}{16}.$$

In Exercises 84 and 86, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x+3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

84. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3)) = g^{-1}(0) = 0$

86. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}(\sqrt[3]{-4})$
 $= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$

In Exercises 88 and 90, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

$$88. (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$$

$$= f^{-1}\left(\frac{x + 5}{2}\right)$$

$$= \frac{x + 5}{2} - 4$$

$$= \frac{x - 3}{2}$$

$$90. (g \circ f)(x) = g(f(x))$$

$$= g(x + 4)$$

$$= 2(x + 4) - 5$$

$$= 2x + 3$$

$$\text{Hence, } (g \circ f)^{-1}(x) = \frac{x - 3}{2}$$

$$(\text{Note: } (g \circ f)^{-1} = f^{-1} \circ g^{-1})$$

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

94. Theorem 5.9: Let f be differentiable on an interval I . If f has an inverse g , then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

96. f is not one-to-one because different x -values yield the same y -value.

$$\text{Example: } f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$$

Not continuous at ± 2 .

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

102. True; if f has a y -intercept.

104. False

$$\text{Let } f(x) = x \text{ or } g(x) = 1/x.$$

106. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/f'(g(x))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

Section 5.4 Exponential Functions: Differentiation and Integration

2. $e^{-2} = 0.1353\dots$

$$\ln 0.1353\dots = -2$$

4. $\ln 0.5 = -0.6931\dots$

$$e^{-0.6931\dots} = \frac{1}{2}$$

6. $e^{\ln 2x} = 12$

$$2x = 12$$

$$x = 6$$

8. $4e^x = 83$

$$e^x = \frac{83}{4}$$

$$x = \ln\left(\frac{83}{4}\right) \approx 3.033$$

10. $-6 + 3e^x = 8$

$$3e^x = 14$$

$$e^x = \frac{14}{3}$$

$$x = \ln\left(\frac{14}{3}\right) \approx 1.540$$

12. $200e^{-4x} = 15$

$$e^{-4x} = \frac{15}{200} = \frac{3}{40}$$

$$-4x = \ln\left(\frac{3}{40}\right)$$

$$x = \frac{1}{4} \ln\left(\frac{40}{3}\right) \approx 0.648$$

14. $\ln x^2 = 10$

$$x^2 = e^{10}$$

$$x = \pm e^5 \approx \pm 148.4132$$

16. $\ln 4x = 1$

$$4x = e^1 = e$$

$$x = \frac{e}{4} \approx 0.680$$

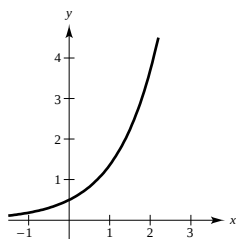
18. $\ln(x - 2)^2 = 12$

$$(x - 2)^2 = e^{12}$$

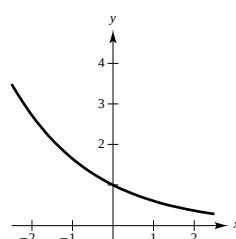
$$x - 2 = e^6$$

$$x = 2 + e^6 \approx 405.429$$

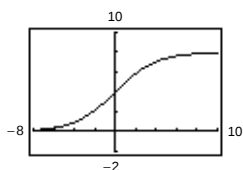
20. $y = \frac{1}{2}e^x$



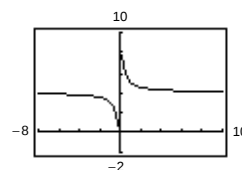
22. $y = e^{-x/2}$



24. (a)


 Horizontal asymptotes: $y = 0$ and $y = 8$

(b)


 Horizontal asymptote: $y = 4$

26. $y = Ce^{-ax}$

 Horizontal asymptote: $y = 0$

 Reflection in the y -axis

Matches (d)

28. $y = \frac{C}{1 + e^{-ax}}$

$$\lim_{x \rightarrow \infty} \frac{C}{1 + e^{-ax}} = C$$

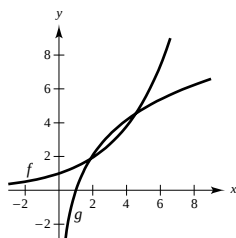
$$\lim_{x \rightarrow -\infty} \frac{C}{1 + e^{-ax}} = 0$$

 Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

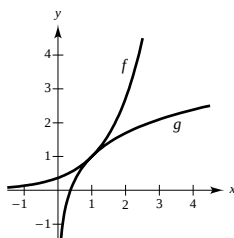
30. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$



32. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



34. In the same way,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

36. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 2.71825396$

$e \approx 2.718281828$

$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

38. (a) $y = e^{2x}$

$y' = 2e^{2x}$

 At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

 At $(0, 1)$, $y' = -2$.

40. $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

42. $y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

44. $y = x^2e^{-x}$

$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x}$

$= xe^{-x}(2 - x)$

46. $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \frac{6}{t^3e^{3/t^2}}$

48. $y = \ln\left(\frac{1 + e^x}{1 - e^x}\right)$

$= \ln(1 + e^x) - \ln(1 - e^x)$

$\frac{dy}{dx} = \frac{e^x}{1 + e^x} + \frac{e^x}{1 - e^x}$

$= \frac{2e^x}{1 - e^{2x}}$

50. $y = \ln\left(\frac{e^x + e^{-x}}{2}\right)$

$= \ln(e^x + e^{-x}) - \ln 2$

$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$= \frac{e^{2x} - 1}{e^{2x} + 1}$

52. $y = \frac{e^x - e^{-x}}{2}$

$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$

54. $y = xe^x - e^x = e^x(x - 1)$

$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$

56. $f(x) = e^3 \ln x$

$$f'(x) = \frac{e^3}{x}$$

58. $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

60. $e^{xy} + x^2 - y^2 = 10$

$$\left(x \frac{dy}{dx} + y\right)e^{xy} + 2x - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = \frac{-ye^{xy} - 2x}{xe^{xy} - 2y}$$

62. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$

$$\begin{aligned} g''(x) &= -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x \\ &= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x \end{aligned}$$

64. $y = e^x(3 \cos 2x - 4 \sin 2x)$

$$\begin{aligned} y' &= e^x(-6 \sin 2x - 8 \cos 2x) + e^x(3 \cos 2x - 4 \sin 2x) \\ &= e^x(-10 \sin 2x - 5 \cos 2x) = -5e^x(2 \sin 2x + \cos 2x) \end{aligned}$$

$$y'' = -5e^x(4 \cos 2x - 2 \sin 2x) - 5e^x(2 \sin 2x + \cos 2x) = -5e^x(5 \cos 2x) = -25e^x \cos 2x$$

$$y'' - 2y' = -25e^x \cos 2x - 2(-5e^x)(2 \sin 2x + \cos 2x) = -5e^x(3 \cos 2x - 4 \sin 2x) = -5y$$

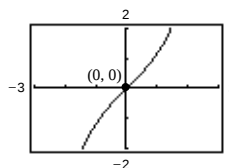
$$\text{Therefore, } y'' - 2y' = -5y \Rightarrow y'' - 2y' + 5y = 0.$$

66. $f(x) = \frac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$



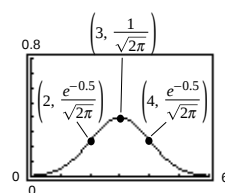
68. $g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}$

$$g'(x) = \frac{-1}{\sqrt{2\pi}}(x-3)e^{-(x-3)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}}(x-2)(x-4)e^{-(x-3)^2/2}$$

$$\text{Relative maximum: } \left(3, \frac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$$

$$\text{Points of inflection: } \left(2, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$$



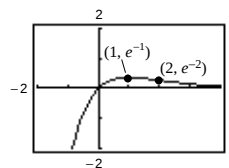
70. $f(x) = xe^{-x}$

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) = 0 \text{ when } x = 1.$$

$$f''(x) = -e^{-x} + (-e^{-x})(1-x) = e^{-x}(x-2) = 0 \text{ when } x = 2.$$

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



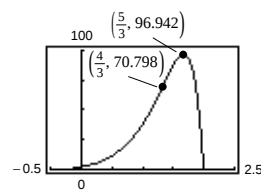
72. $f(x) = -2 + e^{3x}(4 - 2x)$

$$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0 \text{ when } x = \frac{5}{3}.$$

$$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0 \text{ when } x = \frac{4}{3}.$$

Relative maximum: $(\frac{5}{3}, 96.942)$

Point of inflection: $(\frac{4}{3}, 70.798)$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

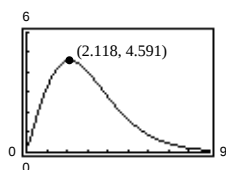
$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

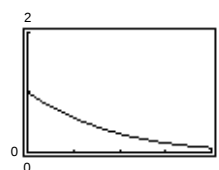
(c) $A(x) = \frac{10x^2}{e^x - 1} e^{x/(1-e^x)}$



The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(b)
$$A(x) = xf(c) = x \left[10 \left(\frac{x}{e^x - 1} \right) e^{-(x/(e^x - 1))} \right]$$
$$= \frac{10x^2}{e^x - 1} e^{x/(1-e^x)}$$

(d) $c = \frac{x}{e^x - 1}$



$$\lim_{x \rightarrow 0^+} c = 1$$

$$\lim_{x \rightarrow \infty} c = 0$$

76. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad (\text{Slope of tangent line})$$

$$-\frac{1}{y'} = e^x \quad (\text{Slope of normal line})$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$

We want $(0, 0)$ to satisfy the equation:

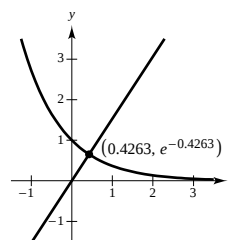
$$-e^{-x_0} = -x_0 e^{x_0}$$

$$1 = x_0 e^{2x_0}$$

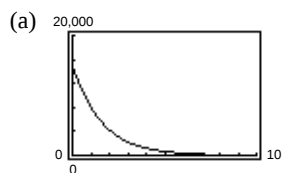
$$x_0 e^{2x_0} - 1 = 0$$

Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$$(0.4263, e^{-0.4263})$$



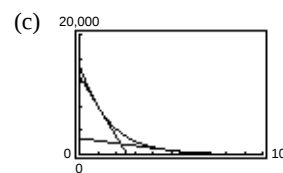
78. $V = 15,000e^{-0.6286t}, 0 \leq t \leq 10$



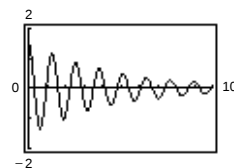
(b) $\frac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1$, $\frac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\frac{dV}{dt} \approx -406.89$.

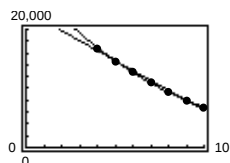


80. $1.56e^{-0.22t} \cos 4.9t \leq 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \geq 7.79$ seconds.



82. (a) $V_1 = -1686.79t + 23,181.79$

$V_2 = 109.52t^2 - 3220.12t + 28,110.36$



(b) The slope represents the rate of decrease in value of the car.

(c) $V_3 = 31,450.77(0.8592)^t = 31,450.77e^{-0.1518t}$

(d) Horizontal asymptote: $\lim_{t \rightarrow \infty} V_3(t) = 0$

As $t \rightarrow \infty$, the value of the car approaches 0.

(e) $\frac{dV_3}{dt} = -4774.2e^{-0.1518t}$

For $t = 5$, $\frac{dV_3}{dt} \approx -2235$ dollars/year.

For $t = 9$, $\frac{dV_3}{dt} \approx -1218$ dollars/year.

84. $f(x) = e^{-x^2/2}, f(0) = 1$

$f'(x) = -xe^{-x^2/2}, f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), f''(0) = -1$

$P_1(x) = 1 + 0(x - 0) = 1, P_1(0) = 1$

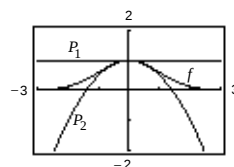
$P_1'(x) = 0, P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \frac{1}{2}(x - 0)^2 = 1 - \frac{x^2}{2}, P_2(0) = 1$

$P_2'(x) = -x, P_2'(0) = 0$

$P_2''(x) = -1, P_2''(0) = -1$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.



86. n^{th} term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Conjecture: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

90. $\int_3^4 e^{3-x} dx = \left[-e^{3-x} \right]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$

94. $\int \frac{e^{1/x^2}}{x^3} dx = -\frac{1}{2} \int e^{1/x^2} \left(\frac{-2}{x^3} \right) dx = -\frac{1}{2} e^{1/x^2} + C$

98. Let $u = \frac{-x^2}{2}$, $du = -x dx$.

$$\int_0^{\sqrt{2}} x e^{-x^2/2} dx = - \int_0^{\sqrt{2}} e^{-x^2/2} (-x) dx = \left[-e^{-x^2/2} \right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

100. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln(e^x + e^{-x}) + C$$

104. $\int \frac{e^{2x} + 2e^x + 1}{e^x} dx = \int (e^x + 2 + e^{-x}) dx$
 $= e^x + 2x - e^{-x} + C$

108. $\int \ln(e^{2x-1}) dx = \int (2x - 1) dx$
 $= x^2 - x + C$

112. $f'(x) = \int (\sin x + e^{2x}) dx = -\cos x + \frac{1}{2} e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2} e^{2x} + 1$$

$$f(x) = \int \left(-\cos x + \frac{1}{2} e^{2x} + 1 \right) dx$$

$$= -\sin x + \frac{1}{4} e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \Rightarrow C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4} e^{2x}$$

88. Let $u = -x^4$, $du = -4x^3 dx$.

$$\int e^{-x^4} (-4x^3) dx = e^{-x^4} + C$$

92. $\int x^2 e^{x^3/2} dx = \frac{2}{3} \int e^{x^3/2} \left(\frac{3x^2}{2} \right) dx = \frac{2}{3} e^{x^3/2} + C$

96. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \ln(1 + e^{2x}) + C$$

102. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x}) dx$.

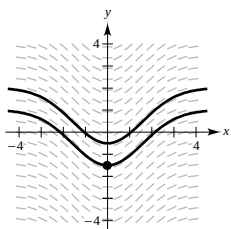
$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx = 2 \int (e^x + e^{-x})^{-2} (e^x - e^{-x}) dx$$

$$= \frac{-2}{e^x + e^{-x}} + C$$

106. $\int e^{\sec 2x} \sec 2x \tan 2x dx = \frac{1}{2} e^{\sec 2x} + C$
 $(u = \sec 2x, du = 2 \sec 2x \tan 2x)$

110. $y = \int (e^x - e^{-x})^2 dx$
 $= \int (e^{2x} - 2 + e^{-2x}) dx$
 $= \frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} + C$

114. (a)



(b) $\frac{dy}{dx} = xe^{-0.2x^2}, \left(0, -\frac{3}{2}\right)$

$$y = \int xe^{-0.2x^2} dx = \frac{1}{-0.4} \int e^{-0.2x^2} (-0.4x) dx$$

$$= -\frac{1}{0.4} e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$$\left(0, -\frac{3}{2}\right): -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \Rightarrow C = 1$$

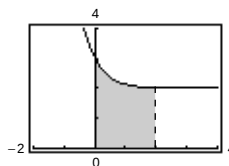
$$y = -2.5e^{-0.2x^2} + 1$$

116. $\int_a^b e^{-x} dx = \left[-e^{-x}\right]_a^b = e^{-a} - e^{-b}$



118. $\int_0^2 (e^{-2x} + 2) dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



120. (a) $\int_0^4 \sqrt{x} e^x dx, n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\int_0^2 2xe^{-x} dx, n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

122. $\int_0^x 0.3^{-0.3t} dt = \frac{1}{2}$

$$\left[-e^{-0.3t}\right]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln \frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

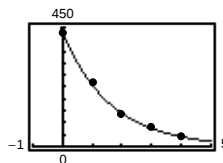
124.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$$

(b)



(c) $\int_0^4 R(t) dt = \int_0^4 428.78e^{-0.6155t} dt \approx 637.2 \text{ liters}$

126. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

128. (a) Log Rule: ($u = e^x + 1$)

(b) Substitution: ($u = x^2$)

130. $\ln \frac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$$\ln e^{a-b} = a - b$$

Therefore, $\ln \frac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\frac{e^a}{e^b} = e^{a-b}$.

Section 5.5 Bases Other than e and Applications

2. $y = \left(\frac{1}{2}\right)^{t/8}$

At $t_0 = 16$, $y = \left(\frac{1}{2}\right)^{16/8} = \frac{1}{4}$

4. $y = \left(\frac{1}{2}\right)^{t/5}$

At $t_0 = 2$, $y = \left(\frac{1}{2}\right)^{2/5} \approx 0.7579$

6. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

8. $\log_a \frac{1}{a} = \log_a 1 - \log_a a = -1$

10. (a) $27^{2/3} = 9$

$$\log_{27} 9 = \frac{2}{3}$$

(b) $16^{3/4} = 8$

$$\log_{16} 8 = \frac{3}{4}$$

12. (a) $\log_3 \frac{1}{9} = -2$

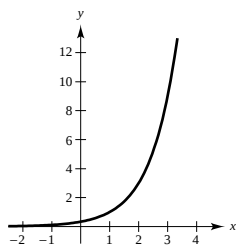
$$3^{-2} = \frac{1}{9}$$

(b) $49^{1/2} = 7$

$$\log_{49} 7 = \frac{1}{2}$$

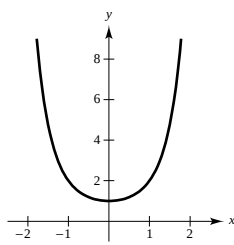
14. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



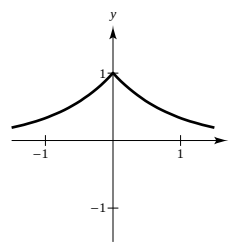
16. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



18. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



20. (a) $\log_3 \frac{1}{81} = x$

$$3^x = \frac{1}{81}$$

$$x = -4$$

(b) $\log_6 36 = x$

$$6^x = 36$$

$$x = 2$$

22. (a) $\log_b 27 = 3$

$$b^3 = 27$$

$$b = 3$$

(b) $\log_b 125 = 3$

$$b^3 = 125$$

$$b = 5$$

24. (a) $\log_3 x + \log_3(x - 2) = 1$

$$\log_3[x(x - 2)] = 1$$

$$x(x - 2) = 3^1$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x + 3) - \log_{10} x = 1$

$$\log_{10} \frac{x + 3}{x} = 1$$

$$\frac{x + 3}{x} = 10^1$$

$$x + 3 = 10x$$

$$3 = 9x$$

$$x = \frac{1}{3}$$

26. $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$

$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

28. $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$

$$(x - 1) \ln 5 = \ln\left(\frac{86}{3}\right)$$

$$x - 1 = \frac{\ln\left(\frac{86}{3}\right)}{\ln 5}$$

$$x = 1 + \frac{\ln\left(\frac{86}{3}\right)}{\ln 5} \approx 3.085$$

30. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

32. $\log_{10}(t - 3) = 2.6$

$$t - 3 = 10^{2.6}$$

$$t = 3 + 10^{2.6} \approx 401.107$$

34. $\log_5 \sqrt{x - 4} = 3.2$

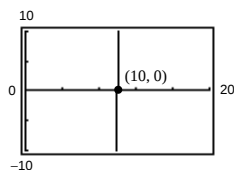
$$\sqrt{x - 4} = 5^{3.2}$$

$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$

$$x = 4 + 5^{6.4} \approx 29,748.593$$

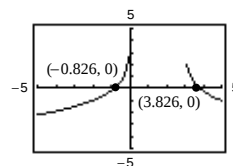
36. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



38. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

Zeros: $x \approx -0.826, 3.826$

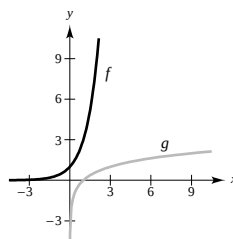


40. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



42. $g(x) = 2^{-x}$

$g'(x) = -(\ln 2) 2^{-x}$

44. $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}] + 6^{-2x}$$

$$= 6^{-2x}[-2x(\ln 6) + 1]$$

$$= 6^{-2x}(1 - 2x \ln 6)$$

46. $f(t) = \frac{3^{2t}}{t}$

$$f'(t) = \frac{t(2 \ln 3) 3^{2t} - 3^{2t}}{t^2}$$

$$= \frac{3^{2t}(2t \ln 3 - 1)}{t^2}$$

48. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$$g'(\alpha) = 5^{-\alpha/2} 2 \cos 2\alpha - \frac{1}{2}(\ln 5) 5^{-\alpha/2} \sin 2\alpha$$

50. $y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$

$$\frac{dy}{dx} = 0 + \frac{1}{x \ln 10} = \frac{1}{x \ln 10}$$

52. $h(x) = \log_3 \frac{x\sqrt{x-1}}{2}$

$$= \log_3 x + \frac{1}{2} \log_3 (x-1) - \log_3 2$$

$$h'(x) = \frac{1}{x \ln 3} + \frac{1}{2} \cdot \frac{1}{(x-1) \ln 3} - 0$$

$$= \frac{1}{\ln 3} \left[\frac{1}{x} + \frac{1}{2(x-1)} \right]$$

$$= \frac{1}{\ln 3} \left[\frac{3x-2}{2x(x-1)} \right]$$

54. $y = \log_{10} \frac{x^2-1}{x}$

$$= \log_{10}(x^2-1) - \log_{10} x$$

$$\frac{dy}{dx} = \frac{2x}{(x^2-1) \ln 10} - \frac{1}{x \ln 10}$$

$$= \frac{1}{\ln 10} \left[\frac{2x}{x^2-1} - \frac{1}{x} \right]$$

$$= \frac{1}{\ln 10} \left[\frac{x^2+1}{x(x^2-1)} \right]$$

56. $f(t) = t^{3/2} \log_2 \sqrt{t+1} = t^{3/2} \frac{1}{2} \frac{\ln(t+1)}{\ln 2}$

$$f'(t) = \frac{1}{2 \ln 2} \left[t^{3/2} \frac{1}{t+1} + \frac{3}{2} t^{1/2} \ln(t+1) \right]$$

58. $y = x^{x-1}$

$$\ln y = (x-1)(\ln x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x-1) \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = y \left[\frac{x-1}{x} + \ln x \right]$$

$$= x^{x-2} (x-1 + x \ln x)$$

60. $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} \left(\frac{1}{1+x} \right) + \ln(1+x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{y}{x} \left[\frac{1}{x+1} - \frac{\ln(x+1)}{x} \right]$$

$$= \frac{(1+x)^{1/x}}{x} \left[\frac{1}{x+1} - \frac{\ln(x+1)}{x} \right]$$

62. $\int 5^{-x} dx = \frac{-5^{-x}}{\ln 5} + C$

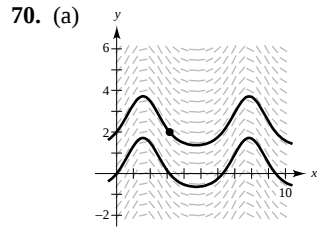
64. $\int_{-2}^0 (3^3 - 5^2) dx = \int_{-2}^0 (27 - 25) dx$

$$= \int_{-2}^0 2 dx$$

$$= \left[2x \right]_{-2}^0 = 4$$

$$\begin{aligned}
 66. \int (3-x) 7^{(3-x)^2} dx &= -\frac{1}{2} \int -2(3-x) 7^{(3-x)^2} dx \\
 &= -\frac{1}{2 \ln 7} [7^{(3-x)^2}] + C
 \end{aligned}$$

$$\begin{aligned}
 68. \int 2^{\sin x} \cos x dx, u &= \sin x, du = \cos x dx \\
 &= \frac{1}{\ln 2} 2^{\sin x} + C
 \end{aligned}$$



$$\begin{aligned}
 (b) \frac{dy}{dx} &= e^{\sin x} \cos x \quad (\pi, 2) \\
 y &= \int e^{\sin x} \cos x dx = e^{\sin x} + C \\
 (\pi, 2): 2 &= e^{\sin \pi} + C = 1 + C \Rightarrow C = 1 \\
 y &= e^{\sin x} + 1
 \end{aligned}$$

$$72. \log_b x = \frac{\ln x}{\ln b} = \frac{\log_{10} x}{\log_{10} b}$$

$$74. f(x) = \log_{10} x$$

$$(a) \text{ Domain: } x > 0$$

$$(b) y = \log_{10} x$$

$$10^y = x$$

$$f^{-1}(x) = 10^x$$

$$(c) \log_{10} 1000 = \log_{10} 10^3 = 3$$

$$\log_{10} 10,000 = \log_{10} 10^4 = 4$$

$$\text{If } 1000 \leq x \leq 10,000, \text{ then } 3 \leq f(x) \leq 4.$$

$$(d) \text{ If } f(x) < 0, \text{ then } 0 < x < 1.$$

$$\begin{aligned}
 (e) f(x) + 1 &= \log_{10} x + \log_{10} 10 \\
 &= \log_{10}(10x)
 \end{aligned}$$

x must have been increased by a factor of 10.

$$\begin{aligned}
 (f) \log_{10} \left(\frac{x_1}{x_2} \right) &= \log_{10} x_1 - \log_{10} x_2 \\
 &= 3n - n = 2n
 \end{aligned}$$

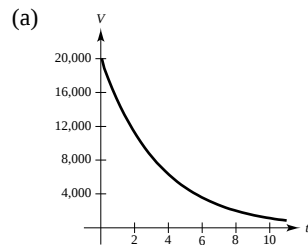
$$\text{Thus, } x_1/x_2 = 10^{2n} = 100^n.$$

$$76. f(x) = a^x$$

$$(a) f(u+v) = a^{u+v} = a^u a^v = f(u) f(v)$$

$$(b) f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$$

$$78. V(t) = 20,000 \left(\frac{3}{4} \right)^t$$

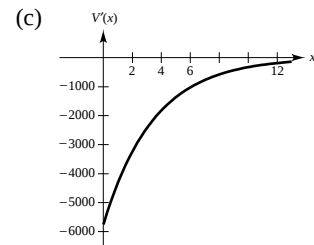


$$V(2) = 20,000 \left(\frac{3}{4} \right)^2 = \$11,250$$

$$(b) \frac{dV}{dt} = 20,000 \left(\ln \frac{3}{4} \right) \left(\frac{3}{4} \right)^t$$

$$\text{When } t = 1: \frac{dV}{dt} \approx -4315.23$$

$$\text{When } t = 4: \frac{dV}{dt} \approx -1820.49$$



Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

80. $P = \$2500$, $r = 6\% = 0.06$, $t = 20$

$$A = 2500 \left(1 + \frac{0.06}{n} \right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

82. $P = \$5000$, $r = 7\% = 0.07$, $t = 25$

$$A = 5000 \left(1 + \frac{0.07}{n} \right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

84. $100,000 = Pe^{0.06t} \Rightarrow P = 100,000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

86. $100,000 = P \left(1 + \frac{0.07}{365} \right)^{365t} \Rightarrow P = 100,000 \left(1 + \frac{0.07}{365} \right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

88. Let $P = \$100$, $0 \leq t \leq 20$.

(a) $A = 100e^{0.03t}$

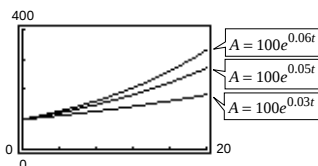
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



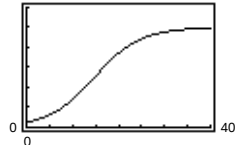
90. (a) $\lim_{n \rightarrow \infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b) $P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})}$

$P'(3) \approx 0.069$

$P'(10) \approx 0.016$

92. (a)



(b) Limiting size: 10,000 fish

(c) $p(t) = \frac{10,000}{1 + 19e^{-t/5}}$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2} \left(\frac{19}{5} \right) (10,000)$$

$$= \frac{38,000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

(d) $p''(t) = -\frac{38,000}{5}(e^{-t/5}) \left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3} \right] = 0$

$19e^{-t/5} = 1$

$\frac{t}{5} = \ln 19$

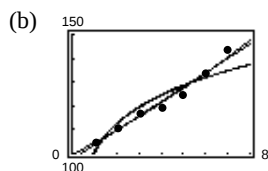
$t = 5 \ln 19 \approx 14.72$

94. (a) $y_1 = 6.0536x + 97.5571$

$$y_2 = 100.0751 + 17.8148 \ln x$$

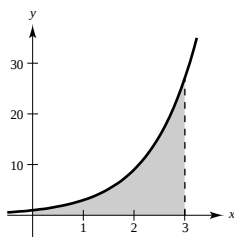
$$y_3 = 99.4557(1.0506)^x$$

$$y_4 = 101.2875x^{0.1471}$$



y_3 seems best.

96. $A = \int_0^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_0^3 = \frac{26}{\ln 3} \approx 23.666$



(c) The slope of 6.0536 is the annual rate of change in the amount given to philanthropy.

 (d) For 1996, $x = 6$ and $y_1' = 6.0536$, $y_2' \approx 2.9691$,

$$y_3' \approx 6.6015, y_4' \approx 3.2321.$$

y_3 is increasing at the greatest rate in 1996.

98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1+x)^{1/x}$	2	2.594	2.705	2.718	2.718

100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$$y = C(k^t)$$

When $t = 0$, $y = 600 \Rightarrow C = 600$.

$$y = 600(k^t)$$

$$\frac{630}{600} = 1.05, \frac{661.50}{630} = 1.05, \frac{694.58}{661.50} \approx 1.05, \frac{729.30}{694.58} \approx 1.05$$

Let $k = 1.05$.

$$y = 600(1.05)^t$$

102. True.

$$\begin{aligned} f(e^{n+1}) - f(e^n) &= \ln e^{n+1} - \ln e^n \\ &= n + 1 - n \\ &= 1 \end{aligned}$$

104. True.

$$\begin{aligned} \frac{d^n y}{dx^n} &= Ce^x \\ &= y \text{ for } n = 1, 2, 3, \dots \end{aligned}$$

106. True.

$$\begin{aligned} f(x) &= g(x)e^x = 0 \Rightarrow \\ g(x) &= 0 \text{ since } e^x > 0 \text{ for all } x. \end{aligned}$$

108. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x \left(\frac{1}{x} \right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x} \left[\frac{\sin x}{x} + \cos x \cdot \ln x \right]$$

$$\text{At } \left(\frac{\pi}{2}, \frac{\pi}{2} \right),$$

$$y' = \left(\frac{\pi}{2} \right)^{\sin(\pi/2)} \left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right) \ln\left(\frac{\pi}{2}\right) \right]$$

$$= \frac{\pi}{2} \left[\frac{2}{\pi} + 0 \right] = 1$$

$$\text{Tangent line: } y - \frac{\pi}{2} = 1 \left(x - \frac{\pi}{2} \right)$$

$$y = x$$

Section 5.6 Differential Equations: Growth and Decay

2. $\frac{dy}{dx} = 4 - x$

$$y = \int (4 - x) dx = 4x - \frac{x^2}{2} + C$$

4. $\frac{dy}{dx} = 4 - y$

$$\frac{dy}{4 - y} = dx$$

$$\int \frac{-1}{4 - y} dy = \int -dx$$

$$\ln|4 - y| dy = -x + C_1$$

$$4 - y = e^{-x+C_1} = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

6. $y' = \frac{\sqrt{x}}{3y}$

$$3yy' = \sqrt{x}$$

$$\int 3yy' dx = \int \sqrt{x} dx$$

$$\frac{3y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$

$$9y^2 - 4x^{3/2} = C$$

8. $y' = x(1 + y)$

$$\frac{y'}{1 + y} = x$$

$$\int \frac{y'}{1 + y} dx = \int x dx$$

$$\int \frac{dy}{1 + y} = \int x dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C_1$$

$$1 + y = e^{(x^2/2)+C_1}$$

$$y = e^{C_1} e^{x^2/2} - 1$$

$$= Ce^{x^2/2} - 1$$

10. $xy + y' = 100x$

$$y' = 100x + xy = x(100 + y)$$

$$\frac{y'}{100 + y} = x$$

$$\int \frac{y'}{100 + y} dx = \int x dx$$

$$\int \frac{1}{100 + y} dy = \int x dx$$

$$-\ln(100 + y) = \frac{x^2}{2} + C_1$$

$$\ln(100 + y) = -\frac{x^2}{2} - C_1$$

$$100 + y = e^{-(x^2/2)-C_1}$$

$$-y = e^{-C_1} e^{-x^2/2} - 100$$

$$y = 100 - Ce^{-x^2/2}$$

$$12. \quad \frac{dP}{dt} = k(10 - t)$$

$$\int \frac{dP}{dt} dt = \int k(10 - t) dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

$$14. \quad \frac{dy}{dx} = kx(L - y)$$

$$\frac{1}{L - y} \frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y} \frac{dy}{dx} dx = \int kx dx$$

$$\int \frac{1}{L - y} dy = \frac{kx^2}{2} + C_1$$

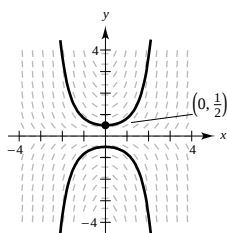
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1} e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

16. (a)



$$(b) \quad \frac{dy}{dx} = xy, \quad \left(0, \frac{1}{2}\right)$$

$$\frac{dy}{y} = x dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right): \frac{1}{2} = C_1 e^0 \Rightarrow C_1 = \frac{1}{2} \Rightarrow y = \frac{1}{2} e^{x^2/2}$$

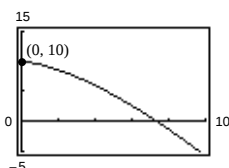
$$18. \quad \frac{dy}{dt} = -\frac{3}{4} \sqrt{t}, \quad (0, 10)$$

$$\int dy = \int -\frac{3}{4} \sqrt{t} dt$$

$$y = -\frac{1}{2} t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \Rightarrow C = 10$$

$$y = -\frac{1}{2} t^{3/2} + 10$$



$$20. \quad \frac{dy}{dt} = \frac{3}{4} y, \quad (0, 10)$$

$$\int \frac{dy}{y} = \int \frac{3}{4} dt$$

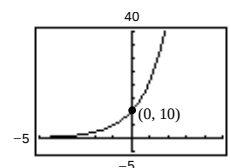
$$\ln y = \frac{3}{4} t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1} e^{(3/4)t} = C e^{3t/4}$$

$$10 = C e^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$



$$22. \quad \frac{dN}{dt} = kN$$

$$N = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 250): C = 250$$

$$(1, 400): 400 = 250e^k \Rightarrow k = \ln \frac{400}{250} = \ln \frac{8}{5}$$

$$\text{When } t = 4, N = 250e^{4 \ln(8/5)} = 250e^{\ln(8/5)^4}$$

$$= 250 \left(\frac{8}{5}\right)^4 = \frac{8192}{5}$$

$$24. \quad \frac{dP}{dt} = kP$$

$$P = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 5000): C = 5000$$

$$(1, 4750): 4750 = 5000e^k \Rightarrow k = \ln \left(\frac{19}{20}\right)$$

$$\text{When } t = 5, P = 5000e^{\ln(19/20)(5)}$$

$$= 5000 \left(\frac{19}{20}\right)^5 \approx 3868.905$$

26. $y = Ce^{kt}$, $(0, 4)$, $\left(5, \frac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

28. $y = Ce^{kt}$, $\left(3, \frac{1}{2}\right)$, $(4, 5)$

$$\frac{1}{2} = Ce^{3k}$$

$$5 = Ce^{4k}$$

$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$

$$10e^{3k} = e^{4k}$$

$$10 = e^k$$

$$k = \ln 10 \approx 2.3026$$

$$y = Ce^{2.3026t}$$

$$5 = Ce^{2.3026(4)}$$

$$C \approx 0.0005$$

$$y = 0.0005e^{2.3026t}$$

30. $y' = \frac{dy}{dt} = ky$

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

$$\frac{dy}{dx} > 0 \text{ when } y > 0. \text{ Quadrants I and II.}$$

34. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $1.5 = Ce^{[\ln(1/2)/1620](1000)} \Rightarrow C \approx 2.30$ which implies that the initial quantity is 2.30 grams. When $t = 10,000$, we have $y = 2.30e^{[\ln(1/2)/1620](10,000)} \approx 0.03$ gram.

36. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $2.0 = Ce^{[\ln(1/2)/5730](10,000)} \Rightarrow C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{[\ln(1/2)/5730](1000)} \approx 5.94$ grams.

38. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $3.2 = Ce^{[\ln(1/2)/5730]1000} \Rightarrow C \approx 3.61$.

Initial quantity: 3.61 grams.

When $t = 10,000$, we have $y \approx 1.08$ grams.

40. Since $y = Ce^{[\ln(1/2)/24,360]t}$, we have $0.4 = Ce^{[\ln(1/2)/24,360](10,000)} \Rightarrow C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{[\ln(1/2)/24,360](1000)} \approx 0.52$ gram.

42. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{5730k}$$

$$k = -\frac{\ln 2}{5730}$$

$$0.15y_0 = y_0e^{(-\ln 2/5730)t}$$

$$\ln 0.15 = -\frac{(\ln 2)t}{5730}$$

$$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15,682.8 \text{ years.}$$

44. Since $A = 20,000e^{0.055t}$, the time to double is given by $40,000 = 20,000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years:

$$A = 20,000e^{(0.055)(10)} \approx \$34,665.06$$

46. Since $A = 10,000e^{rt}$ and $A = 20,000$ when $t = 5$, we have the following.

$$20,000 = 10,000e^{5r}$$

$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

$$\text{Amount after 10 years: } A = 10,000e^{[(\ln 2)/5](10)} = \$40,000$$

48. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$

$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$

$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

50. $500,000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500,000(1.005)^{-480} \approx \$45,631.04$$

52. $500,000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 500,000\left(1 + \frac{0.09}{12}\right)^{-300} \\ \approx \$53,143.92$$

54. (a) $2000 = 1000(1 + 0.6)^t$

$$2 = 1.06^t$$

$$\ln 2 = t \ln 1.06$$

$$t = \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.06}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.06}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.06}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.06}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.06}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.06}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55 \text{ years}$$

(d) $2000 = 1000e^{0.06t}$

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

56. (a) $2000 = 1000(1 + 0.055)^t$

$$2 = 1.055^t$$

$$\ln 2 = t \ln 1.055$$

$$t = \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.055}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.055}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.055}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.055}{12}\right)} \approx 12.63 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.055}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.055}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.055}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.055}{365}\right)} \approx 12.60 \text{ years}$$

(d) $2000 = 1000e^{0.055t}$

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.60 \text{ years}$$

58. $P = Ce^{kt} = Ce^{0.031t}$

$$P(-1) = 11.6 = Ce^{0.031(-1)} \Rightarrow C = 11.9652$$

$$P = 11.9652e^{0.031t}$$

$$P(10) \approx 16.31 \text{ or } 16,310,000 \text{ people in 2010}$$

60. $P = Ce^{kt} = Ce^{-0.004t}$

$$P(-1) = 3.6 = Ce^{-0.004(-1)} \Rightarrow C = 3.5856$$

$$P = 3.5856e^{-0.004t}$$

$$P(10) \approx 3.45 \text{ or } 3,450,000 \text{ people in 2010}$$

62. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

64. $y = Ce^{kt}$, $(0, 742,000)$, $(2, 632,000)$

$$C = 742,000$$

$$632,000 = 742,000e^{2k}$$

$$k = \frac{\ln(632/742)}{2} \approx -0.0802$$

$$y \approx 742,000e^{-0.0802t}$$

$$\text{When } t = 4, y \approx \$538,372.$$

66. (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

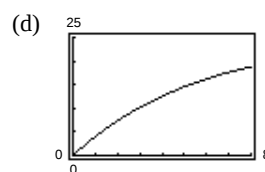
$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

68. $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \Rightarrow 1 - e^k = \frac{4}{25} \Rightarrow e^k = \frac{21}{25} \Rightarrow k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

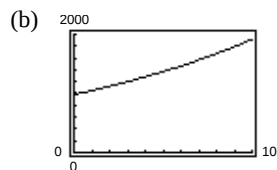
(b) 25,000 units $\left(\lim_{t \rightarrow \infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

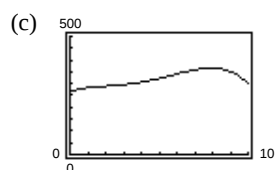


70. (a) $R = 979.3993(1.0694)^t = 979.3993e^{0.0671t}$

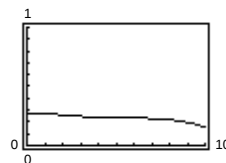
$$I = -0.1385t^4 + 2.1770t^3 - 9.9755t^2 + 23.8513t + 266.4923$$



$$\text{Rate of growth} = R'(t) = 65.7e^{0.0671t}$$



(d) $P(t) = \frac{I}{R}$



$$72. \quad 93 = 10 \log_{10} \frac{I}{10^{-16}} = 10(\log_{10} I + 16)$$

$$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$$

$$80 = 10 \log_{10} \frac{I}{10^{-16}} = 10(\log_{10} I + 16)$$

$$-8 = \log_{10} I \Rightarrow I = 10^{-8}$$

$$\text{Percentage decrease: } \left(\frac{10^{-6.7} - 10^{-8}}{10^{-6.7}} \right) (100) \approx 95\%$$

$$74. \quad \text{Since } \frac{dy}{dt} = k(y - 80)$$

$$\int \frac{1}{y - 80} dy = \int k dt$$

$$\ln(y - 80) = kt + C.$$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1420$.

When $t = 1$, $y = 1120$. Thus,

$$k(1) + \ln 1420 = \ln(1120 - 80)$$

$$k = \ln 1040 - \ln 1420 = \ln \frac{104}{142}.$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5$, $y \approx 379.2^\circ$.

76. True

78. True

Section 5.7 Differential Equations: Separation of Variables

$$2. \text{ Differential equation: } y' = \frac{2xy}{x^2 - y^2}$$

$$\text{Solution: } x^2 + y^2 = Cy$$

$$\text{Check: } 2x + 2yy' = Cy'$$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

$$4. \text{ Differential equation: } y'' + 2y' + 2y = 0$$

$$\text{Solution: } y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$$

$$\text{Check: } y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x +$$

$$2(-C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x + 2(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

$$6. \quad y = \frac{2}{3}(e^{-2x} + e^x)$$

$$y' = \frac{2}{3}(-2e^{-2x} + e^x)$$

$$y'' = \frac{2}{3}(4e^{-2x} + e^x)$$

$$\text{Substituting, } y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x.$$

In Exercises 8–12, the differential equation is $y^{(4)} - 16y = 0$.

8. $y = 3 \cos 2x$

$$y^{(4)} = 48 \cos 2x$$

$$y^{(4)} - 16y = 48 \cos 2x - 48 \cos 2x = 0, \quad \text{Yes.}$$

10. $y = 5 \ln x$

$$y^{(4)} = -\frac{30}{x^4}$$

$$y^{(4)} - 16y = -\frac{30}{x^4} - 80 \ln x \neq 0, \quad \text{No.}$$

12. $y = 3e^{2x} - 4 \sin 2x$

$$y^{(4)} = 48e^{2x} - 64 \sin 2x$$

$$y^{(4)} - 16y = (48e^{2x} - 64 \sin 2x) - 16(3e^{2x} - 4 \sin 2x) = 0, \quad \text{Yes}$$

In 14–18, the differential equation is $xy' - 2y = x^3e^x$.

14. $y = x^2e^x, y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x, \quad \text{Yes.}$$

16. $y = \sin x, y' = \cos x$

$$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x, \quad \text{No.}$$

18. $y = x^2e^x - 5x^2, y' = x^2e^x + 2xe^x - 10x$

$$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x, \quad \text{Yes.}$$

20. $y = A \sin \omega t$

$$\frac{d^2y}{dt^2} = -A\omega^2 \sin \omega t$$

Since $(d^2y/dt^2) + 16y = 0$, we have

$$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0.$$

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

22. $2x^2 - y^2 = C$ passes through (3, 4)

$$2(9) - 16 = C \Rightarrow C = 2$$

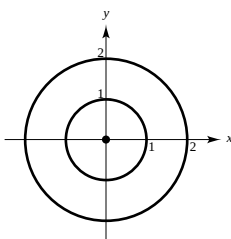
Particular solution: $2x^2 - y^2 = 2$

24. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1, C = 4$, Circles



26. Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$$6x + 4yy' = 0$$

$$2(3x + 2yy') = 0$$

$$3x + 2yy' = 0$$

Initial condition:

$$y(1) = 3: 3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$$

Particular solution: $3x^2 + 2y^2 = 21$

28. Differential equation: $xy'' + y' = 0$ General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2 \left(\frac{1}{x} \right), y'' = -C_2 \left(\frac{1}{x^2} \right)$$

$$xy'' + y' = x \left(-C_2 \frac{1}{x^2} \right) + C_2 \frac{1}{x} = 0$$

Initial conditions: $y(2) = 0, y'(2) = \frac{1}{2}$

$$0 = C_1 + C_2 \ln 2$$

$$y' = \frac{C_2}{x}$$

$$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1, C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$ 30. Differential equation: $9y'' - 12y' + 4y = 0$ General solution: $y = e^{2x/3}(C_1 + C_2 x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2 x) + C_2 e^{2x/3} = e^{2x/3} \left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x \right)$$

$$y'' = \frac{2}{3}e^{2x/3} \left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x \right) + e^{2x/3} \frac{2}{3}C_2 = \frac{2}{3}e^{2x/3} \left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2 x \right)$$

$$9y'' - 12y' + 4y = 9 \left(\frac{2}{3}e^{2x/3} \right) \left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2 x \right) - 12 \left(e^{2x/3} \right) \left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2 x \right) + 4 \left(e^{2x/3} \right) (C_1 + C_2 x) = 0$$

Initial conditions: $y(0) = 4, y(3) = 0$

$$0 = e^2(C_1 + 3C_2)$$

$$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$$

$$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\frac{4}{3}$$

Particular solution: $y = e^{2x/3} \left(4 - \frac{4}{3}x \right)$ 32. $\frac{dy}{dx} = x^3 - 4x$

$$y = \int (x^3 - 4x) dx = \frac{x^4}{4} - 2x^2 + C$$

34. $\frac{dy}{dx} = \frac{e^x}{1 + e^x}$

$$y = \int \frac{e^x}{1 + e^x} dx = \ln(1 + e^x) + C$$

36. $\frac{dy}{dx} = x \cos x^2$

$$y = \int x \cos(x^2) dx = \frac{1}{2} \sin(x^2) + C$$

$$(u = x^2, du = 2x dx)$$

38. $\frac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int (\sec^2 x - 1) dx = \tan x - x + C$$

40. $\frac{dy}{dx} = x\sqrt{5-x}$. Let $u = \sqrt{5-x}, u^2 = 5-x, dx = -2u du$

$$y = \int x\sqrt{5-x} dx = \int (5 - u^2)u(-2u) du$$

$$= \int (-10u^2 + 2u^4) du$$

$$= -\frac{10u^3}{3} + \frac{2u^5}{5} + C$$

$$= -\frac{10}{3}(5-x)^{3/2} + \frac{2}{5}(5-x)^{5/2} + C$$

42. $\frac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2} dx = 5(-2) \int e^{-x/2} \left(-\frac{1}{2}\right) dx$$

$$= -10e^{-x/2} + C$$

44. $\frac{dy}{dx} = \frac{x^2 + 2}{3y^2}$

$$\int 3y^2 dy = \int (x^2 + 2) dx$$

$$y^3 = \frac{x^3}{3} + 2x + C$$

46. $\frac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s ds$$

$$r = 0.025s^2 + C$$

48. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

50. $y \frac{dy}{dx} = 6 \cos \pi x$

$$\int y dy = \int 6 \cos \pi x dx$$

$$\frac{y^2}{2} = \frac{6}{\pi} \sin \pi x + C_1$$

$$y^2 = \frac{12}{\pi} \sin \pi x + C$$

52. $\sqrt{x^2 - 9} \frac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2 - 9}} dx$$

$$y = 5(x^2 - 9)^{1/2} + C$$

54. $4y \frac{dy}{dx} = 3e^x$

$$\int 4y dy = \int 3e^x dx$$

$$2y^2 = 3e^x + C$$

56. $\sqrt{x} + \sqrt{y} y' = 0$

$$\int y^{1/2} dy = - \int x^{1/2} dx$$

$$\frac{2}{3} y^{3/2} = -\frac{2}{3} x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4$,
 $(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

58. $2xy' - \ln x^2 = 0$

$$2x \frac{dy}{dx} = 2 \ln x$$

$$\int dy = \int \frac{\ln x}{x} dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

$y(1) = 2$: $2 = C$

$y = \frac{1}{2}(\ln x)^2 + 2$

60. $y\sqrt{1-x^2} \frac{dy}{dx} = x\sqrt{1-y^2}$

$$\int (1-y^2)^{-1/2} y dy = \int (1-x^2)^{-1/2} x dx$$

$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

$y(0) = 1$: $0 = -1 + C \Rightarrow C = 1$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

62. $\frac{dr}{ds} = e^{r-2s}$

$$\int e^{-r} dr = \int e^{-2s} ds$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$r(0) = 0$: $-1 = -\frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$

$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$

$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1+e^{-2s}}{2}\right)$$

$$r = \ln\left(\frac{2}{1+e^{-2s}}\right)$$

64. $dT + k(T - 70) dt = 0$

$$\int \frac{dT}{T - 70} = -k \int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;

$$140 - 70 = 70 = Ce^0 = C$$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

66. $\frac{dy}{dx} = \frac{2y}{3x}$

$$\int \frac{3}{y} dy = \int \frac{2}{x} dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \frac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \frac{1}{2}x^{2/3}$

68. $m = \frac{dy}{dx} = \frac{y-0}{x-0} = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

70. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

72. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx ty}{\sqrt{t^2 x^2 + t^2 y^2}}$$

$$= \frac{t^2 xy}{t\sqrt{x^2 + y^2}} = t \frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

74. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

76. $f(x, y) = \tan \frac{y}{x}$

$$f(tx, ty) = \tan \frac{ty}{tx} = \tan \frac{y}{x}$$

Homogeneous of degree 0

78. $y' = \frac{(x^3 + y^3)}{xy^2}$

$$xy^2 dy = (x^3 + y^3) dx$$

$$y = vx, \quad dy = x dv + v dx$$

$$x(vx)^2(x dv + v dx) = (x^3 + (vx)^3) dx$$

$$x^4 v^2 dv + x^3 v^3 dx = x^3 dx + v^3 x^3 dx$$

$$xv^2 dv = dx$$

$$\int v^2 dv = \int \frac{1}{x} dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3 \ln|x| + C$$

$$y^3 = 3x^3 \ln|x| + Cx^3$$

80. $y' = \frac{x^2 + y^2}{2xy}$, $y = vx$

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x^2 v}$$

$$2v dx + 2x dv = \frac{1 + v^2}{v} dx$$

$$\int \frac{2v}{v^2 - 1} dv = - \int \frac{dx}{x}$$

$$\ln(v^2 - 1) = -\ln x + \ln C = \ln \frac{C}{x}$$

$$v^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

$$82. \quad y' = \frac{2x + 3y}{x}, y = vx$$

$$v + x \frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x \frac{dv}{dx} = 2 + 2v \Rightarrow \int \frac{dv}{1 + v} = 2 \int \frac{dx}{x}$$

$$\ln|1 + v| = \ln x^2 + \ln C = \ln x^2 C$$

$$1 + v = x^2 C$$

$$1 + \frac{y}{x} = x^2 C$$

$$\frac{y}{x} = x^2 C - 1$$

$$y = Cx^3 - x$$

$$84. \quad -y^2 dx + x(x + y) dy = 0, y = vx$$

$$-x^2 v^2 dx + (x^2 + x^2 v)(v dx + x dv) = 0$$

$$\int \frac{1 + v}{v} dv = - \int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

$$\text{Initial condition: } y(1) = 1, 1 = Ce^{-1} \Rightarrow C = e$$

$$\text{Particular solution: } y = e^{1-y/x}$$

$$86. (2x^2 + y^2) dx + xy dy = 0$$

$$\text{Let } y = vx, dy = x dv + v dx.$$

$$(2x^2 + v^2 x^2) dx + x(vx)(x dv + v dx) = 0$$

$$(2x^2 + 2x^2 v^2) dx + x^3 v dv = 0$$

$$(2 + 2v^2) dx = -xv dv$$

$$\frac{-2}{x} dx = \frac{v}{1 + v^2} dv$$

$$-2 \ln x = \frac{1}{2} \ln(1 + v^2) + C_1$$

$$\ln x^{-2} = \ln(1 + v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1 + v^2)^{1/2}$$

$$\frac{1}{x^2} = C \left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x} (x^2 + y^2)^{1/2}$$

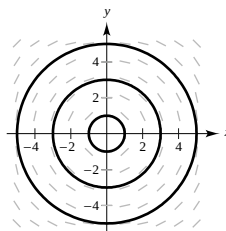
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$$y(1) = 0: 1 = C(1 + 0) \Rightarrow C = 1$$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

$$1 = x\sqrt{x^2 + y^2}$$

$$88. \quad \frac{dy}{dx} = -\frac{x}{y}$$



$$y dy = -x dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

90. $\frac{dy}{dx} = 0.25x(4 - y)$

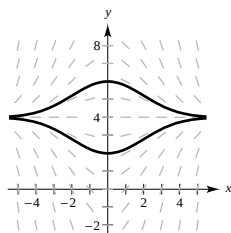
$$\frac{dy}{4 - y} = 0.25x \, dx$$

$$\begin{aligned} \int \frac{dy}{y - 4} &= \int -0.25x \, dx \\ &= -\frac{1}{4} \int x \, dx \end{aligned}$$

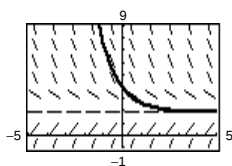
$$\ln |y - 4| = -\frac{1}{8}x^2 + C_1$$

$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

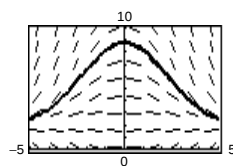
$$y = 4 + Ce^{-(1/8)x^2}$$



92. $\frac{dy}{dx} = 2 - y, y(0) = 4$



94. $\frac{dy}{dx} = 0.2x(2 - y), y(0) = 9$



96. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = 20, y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln \frac{4}{5}$$

Particular solution: $y = 20e^{t \ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t \ln(4/5)}$$

$$\frac{1}{4} = e^{t \ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

98. $\frac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$:
Matches (b).

100. $\frac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

102. From Exercise 101,

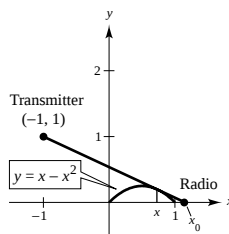
$$w = 1200 - Ce^{-kt}, k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

104. Let the radio receiver be located at $(x_0, 0)$.
The tangent line to $y = x - x^2$ joins $(-1, 1)$
and $(x_0, 0)$.



- (a) If (x, y) is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y - 1}{x + 1} = \frac{x - x^2 - 1}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4 + 8}}{2} \right) = -1 + \sqrt{3}$$

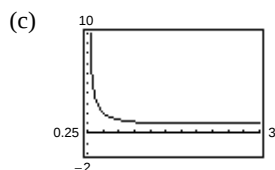
$$y = x - x^2 = 3\sqrt{3} - 5$$

$$\text{Then } \frac{1 - 0}{-1 - x_0} = \frac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \frac{6 - 3\sqrt{3}}{-\sqrt{3}}$$

$$\sqrt{3} = (1 + x_0)(6 - 3\sqrt{3})$$

$$= 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3})$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

- (b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y - h}{x + 1} = \frac{x - x^2 - h}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{(-2 \pm \sqrt{4 + 4(h + 1)})}{2}$$

$$= -1 + \sqrt{2 + h}$$

$$y = x - x^2$$

$$= 3\sqrt{2 + h} - h - 4$$

$$\text{Then, } \frac{h - 0}{-1 - x_0} = \frac{h - (3\sqrt{2 + h} - h - 4)}{-1 - (-1 + \sqrt{2 + h})}$$

$$= \frac{2h + 4 - 3\sqrt{2 + h}}{-\sqrt{2 + h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}}$$

$$x_0 = \frac{h\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}} - 1$$

106. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

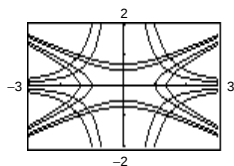
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \frac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x} dx$$

$$\ln y = -2 \ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$



108. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

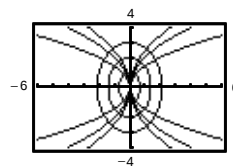
$$y' = \frac{C}{y} = \frac{y^2}{2x} \left(\frac{1}{y} \right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y dy = -\int 2x dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



110. Given family (exponential functions): $y = Ce^x$

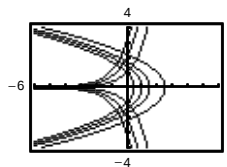
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y \, dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



112. The number of initial conditions matches the number of constants in the general solution.

114. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

116. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

118. True

$$x^2 + y^2 = 2Cy$$

$$x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y}$$

$$\frac{dy}{dx} = \frac{K - x}{y}$$

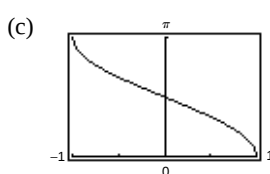
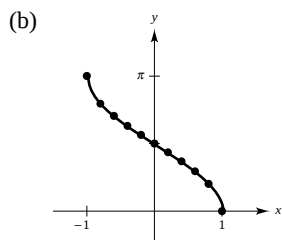
$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

2. $y = \arccos x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.634	0



(d) Intercepts: $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

4. $\left(_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$$\left(_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$$

$$\left(-\sqrt{3}, _\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$$

6. $\arcsin 0 = 0$

8. $\arccos 0 = \frac{\pi}{2}$

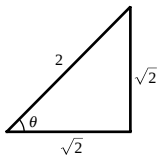
10. $\operatorname{arc cot}(-\sqrt{3}) = \frac{5\pi}{6}$

12. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

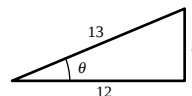
14. $\arcsin(-0.39) \approx -0.40$

16. $\arctan(-3) \approx -1.25$

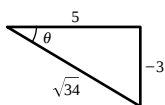
18. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$



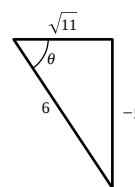
(b) $\cos\left(\arcsin \frac{5}{13}\right) = \frac{12}{13}$



20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \frac{\sqrt{34}}{5}$



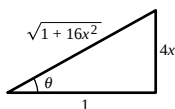
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right] = -\frac{5\sqrt{11}}{11}$



22. $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

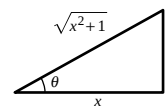
$y = \sec \theta = \sqrt{1 + 16x^2}$



24. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

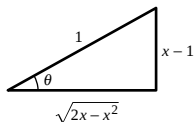
$y = \cos \theta = \frac{x}{\sqrt{x^2 + 1}}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

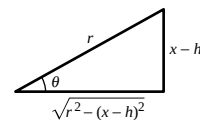
$y = \sec \theta = \frac{1}{\sqrt{2x - x^2}}$



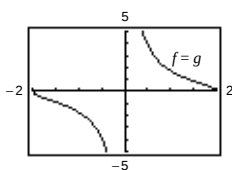
28. $y = \cos\left(\arcsin \frac{x - h}{r}\right)$

$\theta = \arcsin \frac{x - h}{r}$

$y = \cos \theta = \frac{\sqrt{r^2 - (x - h)^2}}{r}$



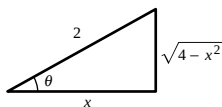
30.

Asymptote: $x = 0$

$\arccos \frac{x}{2} = \theta$

$\cos \theta = \frac{x}{2}$

$\tan \theta = \frac{\sqrt{4 - x^2}}{x}$



32. $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \frac{1}{2}(\tan(-1) + 5) \approx 1.721$

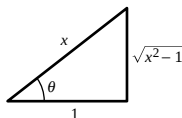
34. $\arccos x = \operatorname{arcsec} x$

$$x = \cos(\operatorname{arcsec} x)$$

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$



36. (a) $\arcsin(-x) = -\arcsin x, |x| \leq 1.$

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \Rightarrow x = -\sin y \Rightarrow x = \sin(-y).$$

Thus, $-y = \arcsin x \Rightarrow y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x, |x| \leq 1.$

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \Rightarrow x = -\cos y \Rightarrow x = \cos(\pi - y).$$

Thus, $\pi - y = \arccos x \Rightarrow y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

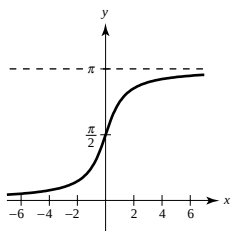
38. $f(x) = \arctan x + \frac{\pi}{2}$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.



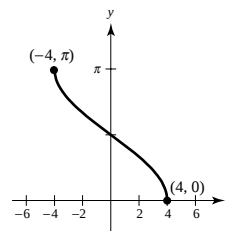
40. $f(x) = \arccos\left(\frac{x}{4}\right)$

$$\frac{x}{4} = \cos y$$

$$x = 4 \cos y$$

Domain: $[-4, 4]$

Range: $[0, \pi]$



42. $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1-t^4}}$$

46. $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}$$

50. $f(x) = \arcsin x + \arccos x = \frac{\pi}{2}$

$$f'(x) = 0$$

44. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$$

48. $h(x) = x^2 \arctan x$

$$h'(x) = 2x \arctan x + \frac{x^2}{1+x^2}$$

52. $y = \ln(t^2 + 4) - \frac{1}{2} \arctan \frac{t}{2}$

$$\begin{aligned} y' &= \frac{2t}{t^2+4} - \frac{1}{2} \cdot \frac{1}{1+\left(\frac{t}{2}\right)^2} \left(\frac{1}{2}\right) \\ &= \frac{2t}{t^2+4} - \frac{1}{t^2+4} = \frac{2t-1}{t^2+4} \end{aligned}$$

54. $y = \frac{1}{2} \left[x\sqrt{4-x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]$

$$\begin{aligned} y' &= \frac{1}{2} \left[x \frac{1}{2} (4-x^2)^{-1/2} (-2x) + \sqrt{4-x^2} + 2 \frac{1}{\sqrt{1-(x/2)^2}} \right] \\ &= \frac{1}{2} \left[\frac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} + \frac{4}{\sqrt{4-x^2}} \right] \\ &= \sqrt{4-x^2} \end{aligned}$$

56. $y = x \arctan 2x - \frac{1}{4} \ln(1+4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4} \left(\frac{8x}{1+4x^2} \right) = \arctan(2x)$$

58. $y = 25 \arcsin \frac{x}{5} - x\sqrt{25 - x^2}$

$$\begin{aligned} y' &= 5 \frac{1}{\sqrt{1 - (x/5)^2}} - \sqrt{25 - x^2} - x \frac{1}{2} (25 - x^2)^{-1/2} (-2x) \\ &= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}} \\ &= \frac{2x^2}{\sqrt{25 - x^2}} \end{aligned}$$

60. $y = \arctan \frac{x}{2} - \frac{1}{2(x^2 + 4)}$

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{1 + (x/2)^2} + \frac{1}{2} (x^2 + 4)^{-2} (2x) \\ &= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2} \\ &= \frac{2x^2 + 8 + x}{(x^2 + 4)^2} \end{aligned}$$

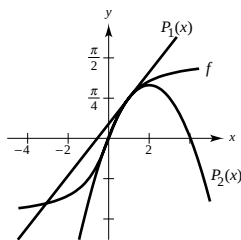
62. $f(x) = \arctan x$, $a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2} f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$



64. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0 \text{ when } \sqrt{1 - x^2} = \frac{1}{2} \text{ or}$$

$$x = \pm \frac{\sqrt{3}}{2}.$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

$$\text{Relative minimum: } \left(\frac{\sqrt{3}}{2}, -0.68\right)$$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

$$\text{Relative maximum: } \left(-\frac{\sqrt{3}}{2}, 0.68\right)$$

66. $f(x) = \arcsin x - 2 \arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$

$$1 + x^2 = 2\sqrt{1 - x^2}$$

$$1 + 2x^2 + x^4 = 4(1 - x^2)$$

$$x^4 + 6x^2 - 3 = 0$$

$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

68. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

70. The derivatives are algebraic. See Theorem 5.18.

72. (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9} \frac{dx}{dt}$

$$\text{If } x = 10, \frac{d\theta}{dt} \approx 11.001 \text{ rad/hr.}$$

$$\text{If } x = 3, \frac{d\theta}{dt} \approx 66.667 \text{ rad/hr.}$$

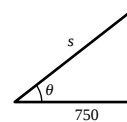
A lower altitude results in a greater rate of change of θ .

74. $\cos \theta = \frac{750}{s}$

$$\theta = \arccos\left(\frac{750}{s}\right)$$

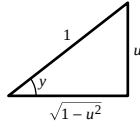
$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}} \left(\frac{-750}{s^2}\right) \frac{ds}{dt}$$

$$= \frac{750}{s\sqrt{s^2 - 750^2}} \frac{ds}{dt}$$



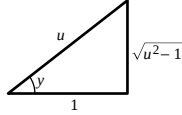
76. (a) Let
- $y = \arcsin u$
- . Then

$$\begin{aligned}\sin y &= u \\ \cos y \cdot y' &= u' \\ \frac{dy}{dx} &= \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}.\end{aligned}$$



- (c) Let
- $y = \operatorname{arccsc} u$
- . Then

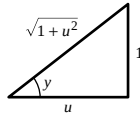
$$\begin{aligned}\sec y &= u \\ \sec y \tan y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}.\end{aligned}$$



Note: The absolute value sign in the formula for the derivative of $\operatorname{arccsc} u$ is necessary because the inverse secant function has a positive slope at every value in its domain.

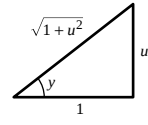
- (e) Let
- $y = \operatorname{arccot} u$
- . Then

$$\begin{aligned}\cot y &= u \\ -\csc^2 y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.\end{aligned}$$



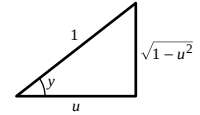
- (b) Let
- $y = \arctan u$
- . Then

$$\begin{aligned}\tan y &= u \\ \sec^2 y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}.\end{aligned}$$



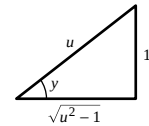
- (d) Let
- $y = \arccos u$
- . Then

$$\begin{aligned}\cos y &= u \\ -\sin y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}.\end{aligned}$$



- (f) Let
- $y = \operatorname{arccsc} u$
- . Then

$$\begin{aligned}\csc y &= u \\ -\csc y \cot y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}.\end{aligned}$$



Note: The absolute value sign in the formula for the derivative of $\operatorname{arccsc} u$ is necessary because the inverse cosecant function has a negative slope at every value in its domain.

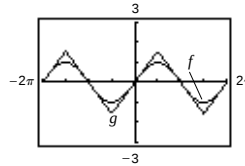
- 78.
- $f(x) = \sin x$

$$g(x) = \arcsin(\sin x)$$

- (a) The range of
- $y = \arcsin x$
- is
- $-\pi/2 \leq y \leq \pi/2$
- .

- (b) Maximum:
- $\pi/2$

$$\text{Minimum: } -\pi/2$$



80. False

$$\text{The range of } y = \arcsin x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

82. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

$$2. \int \frac{3}{\sqrt{1-4x^2}} dx = \frac{3}{2} \int \frac{2}{\sqrt{1-4x^2}} dx = \frac{3}{2} \arcsin(2x) + C$$

$$4. \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$$

$$6. \int \frac{4}{1+9x^2} dx = \frac{4}{3} \int \frac{3}{1+9x^2} dx = \frac{4}{3} \arctan(3x) + C$$

$$8. \int_{\sqrt{3}}^3 \frac{1}{9+x^2} dx = \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{\sqrt{3}}^3 = \frac{\pi}{36}$$

$$10. \int \frac{1}{4+(x-1)^2} dx = \frac{1}{2} \arctan\left(\frac{x-1}{2}\right) + C$$

$$12. \int \frac{x^4-1}{x^2+1} dx = \int (x^2-1) dx = \frac{1}{3}x^3 - x + C$$

14. Let $u = t^2$, $du = 2t dt$.

$$\int \frac{t}{t^4 + 16} dt = \frac{1}{2} \int \frac{1}{(4)^2 + (t^2)^2} (2t) dt = \frac{1}{8} \arctan \frac{t^2}{4} + C$$

16. Let $u = x^2$, $du = 2x dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^4 - 4}} dx &= \frac{1}{2} \int \frac{1}{x^2 \sqrt{(x^2)^2 - 2^2}} (2x) dx \\ &= \frac{1}{4} \operatorname{arcsec} \frac{x^2}{2} + C \end{aligned}$$

18. Let $u = \arccos x$, $du = -\frac{1}{\sqrt{1-x^2}} dx$.

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}} dx &= - \int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}} dx \\ &= \left[-\frac{1}{2} \arccos^2 x \right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925 \end{aligned}$$

20. Let $u = 1 + x^2$, $du = 2x dx$.

$$\begin{aligned} \int_{-\sqrt{3}}^0 \frac{x}{1+x^2} dx &= \frac{1}{2} \int_{-\sqrt{3}}^0 \frac{1}{1+x^2} (2x) dx \\ &= \left[\frac{1}{2} \ln(1+x^2) \right]_{-\sqrt{3}}^0 = -\ln 2 \end{aligned}$$

22. $\int_1^2 \frac{1}{3 + (x-2)^2} dx = \int_1^2 \frac{1}{(\sqrt{3})^2 + (x-2)^2} dx = \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) \right]_1^2 = \frac{\sqrt{3}\pi}{18}$

24. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx = \arctan(\sin x) \Big|_0^{\pi/2} = \frac{\pi}{4}$

26. $\int \frac{3}{2\sqrt{x}(1+x)} dx$. $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$, $dx = 2u du$

$$\begin{aligned} \frac{3}{2} \int \frac{2u du}{u(1+u^2)} &= 3 \int \frac{du}{1+u^2} = 3 \arctan u + C \\ &= 3 \arctan \sqrt{x} + C \end{aligned}$$

28. $\int \frac{4x+3}{\sqrt{1-x^2}} dx = (-2) \int \frac{-2x}{\sqrt{1-x^2}} dx + 3 \int \frac{1}{\sqrt{1-x^2}} dx = -4\sqrt{1-x^2} + 3 \arcsin x + C$

30. $\int \frac{x-2}{(x+1)^2+4} dx = \frac{1}{2} \int \frac{2x+2}{(x+1)^2+4} dx - \int \frac{3}{(x+1)^2+4} dx$

$$= \frac{1}{2} \ln(x^2+2x+5) - \frac{3}{2} \arctan\left(\frac{x+1}{2}\right) + C$$

32. $\int_{-2}^2 \frac{dx}{x^2+4x+13} = \int_{-2}^2 \frac{dx}{(x+2)^2+9} = \left[\frac{1}{3} \arctan\left(\frac{x+2}{3}\right) \right]_{-2}^2 = \frac{1}{3} \arctan\left(\frac{4}{3}\right)$

34. $\int \frac{2x-5}{x^2+2x+2} dx = \int \frac{2x+2}{x^2+2x+2} dx - 7 \int \frac{1}{1+(x+1)^2} dx = \ln|x^2+2x+2| - 7 \arctan(x+1) + C$

36. $\int \frac{2}{\sqrt{-x^2+4x}} dx = \int \frac{2}{\sqrt{4-(x^2-4x+4)}} dx$

$$\begin{aligned} &= \int \frac{2}{\sqrt{4-(x-2)^2}} dx \\ &= 2 \arcsin\left(\frac{x-2}{2}\right) + C \end{aligned}$$

38. Let $u = x^2 - 2x$, $du = (2x-2) dx$.

$$\begin{aligned} \int \frac{x-1}{\sqrt{x^2-2x}} dx &= \frac{1}{2} \int (x^2-2x)^{-1/2} (2x-2) dx \\ &= \sqrt{x^2-2x} + C \end{aligned}$$

40. $\int \frac{1}{(x-1)\sqrt{x^2-2x}} dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}} dx = \operatorname{arcsec}|x-1| + C$

42. Let $u = x^2 - 4$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}} dx = \frac{1}{2} \arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

44. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u du = dx$

$$\begin{aligned} \int \frac{\sqrt{x - 2}}{x + 1} dx &= \int \frac{2u^2}{u^2 + 3} du = \int \frac{2u^2 + 6 - 6}{u^2 + 3} du = 2 \int du - 6 \int \frac{1}{u^2 + 3} du \\ &= 2u - \frac{6}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3} \arctan \sqrt{\frac{x - 2}{3}} + C \end{aligned}$$

46. The term is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$: $x^2 + 3x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

48. (a) $\int e^{x^2} dx$ cannot be evaluated using the basic integration rules.

(b) $\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$, $u = x^2$

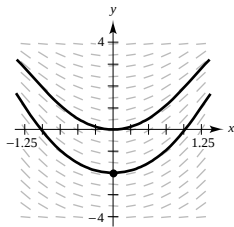
(c) $\int \frac{1}{x^2} e^{1/x} dx = -e^{1/x} + C$, $u = \frac{1}{x}$

50. (a) $\int \frac{1}{1 + x^4} dx$ cannot be evaluated using the basic integration rules.

(b) $\int \frac{x}{1 + x^4} dx = \frac{1}{2} \int \frac{2x}{1 + (x^2)^2} dx = \frac{1}{2} \arctan(x^2) + C$, $u = x^2$

(c) $\int \frac{x^3}{1 + x^4} dx = \frac{1}{4} \int \frac{4x^3}{1 + x^4} dx = \frac{1}{4} \ln(1 + x^4) + C$, $u = 1 + x^4$

52. (a)



(b) $\frac{dy}{dx} = x\sqrt{16 - y^2}$, $(0, -2)$

$$\frac{dy}{\sqrt{16 - y^2}} = x dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

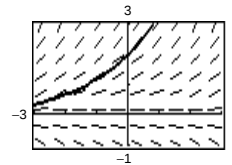
$$(0, -2): \arcsin\left(-\frac{2}{4}\right) = C \Rightarrow C = -\frac{\pi}{6}$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

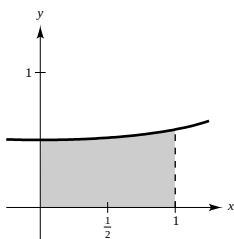
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4 \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

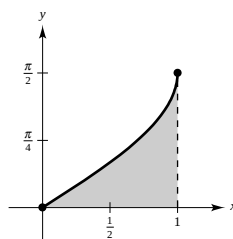
54. $\frac{dy}{dx} = \frac{2y}{\sqrt{16 - x^2}}$, $y(0) = 2$



$$56. A = \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$$



$$58. \int_0^1 \arcsin x \, dx \approx 0.571$$



$$60. F(x) = \frac{1}{2} \int_x^{x+2} \frac{2}{t^2 + 1} dt$$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

$$(b) F(x) = \left[\arctan t \right]_x^{x+2} = \arctan(x+2) - \arctan x$$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2) - (x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

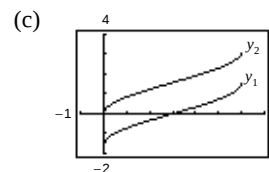
$$62. \int \frac{1}{\sqrt{6x-x^2}} dx$$

$$(a) 6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$$

$$\int \frac{1}{\sqrt{6x-x^2}} dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

$$(b) u = \sqrt{x}, u^2 = x, 2u \, du = dx$$

$$\int \frac{1}{\sqrt{6u^2-u^4}} (2u \, du) = \int \frac{2}{\sqrt{6-u^2}} du = 2 \arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2 \arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

$$64. \text{ Let } f(x) = \arctan x - \frac{x}{1+x^2}$$

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$, $\arctan x - \frac{x}{1+x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1+x^2}.$$

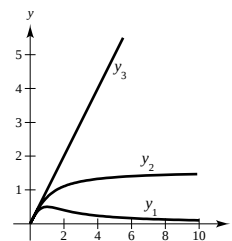
$$\text{Let } g(x) = x - \arctan x$$

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1+x^2} < \arctan x < x.$$



Section 5.10 Hyperbolic Functions

$$2. (a) \cosh(0) = \frac{e^0 + e^0}{2} = 1$$

$$(b) \operatorname{sech}(1) = \frac{2}{e + e^{-1}} \approx 0.648$$

$$6. (a) \operatorname{csch}^{-1}(2) = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.481$$

$$(b) \operatorname{coth}^{-1}(3) = \frac{1}{2} \ln\left(\frac{4}{2}\right) \approx 0.347$$

$$8. \frac{1 + \cosh 2x}{2} = \frac{1 + (e^{2x} + e^{-2x})/2}{2} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$$

$$10. 2 \sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

$$\begin{aligned} 12. 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right) &= 2 \left[\frac{e^{(x+y)/2} + e^{-(x+y)/2}}{2} \right] \left[\frac{e^{(x-y)/2} + e^{-(x-y)/2}}{2} \right] \\ &= 2 \left[\frac{e^x + e^y + e^{-y} + e^{-x}}{4} \right] = \frac{e^x + e^{-x}}{2} + \frac{e^y + e^{-y}}{2} \\ &= \cosh x + \cosh y \end{aligned}$$

$$14. \quad \tanh x = \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \Rightarrow \operatorname{sech}^2 x = \frac{3}{4} \Rightarrow \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\cosh x = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

$$\coth x = \frac{1}{1/2} = 2$$

$$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$$

$$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$$

Putting these in order:

$$\sinh x = \frac{\sqrt{3}}{3} \quad \operatorname{csch} x = \sqrt{3}$$

$$\cosh x = \frac{2\sqrt{3}}{3} \quad \operatorname{sech} x = \frac{\sqrt{3}}{2}$$

$$\tanh x = \frac{1}{2} \quad \coth x = 2$$

$$16. y = \coth(3x)$$

$$y' = -3 \operatorname{csch}^2(3x)$$

$$18. g(x) = \ln(\cosh x)$$

$$g'(x) = \frac{1}{\cosh x} (\sinh x) = \tanh x$$

$$20. y = x \cosh x - \sinh x$$

$$y' = x \sinh x + \cosh x - \cosh x = x \sinh x$$

$$22. h(t) = t - \coth t$$

$$h'(t) = 1 + \operatorname{csch}^2 t = \coth^2 t$$

24. $g(x) = \operatorname{sech}^2 3x$

$$g'(x) = -2 \operatorname{sech}(3x) \operatorname{sech}(3x) \tanh(3x)(3)$$

$$= -6 \operatorname{sech}^2 3x \tanh 3x$$

26. $f(x) = e^{\sinh x}$

$f'(x) = (\cosh x)(e^{\sinh x})$

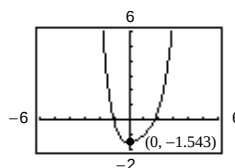
28. $y = \operatorname{sech}(x + 1)$

$y' = -\operatorname{sech}(x + 1) \tanh(x + 1)$

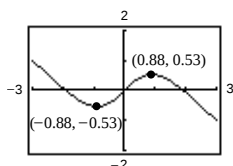
30. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

$f'(x) = x \cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x \cosh(x - 1)$

$f'(x) = 0$ for $x = 0$. By the First Derivative Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



32. $h(x) = 2 \tanh x - x$



Relative maximum: $(0.88, 0.53)$

Relative minimum: $(-0.88, -0.53)$

34. $y = a \cosh x$

$y' = a \sinh x$

$y'' = a \cosh x$

Therefore, $y'' - y = 0$.

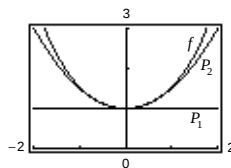
36. $f(x) = \cosh x$ $f(1) = \cosh(0) \approx 1$

$f'(x) = \sinh x$ $f'(1) = \sinh(0) \approx 0$

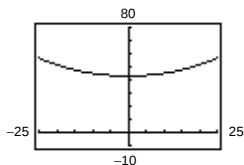
$f''(x) = \cosh x$ $f''(1) = \cosh(0) \approx 1$

$P_1(x) = f(0) + f'(0)(x - 0) = 1$

$P_2(x) = 1 + \frac{1}{2}x^2$



38. (a) $y = 18 + 25 \cosh \frac{x}{25}$, $-25 \leq x \leq 25$



(b) At $x = \pm 25$, $y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0$, $y = 18 + 25 = 43$.

(c) $y' = \sinh \frac{x}{25}$. At $x = 25$, $y' = \sinh(1) \approx 1.175$

40. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{\cosh \sqrt{x}}{\sqrt{x}} dx = 2 \int \cosh \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh \sqrt{x} + C$$

42. Let $u = \cosh x$, $du = \sinh x dx$.

$$\int \frac{\sinh x}{1 + \sinh^2 x} dx = \int \frac{\sinh x}{\cosh^2 x} dx = \frac{-1}{\cosh x} + C$$

$$= -\operatorname{sech} x + C$$

44. Let $u = 2x - 1$, $du = 2 dx$.

$$\int \operatorname{sech}^2(2x - 1) dx = \frac{1}{2} \int \operatorname{sech}^2(2x - 1)(2) dx$$

$$= \frac{1}{2} \tanh(2x - 1) + C$$

46. Let $u = \operatorname{sech} x$, $du = -\operatorname{sech} x \tanh x dx$.

$$\int \operatorname{sech}^3 x \tanh x dx = - \int \operatorname{sech}^2 x (-\operatorname{sech} x \tanh x) dx$$

$$= -\frac{1}{3} \operatorname{sech}^3 x + C$$

$$\begin{aligned}
 48. \int \cosh^2 x \, dx &= \int \frac{1 + \cosh 2x}{2} \, dx \\
 &= \frac{1}{2} \left[x + \frac{\sinh 2x}{2} \right] + C \\
 &= \frac{1}{2}x + \frac{1}{4} \sinh 2x + C
 \end{aligned}$$

$$50. \int_0^4 \frac{1}{\sqrt{25-x^2}} \, dx = \left[\arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5}$$

$$52. \int \frac{2}{x\sqrt{1+4x^2}} \, dx = 2 \int \frac{1}{(2x)\sqrt{1+(2x)^2}} (2) \, dx = -2 \ln \left(\frac{1 + \sqrt{1+4x^2}}{|2x|} \right) + C$$

$$54. \text{ Let } u = \sinh x, \, du = \cosh x \, dx.$$

$$\begin{aligned}
 \int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}} \, dx &= \arcsin \left(\frac{\sinh x}{3} \right) + C \\
 &= \arcsin \left(\frac{e^x - e^{-x}}{6} \right) + C
 \end{aligned}$$

$$56. \, y = \tanh^{-1} \left(\frac{x}{2} \right)$$

$$y' = \frac{1}{1 - (x/2)^2} \left(\frac{1}{2} \right) = \frac{2}{4 - x^2}$$

$$58. \, y = \operatorname{sech}^{-1}(\cos 2x), \, 0 < x < \frac{\pi}{4}$$

$$y' = \frac{-1}{\cos 2x \sqrt{1 - \cos^2 2x}} (-2 \sin 2x) = \frac{2 \sin 2x}{\cos 2x |\sin 2x|} = \frac{2}{\cos 2x} = 2 \sec 2x,$$

since $\sin 2x \geq 0$ for $0 < x < \pi/4$.

$$60. \, y = (\operatorname{csch}^{-1} x)^2$$

$$y' = 2 \operatorname{csch}^{-1} x \left(\frac{-1}{|x| \sqrt{1+x^2}} \right) = \frac{-2 \operatorname{csch}^{-1} x}{|x| \sqrt{1+x^2}}$$

$$62. \, y = x \tanh^{-1} x + \ln \sqrt{1-x^2} = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) \quad 64. \text{ See page 401, Theorem 5.22.}$$

$$y' = x \left(\frac{1}{1-x^2} \right) + \tanh^{-1} x + \frac{-x}{1-x^2} = \tanh^{-1} x$$

$$66. \text{ Equation of tangent line through } P = (x_0, y_0):$$

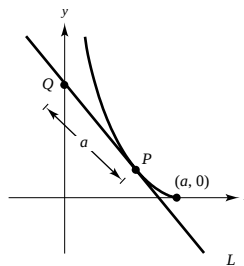
$$y - a \operatorname{sech}^{-1} \frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0} (x - x_0)$$

When $x = 0$,

$$y = a \operatorname{sech}^{-1} \frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a \operatorname{sech}^{-1} \frac{x_0}{a}.$$

Hence, Q is the point $[0, a \operatorname{sech}^{-1}(x_0/a)]$.

Distance from P to Q : $d = \sqrt{x_0^2 + (-\sqrt{a^2 - x_0^2})^2} = a$



$$\begin{aligned}
 68. \int \frac{x}{9-x^4} \, dx &= -\frac{1}{2} \int \frac{-2x}{9-(x^2)^2} \, dx = -\frac{1}{2} \left(\frac{1}{6} \right) \ln \left| \frac{3-x^2}{3+x^2} \right| + C \\
 &= -\frac{1}{12} \ln \left| \frac{3-x^2}{3+x^2} \right| + C
 \end{aligned}$$

70. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x} dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx = \frac{2}{3} \int \frac{1}{\sqrt{1+(x^{3/2})^2}} \left(\frac{3}{2}\sqrt{x}\right) dx = \frac{2}{3} \sinh^{-1}(x^{3/2}) + C = \frac{2}{3} \ln(x^{3/2} + \sqrt{1+x^3}) + C$$

72.
$$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}}$$
$$= -\frac{1}{2} \ln\left(\frac{2 + \sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

74.
$$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}} dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}} dx$$
$$= \frac{1}{\sqrt{2}} \int \frac{1}{(x+1)\sqrt{(x+1)^2+(\sqrt{3})^2}} dx = -\frac{1}{\sqrt{6}} \ln\left(\frac{\sqrt{3} + \sqrt{(x+1)^2+3}}{x+1}\right) + C$$

76. Let $u = 2(x-1)$, $du = 2 dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}} dx = \int \frac{2}{2(x-1)\sqrt{(\sqrt{3})^2-[2(x-1)]^2}} dx = -\frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{3} + \sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

78.
$$y = \int \frac{1-2x}{4x-x^2} dx = \int \frac{4-2x}{4x-x^2} dx + 3 \int \frac{1}{(x-2)^2-4} dx$$
$$= \ln|4x-x^2| + \frac{3}{4} \ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C = \ln|4x-x^2| + \frac{3}{4} \ln\left|\frac{x-4}{x}\right| + C$$

80.
$$A = \int_0^2 \tanh 2x dx$$
$$= \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$$
$$= \frac{1}{2} \int_0^2 \frac{1}{e^{2x} + e^{-2x}} (2)(e^{2x} - e^{-2x}) dx$$
$$= \left[\frac{1}{2} \ln(e^{2x} + e^{-2x}) \right]_0^2$$
$$= \frac{1}{2} \ln(e^4 + e^{-4}) - \frac{1}{2} \ln 2$$
$$= \ln \sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654$$

82.
$$A = \int_3^5 \frac{6}{\sqrt{x^2-4}} dx$$
$$= \left[6 \ln(x + \sqrt{x^2-4}) \right]_3^5$$
$$= 6 \ln(5 + \sqrt{21}) - 6 \ln(3 + \sqrt{5})$$
$$= 6 \ln\left(\frac{5 + \sqrt{21}}{3 + \sqrt{5}}\right) \approx 3.626$$

84. (a) $v(t) = -32t$

(b)
$$s(t) = \int v(t) dt = \int (-32t) dt = -16t^2 + C$$
$$s(0) = -16(0)^2 + C = 400 \Rightarrow C = 400$$
$$s(t) = -16t^2 + 400$$

—CONTINUED—

84. —CONTINUED—

$$(c) \quad \frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = - \int dt$$

$$\text{Let } u = \sqrt{k} v, \text{ then } du = \sqrt{k} dv.$$

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln \left| \frac{\sqrt{32} + \sqrt{k} v}{\sqrt{32} - \sqrt{k} v} \right| = -t + C$$

$$\text{Since } v(0) = 0, C = 0.$$

$$\ln \left| \frac{\sqrt{32} + \sqrt{k} v}{\sqrt{32} - \sqrt{k} v} \right| = -2\sqrt{32k} t$$

$$\frac{\sqrt{32} + \sqrt{k} v}{\sqrt{32} - \sqrt{k} v} = e^{-2\sqrt{32k} t}$$

$$\sqrt{32} + \sqrt{k} v = e^{-2\sqrt{32k} t} (\sqrt{32} - \sqrt{k} v)$$

$$v(\sqrt{k} + \sqrt{k} e^{-2\sqrt{32k} t}) = \sqrt{32}(e^{-2\sqrt{32k} t} - 1)$$

$$v = \frac{\sqrt{32}(e^{-2\sqrt{32k} t} - 1)}{\sqrt{k}(e^{-2\sqrt{32k} t} + 1)} \cdot \frac{e^{\sqrt{32k} t}}{e^{\sqrt{32k} t}}$$

$$= \frac{\sqrt{32}}{\sqrt{k}} \left[\frac{-(e^{\sqrt{32k} t} - e^{-\sqrt{32k} t})}{e^{\sqrt{32k} t} + e^{-\sqrt{32k} t}} \right]$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k} t)$$

$$(d) \quad \lim_{t \rightarrow \infty} \left[-\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k} t) \right] = -\frac{\sqrt{32}}{\sqrt{k}}.$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct) dt = (1/c) \ln \cosh(ct)$ (which can be verified by differentiation), then

$$\begin{aligned} s(t) &= \int -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k} t) dt \\ &= -\frac{\sqrt{32}}{\sqrt{k}} \frac{1}{\sqrt{32k}} \ln[\cosh(\sqrt{32k} t)] + C \\ &= -\frac{1}{k} \ln[\cosh(\sqrt{32k} t)] + C. \end{aligned}$$

When $t = 0$,

$$s(0) = C$$

$$= 400 \Rightarrow 400 - (1/k) \ln[\cosh(\sqrt{32k} t)].$$

When $k = 0.01$,

$$s_2(t) = 400 - 100 \ln(\cosh \sqrt{0.32} t)$$

$$s_1(t) = -16t^2 + 400.$$

$s_1(t) = 0$ when $t = 5$ seconds.

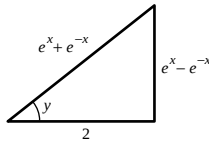
$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.

86. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$



Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

88. $y = \operatorname{sech}^{-1} x$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

90. $y = \sinh^{-1} x$

$$\sinh y = x$$

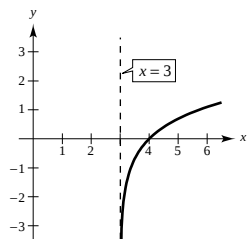
$$(\cosh y)y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

Review Exercises for Chapter 5

2. $f(x) = \ln(x - 3)$

Horizontal shift 3 units to the right

Vertical asymptote: $x = 3$ 

4. $\ln[(x^2 + 1)(x - 1)] = \ln(x^2 + 1) + \ln(x - 1)$

6. $3[\ln x - 2 \ln(x^2 + 1)] + 2 \ln 5 = 3 \ln x - 6 \ln(x^2 + 1) + \ln 5^2 = \ln x^3 - \ln(x^2 + 1)^6 + \ln 25 = \ln \left[\frac{25x^3}{(x^2 + 1)^6} \right]$

8. $\ln x + \ln(x - 3) = 0$

$$\ln x(x - 3) = 0$$

$$x(x - 3) = e^0$$

$$x^2 - 3x - 1 = 0$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

$$x = \frac{3 + \sqrt{13}}{2} \text{ only since } \frac{3 - \sqrt{13}}{2} < 0.$$

10. $h(x) = \ln \frac{x(x-1)}{x-2} = \ln x + \ln(x-1) - \ln(x-2)$

$$h'(x) = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x-2} = \frac{x^2 - 4x + 2}{x^3 - 3x^2 + 2x}$$

12. $f(x) = \ln[x(x^2 - 2)^{2/3}] = \ln x + \frac{2}{3} \ln(x^2 - 2)$

$$f'(x) = \frac{1}{x} + \frac{2}{3} \left(\frac{2x}{x^2 - 2} \right) = \frac{7x^2 - 6}{3x^3 - 6x}$$

14. $y = \frac{1}{b^2}[a + bx - a \ln(a + bx)]$

$$\frac{dy}{dx} = \frac{1}{b^2} \left(b - \frac{ab}{a + bx} \right) = \frac{x}{a + bx}$$

16. $y = -\frac{1}{ax} + \frac{b}{a^2} \ln \frac{a + bx}{x}$

$$= -\frac{1}{ax} + \frac{b}{a^2} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(-\frac{1}{x^2} \right) + \frac{b}{a^2} \left[\frac{b}{a + bx} - \frac{1}{x} \right]$$

$$= \frac{1}{ax^2} + \frac{b}{a^2} \left[\frac{-a}{x(a + bx)} \right] = \frac{1}{ax^2} - \frac{b}{ax(a + bx)}$$

$$= \frac{(a + bx) - bx}{ax^2(a + bx)} = \frac{1}{x^2(a + bx)}$$

18. $u = x^2 - 1, du = 2x dx$

$$\int \frac{x}{x^2 - 1} dx = \frac{1}{2} \int \frac{2x}{x^2 - 1} dx = \frac{1}{2} \ln|x^2 - 1| + C$$

20. $u = \ln x, du = \frac{1}{x} dx$

$$\int \frac{\ln \sqrt{x}}{x} dx = \frac{1}{2} \int (\ln x) \left(\frac{1}{x} \right) dx = \frac{1}{4} (\ln x)^2 + C$$

22. $\int_1^e \frac{\ln x}{x} dx = \int_1^e (\ln x)^1 \left(\frac{1}{x} \right) dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^e = \frac{1}{2}$

24. $\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx = \left[\ln \left| \cos\left(\frac{\pi}{4} - x\right) \right| \right]_0^{\pi/4}$

$$= 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \ln 2$$

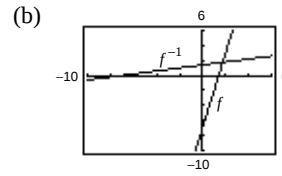
26. (a) $f(x) = 5x - 7$

$$y = 5x - 7$$

$$\frac{y+7}{5} = x$$

$$\frac{x+7}{5} = y$$

$$f^{-1}(x) = \frac{x+7}{5}$$



(c) $f^{-1}(f(x)) = f^{-1}(5x - 7) = \frac{(5x - 7) + 7}{5} = x$

$$f(f^{-1}(x)) = f\left(\frac{x+7}{5}\right) = 5\left(\frac{x+7}{5}\right) - 7 = x$$

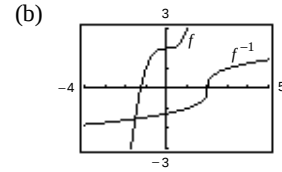
28. (a) $f(x) = x^3 + 2$

$$y = x^3 + 2$$

$$\sqrt[3]{y-2} = x$$

$$\sqrt[3]{x-2} = y$$

$$f^{-1}(x) = \sqrt[3]{x-2}$$



(c) $f^{-1}(f(x)) = f^{-1}(x^3 + 2) = \sqrt[3]{(x^3 + 2) - 2} = x$

$$f(f^{-1}(x)) = f(\sqrt[3]{x-2}) = (\sqrt[3]{x-2})^3 + 2 = x$$

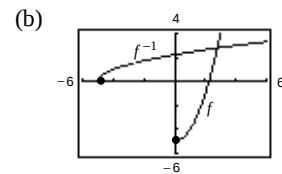
30. (a) $f(x) = x^2 - 5, x \geq 0$

$$y = x^2 - 5$$

$$\sqrt{y+5} = x$$

$$\sqrt{x+5} = y$$

$$f^{-1}(x) = \sqrt{x+5}$$



(c) $f^{-1}(f(x)) = f^{-1}(x^2 - 5) = \sqrt{(x^2 - 5) + 5} = x$ for $x \geq 0$.

$$f(f^{-1}(x)) = f(\sqrt{x+5}) = (\sqrt{x+5})^2 - 5 = x$$

32. $f(x) = x\sqrt{x-3}$

$$f(4) = 4$$

$$f'(x) = \sqrt{x-3} + \frac{1}{2}x(x-3)^{-1/2}$$

$$f'(4) = 1 + 2 = 3$$

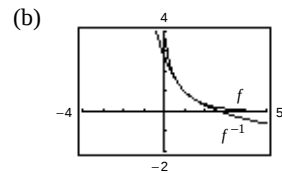
$$(f^{-1})'(4) = \frac{1}{f'(4)} = \frac{1}{3}$$

34. $f(x) = \ln x$

$$f^{-1}(x) = e^x$$

$$(f^{-1})'(x) = e^x$$

$$(f^{-1})'(0) = e^0 = 1$$

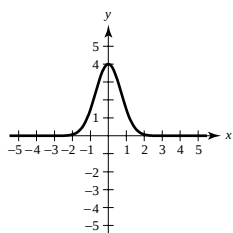


(c) $f^{-1}(f(x)) = f^{-1}(e^{1-x}) = 1 - \ln(e^{1-x})$

$$= 1 - (1 - x) = x$$

$$f(f^{-1}(x)) = f(1 - \ln x) = e^{1 - (1 - \ln x)} = e^{\ln x} = x$$

38. $y = 4e^{-x^2}$



40. $g(x) = \ln\left(\frac{e^x}{1 + e^x}\right)$

$$= \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x)$$

$$g'(x) = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x}$$

42. $h(z) = e^{-z^2/2}$

$$h'(z) = -ze^{-z^2/2}$$

44. $y = 3e^{-3/t}$

$$y' = 3e^{-3/t}(3t^{-2}) = \frac{9e^{-3/t}}{t^2}$$

46. $f(\theta) = \frac{1}{2}e^{\sin 2\theta}$

$$f'(\theta) = \cos 2\theta e^{\sin 2\theta}$$

48. $\cos x^2 = xe^y$

$$-2x \sin x^2 = xe^y \frac{dy}{dx} + e^y$$

$$\frac{dy}{dx} = -\frac{2x \sin x^2 + e^y}{xe^y}$$

50. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\int \frac{e^{1/x}}{x^2} dx = -\int e^{1/x} \left(-\frac{1}{x^2}\right) dx = -e^{1/x} + C$$

52. Let $u = e^{2x} + e^{-2x}$, $du = (2e^{2x} - e^{-2x}) dx$.

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \frac{1}{2} \int \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}} dx$$

$$= \frac{1}{2} \ln(e^{2x} + e^{-2x}) + C$$

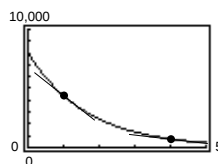
54. Let $u = x^3 + 1$, $du = 3x^2 dx$.

$$\int x^2 e^{x^3+1} dx = \frac{1}{3} \int e^{x^3+1} (3x^2) dx = \frac{1}{3} e^{x^3+1} + C$$

56. $\int \frac{e^{2x}}{e^{2x} + 1} dx = \frac{1}{2} \int \frac{1}{e^{2x} + 1} 2e^{2x} dx$

$$= \frac{1}{2} \ln(e^{2x} + 1) + C$$

58. (a), (c)



(b) $V = 8000e^{-0.6t}$, $0 \leq t \leq 5$

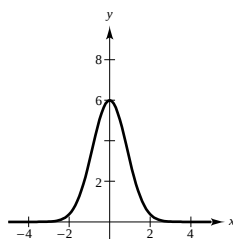
$$V'(t) = -4800e^{-0.6t}$$

$$V'(1) = -2634.3 \text{ dollars/year}$$

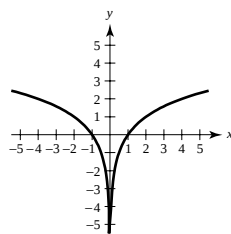
$$V'(4) = -435.4 \text{ dollars/year}$$

60. Area $= \int_0^2 2e^{-x} dx = \left[-2e^{-x}\right]_0^2 = -2e^{-2} + 2 = 2 - \frac{2}{e^2} \approx 1.729$

62. $g(x) = 6(2^{-x^2})$



64. $y = \log_4 x^2$



66. $f(x) = 4^x e^x$

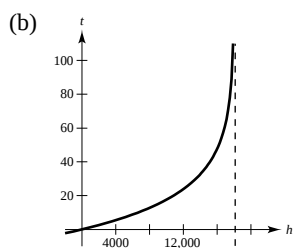
$$f'(x) = 4^x e^x + (\ln 4) 4^x e^x = 4^x e^x (1 + \ln 4)$$

70. $h(x) = \log_5 \frac{x}{x-1} = \log_5 x - \log_5 (x-1)$

$$h'(x) = \frac{1}{\ln 5} \left[\frac{1}{x} - \frac{1}{x-1} \right] = \frac{1}{\ln 5} \left[\frac{-1}{x(x-1)} \right]$$

74. $t = 50 \log_{10} \left(\frac{18,000}{18,000 - h} \right)$

(a) Domain: $0 \leq h < 18,000$



Vertical asymptote: $h = 18,000$

76. $2P = Pe^{10r}$

$$2 = e^{10r}$$

$$\ln 2 = 10r$$

$$r = \frac{\ln 2}{10} \approx 6.93\%$$

80. (a) $\frac{dy}{ds} = -0.012y, s > 50$

$$\frac{-1}{0.012} \int \frac{dy}{y} = \int ds$$

$$\frac{-1}{0.012} \ln y = s + C_1$$

$$y = Ce^{-0.012s}$$

$$\text{When } s = 50, y = 28 = Ce^{-0.012(50)} \Rightarrow C = 28e^{0.6}$$

$$y = 28e^{0.6 - 0.012s}, s > 50$$

82. $\frac{dy}{dx} = \frac{e^{-2x}}{1 + e^{-2x}}$

$$\int dy = \int \frac{e^{-2x}}{1 + e^{-2x}} dx = -\frac{1}{2} \int \frac{-2e^{-2x}}{1 + e^{-2x}} dx$$

$$y = -\frac{1}{2} \ln(1 + e^{-2x}) + C$$

68. $y = x(4^{-x})$

$$y' = 4^{-x} - x \cdot 4^{-x} \ln 4$$

72. $\int \frac{2^{-1/t}}{t^2} dt = \frac{1}{\ln 2} 2^{-1/t} + C$

(c) $t = 50 \log_{10} \left(\frac{18,000}{18,000 - h} \right)$

$$10^{t/50} = \frac{18,000}{18,000 - h}$$

$$18,000 - h = 18,000(10^{-t/50})$$

$$h = 18,000(1 - 10^{-t/50})$$

As $h \rightarrow 18,000$, $t \rightarrow \infty$.

(d) $t = 50 \log_{10} 18,000 - 50 \log_{10}(18,000 - h)$

$$\frac{dt}{dh} = \frac{50}{(\ln 10)(18,000 - h)}$$

$$\frac{d^2t}{dh^2} = \frac{50}{(\ln 10)(18,000 - h)^2}$$

No critical numbers

As t increases, the rate of change of the altitude is increasing.

78. $y = 5 \left(\frac{1}{2} \right)^{t/1620}$

$$y(600) = 5 \left(\frac{1}{2} \right)^{600/1620} \approx 3.868 \text{ grams}$$

(b)

Speed(s)	50	55	60	65	70
Miles per Gallon (y)	28	26.4	24.8	23.4	22.0

84. $y' - e^y \sin x = 0$

$$\frac{dy}{dx} = e^y \sin x$$

$$\int e^{-y} dy = \int \sin x dx$$

$$-e^{-y} = -\cos x + C_1$$

$$e^y = \frac{1}{\cos x + C} \quad (C = -C_1)$$

$$y = \ln \left| \frac{1}{\cos x + C} \right| = -\ln |\cos x + C|$$

86. $\frac{dy}{dx} = \frac{3(x+y)}{x}$ (homogeneous differential equation)

$$3(x+y) dx - x dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$3(x+vx) dx - x(x dv + v dx) = 0$$

$$(3x + 2vx) dx - x^2 dv = 0$$

$$(3 + 2v) dx = x dv$$

$$\int \frac{1}{x} dx = \int \frac{1}{3 + 2v} dv$$

$$\ln|x| = \frac{1}{2} \ln|3 + 2v| + C_1 = \ln(3 + 2v)^{1/2} + \ln C_2$$

$$x = C_2(3 + 2v)^{1/2}$$

$$x^2 = C(3 + 2v) = C\left(3 + 2\left(\frac{y}{x}\right)\right)$$

$$x^3 = C(3x + 2y) = 3Cx + 2Cy$$

$$y = \frac{x^3 - 3Cx}{2C}$$

88. $\frac{dv}{dt} = kv - 9.8$

(a) $\int \frac{dv}{kv - 9.8} = \int dt$

$$\frac{1}{k} \ln|kv - 9.8| = t + C_1$$

$$\ln|kv - 9.8| = kt + C_2$$

$$kv - 9.8 = e^{kt+C_2} = C_3 e^{kt}$$

$$v = \frac{1}{k} \left[9.8 + C_3 e^{kt} \right]$$

At $t = 0$, $v_0 = \frac{1}{k}(9.8 + C_3) \Rightarrow C_3 = kv_0 - 9.8$

$$v = \frac{1}{k} [9.8 + (kv_0 - 9.8)e^{kt}]$$

Note that $k < 0$ since the object is moving downward.

(b) $\lim_{t \rightarrow \infty} v(t) = \frac{9.8}{k}$

(c) $s(t) = \int \frac{1}{k} [9.8 + (kv_0 - 9.8)e^{kt}] dt$

$$= \frac{1}{k} \left[9.8t + \frac{1}{k} (kv_0 - 9.8)e^{kt} \right] + C$$

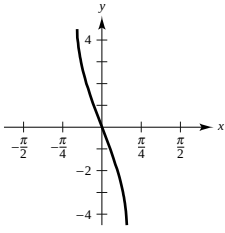
$$= \frac{9.8t}{k} + \frac{1}{k^2} (kv_0 - 9.8)e^{kt} + C$$

$$s(0) = \frac{1}{k^2} (kv_0 - 9.8) + C \Rightarrow C = s_0 - \frac{1}{k^2} (kv_0 - 9.8)$$

$$s(t) = \frac{9.8t}{k} + \frac{1}{k^2} (kv_0 - 9.8)e^{kt} + s_0 - \frac{1}{k^2} (kv_0 - 9.8)$$

$$= \frac{9.8t}{k} + \frac{1}{k^2} (kv_0 - 9.8)(e^{kt} - 1) + s_0$$

90. $h(x) = -3 \arcsin(2x)$



94. $y = \arctan(x^2 - 1)$

$$y' = \frac{2x}{1 + (x^2 - 1)^2} = \frac{2x}{x^4 - 2x^2 + 2}$$

98. $y = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \frac{x}{2}, 2 < x < 4$

$$y' = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{(|x|/2)\sqrt{(x/2)^2 - 1}} = \frac{x}{\sqrt{x^2 - 4}} - \frac{4}{|x|\sqrt{x^2 - 4}} = \frac{x^2 - 4}{|x|\sqrt{x^2 - 4}} = \frac{\sqrt{x^2 - 4}}{x}$$

100. Let $u = 5x, du = 5 dx$.

$$\int \frac{1}{3 + 25x^2} dx = \frac{1}{5} \int \frac{1}{(\sqrt{3})^2 + (5x)^2} (5) dx = \frac{1}{5\sqrt{3}} \arctan \frac{5x}{\sqrt{3}} + C$$

102. $\int \frac{1}{16 + x^2} dx = \frac{1}{4} \arctan \frac{x}{4} + C$

104. $\int \frac{4 - x}{\sqrt{4 - x^2}} dx = 4 \int \frac{1}{\sqrt{4 - x^2}} dx + \frac{1}{2} \int (4 - x^2)^{-1/2} (-2x) dx = 4 \arcsin \frac{x}{2} + \sqrt{4 - x^2} + C$

106. Let $u = \arcsin x, du = \frac{1}{\sqrt{1 - x^2}} dx$.

$$\int \frac{\arcsin x}{\sqrt{1 - x^2}} dx = \frac{1}{2} (\arcsin x)^2 + C$$

110. $y = x \tanh^{-1} 2x$

$$y' = x \left(\frac{2}{1 - 4x^2} \right) + \tanh^{-1} 2x = \frac{2x}{1 - 4x^2} + \tanh^{-1} 2x$$

92. (a) Let $\theta = \operatorname{arccot} 2$

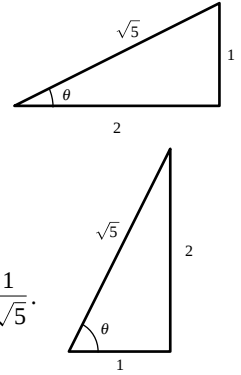
$$\cot \theta = 2$$

$$\tan(\operatorname{arccot} 2) = \tan \theta = \frac{1}{2}.$$

(b) Let $\theta = \operatorname{arcsec} \sqrt{5}$

$$\sec \theta = \sqrt{5}$$

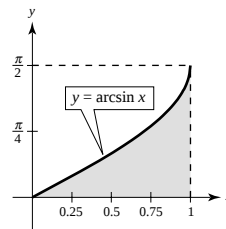
$$\cos(\operatorname{arcsec} \sqrt{5}) = \cos \theta = \frac{1}{\sqrt{5}}.$$



96. $y = \frac{1}{2} \arctan e^{2x}$

$$y' = \frac{1}{2} \left(\frac{1}{1 + e^{4x}} \right) (2e^{2x}) = \frac{e^{2x}}{1 + e^{4x}}$$

108.



Since the area of region A is $1 \left(\int_0^{\pi/2} \sin y dy \right)$,

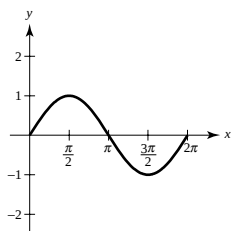
the shaded area is $\int_0^1 \arcsin x dx = \frac{\pi}{2} - 1 \approx 0.571$.

112. Let $u = x^3, du = 3x^2 dx$.

$$\int x^2 (\operatorname{sech} x^3)^2 dx = \frac{1}{3} \int (\operatorname{sech} x^3)^2 (3x^2) dx = \frac{1}{3} \tanh x^3 + C$$

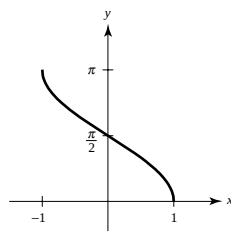
Problem Solving for Chapter 5

2. (a)



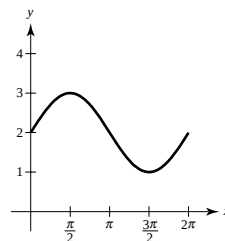
$$\int_0^{\pi} \sin x \, dx = -\int_{\pi}^{2\pi} \sin x \, dx \Rightarrow \int_0^{2\pi} \sin x \, dx = 0$$

(c)



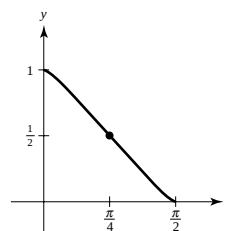
$$\int_{-1}^1 \arccos x \, dx = 2\left(\frac{\pi}{2}\right) = \pi$$

(b)



$$\int_0^{2\pi} (\sin x + 2) \, dx = 2(2\pi) = 4\pi$$

(d)

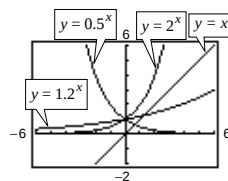


$$y = \frac{1}{1 + (\tan x)\sqrt{2}} \text{ is symmetric with respect to the point } \left(\frac{\pi}{4}, \frac{1}{2}\right). \quad \int_0^{\pi/2} \frac{1}{1 + (\tan x)\sqrt{2}} \, dx = \frac{\pi}{2} \left(\frac{1}{2}\right) = \frac{\pi}{4}$$

4. $y = 0.5^x$ and $y = 1.2^x$ intersect $y = x$. $y = 2^x$ does not intersect $y = x$.Suppose $y = x$ is tangent to $y = a^x$ at (x, y) .

$$a^x = x \Rightarrow a = x^{1/x}.$$

$$y' = a^x \ln a = 1 \Rightarrow x \ln x^{1/x} = 1 \Rightarrow \ln x = 1 \Rightarrow x = e, a = e^{1/e}$$

For $0 < a \leq e^{1/e} \approx 1.445$, the curve $y = a^x$ intersects $y = x$.6. (a) $y = f(x) = \arcsin x$

$$\sin y = x$$

$$\text{Area } A = \int_{\pi/6}^{\pi/4} \sin y \cdot dy = -\cos y \Big|_{\pi/6}^{\pi/4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - \sqrt{2}}{2} \approx 0.1589$$

$$\text{Area } B = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) = \frac{\pi}{12} \approx 0.2618$$

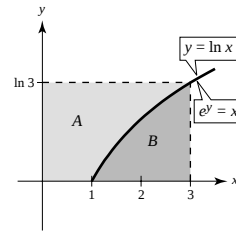
$$\begin{aligned} \text{(b)} \quad \int_{1/2}^{\sqrt{2}/2} \arcsin x \, dx &= \text{Area}(C) = \left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) - A - B \\ &= \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{3} - \sqrt{2}}{2} - \frac{\pi}{12} \\ &= \pi\left(\frac{\sqrt{2}}{8} - \frac{1}{12}\right) + \frac{\sqrt{2} - \sqrt{3}}{2} \approx 0.1346 \end{aligned}$$

—CONTINUED—

6. —CONTINUED—

$$\begin{aligned} \text{(c) Area } A &= \int_0^{\ln 3} e^y dy \\ &= e^y \Big|_0^{\ln 3} = 3 - 1 = 2 \end{aligned}$$

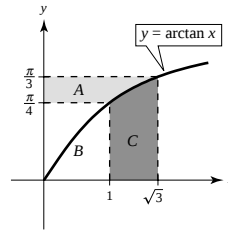
$$\text{Area } B = \int_1^3 \ln x dx = 3(\ln 3) - A = 3 \ln 3 - 2 = \ln 27 - 2 \approx 1.2958$$



$$\text{(d) } \tan y = x$$

$$\begin{aligned} \text{Area } A &= \int_{\pi/4}^{\pi/3} \tan y dy \\ &= -\ln|\cos y| \Big|_{\pi/4}^{\pi/3} \\ &= -\ln \frac{1}{2} + \ln \frac{\sqrt{2}}{2} = \ln \sqrt{2} = \frac{1}{2} \ln 2 \end{aligned}$$

$$\begin{aligned} \text{Area } C &= \int_1^{\sqrt{3}} \arctan x dx = \left(\frac{\pi}{3}\right)(\sqrt{3}) - \frac{1}{2} \ln 2 - \left(\frac{\pi}{4}\right)(1) \\ &= \frac{\pi}{12}(4\sqrt{3} - 3) - \frac{1}{2} \ln 2 \approx 0.6818 \end{aligned}$$



$$\begin{aligned} 8. \quad y &= e^x \\ y' &= e^x \\ y - b &= e^a(x - a) \\ y &= e^a x - ae^a + b \quad \text{Tangent line} \\ \text{If } y &= 0, \\ e^a x &= ae^a - b \\ bx &= ab - b \quad (b = e^a) \\ x &= a - 1 \\ c &= a - 1 \end{aligned}$$

$$\text{Thus, } a - c = a - (a - 1) = 1.$$

$$10. \text{ Let } u = \tan x, du = \sec^2 x dx$$

$$\begin{aligned} \text{Area} &= \int_0^{\pi/4} \frac{1}{\sin^2 x + 4 \cos^2 x} dx = \int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 4} dx \\ &= \int_0^1 \frac{du}{u^2 + 4} \\ &= \left[\frac{1}{2} \arctan\left(\frac{u}{2}\right) \right]_0^1 \\ &= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \end{aligned}$$

$$12. \text{ (a) } \frac{dy}{dt} = y(1 - y), y(0) = \frac{1}{4}$$

$$\int \left(\frac{1}{y} + \frac{1}{1 - y} \right) dy = \int dt$$

$$\ln|y| - \ln|1 - y| = t + C$$

$$\ln \left| \frac{y}{1 - y} \right| = t + C$$

$$\frac{y}{1 - y} = e^{t+C} = C_1 e^t$$

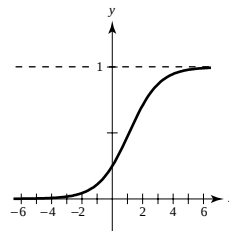
$$y = C_1 e^t - y C_1 e^t$$

$$y = \frac{C_1 e^t}{1 + C_1 e^t} = \frac{1}{1 + C_2 e^{-t}}$$

$$y(0) = \frac{1}{4} = \frac{1}{1 + C_2} \Rightarrow C_2 = 3$$

$$\text{Hence, } y = \frac{1}{1 + 3e^{-t}}.$$

(b)



$$\frac{dy}{dt} = y(1 - y) = y - y^2$$

$$\frac{d^2y}{dt^2} = y'' = y' - 2yy' \Rightarrow y'' = 0 \text{ for } y = \frac{1}{2}$$

$$\frac{d^2y}{dt^2} > 0 \text{ if } 0 < y < \frac{1}{2} \text{ and } \frac{d^2y}{dt^2} < 0 \text{ if } \frac{1}{2} < y < 1.$$

Thus, the rate of growth is maximum at $y = \frac{1}{2}$, the point of inflection.

12. —CONTINUED—

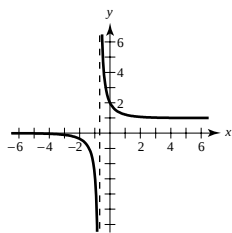
$$(c) \quad y' = y(1 - y), y(0) = 2$$

$$\text{As before, } y = \frac{1}{1 + C_2 e^{-t}}$$

$$y(0) = 2 = \frac{1}{1 + C_2} \Rightarrow C_2 = -\frac{1}{2}$$

$$\text{Thus, } y = \frac{1}{1 - \frac{1}{2}e^{-t}} = \frac{2}{2 - e^{-t}}.$$

The graph is different:



$$14. (a) \quad u = 985.93 - \left(985.93 - \frac{(120,000)(0.095)}{12} \right) \left(1 + \frac{0.095}{12} \right)^{12t}$$

$$v = \left(985.93 - \frac{(120,000)(0.095)}{12} \right) \left(1 + \frac{0.095}{12} \right)^{12t}$$

(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

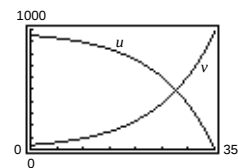
(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \Rightarrow \frac{du}{dt} = -\frac{dv}{dt}$$

$$u'(15) = -v'(15) = -14.06.$$

(d) $t = 12.7$ years

Again, the larger part goes for interest.



CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1	The Natural Logarithmic Function: Differentiation	218
Section 5.2	The Natural Logarithmic Function: Integration	223
Section 5.3	Inverse Functions	227
Section 5.4	Exponential Functions: Differentiation and Integration . .	233
Section 5.5	Bases Other than e and Applications	240
Section 5.6	Differential Equations: Growth and Decay	246
Section 5.7	Differential Equations: Separation of Variables	251
Section 5.8	Inverse Trigonometric Functions: Differentiation	259
Section 5.9	Inverse Trigonometric Functions: Integration	263
Section 5.10	Hyperbolic Functions	267
Review Exercises		272
Problem Solving		278

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t} dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t} dt = -\int_{0.5}^1 \frac{1}{t} dt$

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t} dt \approx 3.8067$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t} dt \approx -0.2231$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

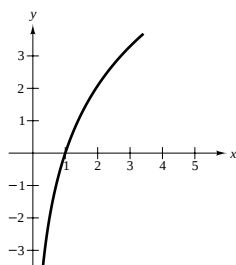
9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

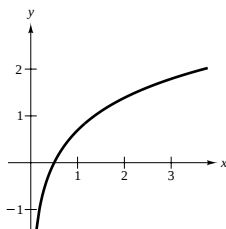
11. $f(x) = 3 \ln x$

Domain: $x > 0$



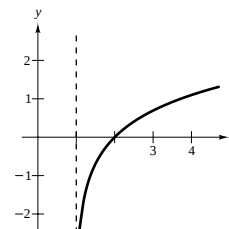
13. $f(x) = \ln 2x$

Domain: $x > 0$



15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

$$25. \ln\left(\frac{x^2 - 1}{x^3}\right)^3 = 3[\ln(x^2 - 1) - \ln x^3]$$

$$= 3[\ln(x + 1) + \ln(x - 1) - 3 \ln x]$$

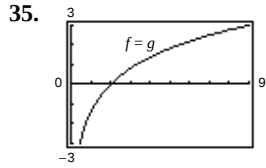
$$27. \ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$$

$$= \ln z + 2 \ln(z - 1)$$

$$29. \ln(x - 2) - \ln(x + 2) = \ln \frac{x - 2}{x + 2}$$

$$31. \frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)] = \frac{1}{3} \ln \frac{x(x + 3)^2}{x^2 - 1} = \ln \sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$$

$$33. 2 \ln 3 - \frac{1}{2} \ln(x^2 + 1) = \ln 9 - \ln \sqrt{x^2 + 1} = \ln \frac{9}{\sqrt{x^2 + 1}}$$



$$37. \lim_{x \rightarrow 3^+} \ln(x - 3) = -\infty$$

$$39. \lim_{x \rightarrow 2^-} \ln[x^2(3 - x)] = \ln 4 \approx 1.3863$$

$$41. y = \ln x^3 = 3 \ln x$$

$$y' = \frac{3}{x}$$

$$\text{At } (1, 0), y' = 3.$$

$$43. y = \ln x^2 = 2 \ln x$$

$$y' = \frac{2}{x}$$

$$\text{At } (1, 0), y' = 2.$$

$$45. g(x) = \ln x^2 = 2 \ln x$$

$$g'(x) = \frac{2}{x}$$

$$47. y = (\ln x)^4$$

$$\frac{dy}{dx} = 4(\ln x)^3 \left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

$$49. y = \ln x \sqrt{x^2 - 1} = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2 - 1} \right) = \frac{2x^2 - 1}{x(x^2 - 1)}$$

$$51. f(x) = \ln \frac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$$

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$$

$$53. g(t) = \frac{\ln t}{t^2}$$

$$g'(t) = \frac{t^2(1/t) - 2t \ln t}{t^4} = \frac{1 - 2 \ln t}{t^3}$$

$$55. y = \ln(\ln x^2)$$

$$\frac{dy}{dx} = \frac{1}{\ln x^2} \frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

$$57. y = \ln \sqrt{\frac{x + 1}{x - 1}} = \frac{1}{2} [\ln(x + 1) - \ln(x - 1)]$$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x + 1} - \frac{1}{x - 1} \right] = \frac{1}{1 - x^2}$$

$$59. f(x) = \ln \frac{\sqrt{4 + x^2}}{x} = \frac{1}{2} \ln(4 + x^2) - \ln x$$

$$f'(x) = \frac{x}{4 + x^2} - \frac{1}{x} = \frac{-4}{x(x^2 + 4)}$$

$$61. y = \frac{-\sqrt{x^2 + 1}}{x} + \ln(x + \sqrt{x^2 + 1})$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-x(x/\sqrt{x^2 + 1}) + \sqrt{x^2 + 1}}{x^2} + \left(\frac{1}{x + \sqrt{x^2 + 1}}\right)\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{x^2\sqrt{x^2 + 1}} + \left(\frac{1}{x + \sqrt{x^2 + 1}}\right)\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right) = \frac{1}{x^2\sqrt{x^2 + 1}} + \frac{1}{\sqrt{x^2 + 1}} = \frac{1 + x^2}{x^2\sqrt{x^2 + 1}} = \frac{\sqrt{x^2 + 1}}{x^2} \end{aligned}$$

$$63. y = \ln|\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

$$65. y = \ln\left|\frac{\cos x}{\cos x - 1}\right|$$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

$$67. y = \ln\left|\frac{-1 + \sin x}{2 + \sin x}\right|$$

$$= \ln|-1 + \sin x| - \ln|2 + \sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1 + \sin x} - \frac{\cos x}{2 + \sin x}$$

$$= \frac{3 \cos x}{(\sin x - 1)(\sin x + 2)}$$

$$69. f(x) = \sin 2x \ln x^2 = 2 \sin 2x \ln x$$

$$f'(x) = (2 \sin 2x)\left(\frac{1}{x}\right) + 4 \cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x \cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x \cos 2x \ln x^2)$$

$$71. (a) y = 3x^2 - \ln x, (1, 3)$$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

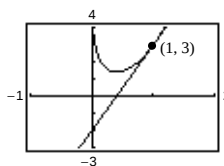
$$\text{When } x = 1, \frac{dy}{dx} = 5.$$

$$\text{Tangent line: } y - 3 = 5(x - 1)$$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



$$73. x^2 - 3 \ln y + y^2 = 10$$

$$2x - \frac{3}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx} \left(\frac{3}{y} - 2y \right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

$$75. y = 2(\ln x) + 3$$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

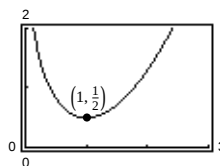
$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

$$77. y = \frac{x^2}{2} - \ln x$$

Domain: $x > 0$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

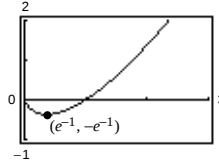
Relative minimum: $(1, \frac{1}{2})$ 

79. $y = x \ln x$

 Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

 Relative minimum: $(e^{-1}, -e^{-1})$


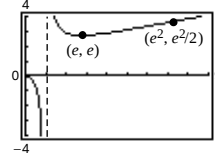
81. $y = \frac{x}{\ln x}$

 Domain: $0 < x < 1, x > 1$

$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x) \ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

 Relative minimum: (e, e)

 Point of inflection: $(e^2, e^2/2)$


83. $f(x) = \ln x, \quad f(1) = 0$

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

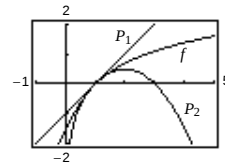
$$= (x - 1) - \frac{1}{2}(x - 1)^2, \quad P_2(1) = 0$$

$$P_1'(x) = 1, \quad P_1'(1) = 1$$

$$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$$

$$P_2''(x) = -1, \quad P_2''(1) = -1$$

The values of f, P_1, P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



85. Find x such that $\ln x = -x$.

$$f(x) = (\ln x) + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{1 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	-0.1931	-0.0076	-0.0001

 Approximate root: $x = 0.567$

87. $y = x\sqrt{x^2 - 1}$

$$\ln y = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$\frac{dy}{dx} = y \left[\frac{2x^2 - 1}{x(x^2 - 1)} \right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$89. \quad y = \frac{x^2 \sqrt{3x-2}}{(x-1)^2}$$

$$\ln y = 2 \ln x + \frac{1}{2} \ln(3x-2) - 2 \ln(x-1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y \left[\frac{3x^2 - 15x + 8}{2x(3x-2)(x-1)} \right]$$

$$= \frac{3x^3 - 15x^2 + 8x}{2(x-1)^3 \sqrt{3x-2}}$$

$$91. \quad y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2} \ln(x-1) - \frac{1}{2} \ln(x+1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{3}{2} \left(\frac{1}{x-1} \right) - \frac{1}{2} \left(\frac{1}{x+1} \right)$$

$$\frac{dy}{dx} = \frac{y}{2} \left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1} \right]$$

$$= \frac{y}{2} \left[\frac{4x^2 + 4x - 2}{x(x^2 - 1)} \right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}$$

93. Answers will vary. See Theorem 5.1 and 5.2.

95. $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

97. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \frac{2}{x} = 0$ for $x = 2$.

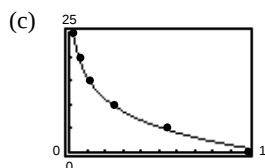
$$99. \quad \beta = 10 \log_{10} \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} \ln \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} [\ln I + 16 \ln 10] = 160 + 10 \log_{10} I$$

$$\beta(10^{-10}) = \frac{10}{\ln 10} [\ln 10^{-10} + 16 \ln 10] = \frac{10}{\ln 10} [-10 \ln 10 + 16 \ln 10] = \frac{10}{\ln 10} [6 \ln 10] = 60 \text{ decibels}$$

101. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474 \ln p.$$



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$$1 = -6.4474 \frac{1}{p} \frac{dp}{dh} \quad (\text{implicit differentiation})$$

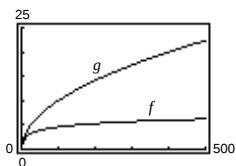
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

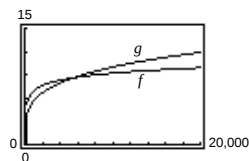
103. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x .

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{4\sqrt[3]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for “large” values of x . $f(x) = \ln x$ increases very slowly for “large” values of x .

105. False

$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

Section 5.2 The Natural Logarithmic Function: Integration

$$1. \int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln |x| + C$$

$$3. u = x + 1, du = dx$$

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

$$5. u = 3 - 2x, du = -2 dx$$

$$\begin{aligned} \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{1}{3-2x} (-2) dx \\ &= -\frac{1}{2} \ln|3-2x| + C \end{aligned}$$

$$7. u = x^2 + 1, du = 2x dx$$

$$\begin{aligned} \int \frac{x}{x^2+1} dx &= \frac{1}{2} \int \frac{1}{x^2+1} (2x) dx \\ &= \frac{1}{2} \ln(x^2+1) + C \\ &= \ln \sqrt{x^2+1} + C \end{aligned}$$

$$\begin{aligned} 9. \int \frac{x^2-4}{x} dx &= \int \left(x - \frac{4}{x} \right) dx \\ &= \frac{x^2}{2} - 4 \ln|x| + C \end{aligned}$$

$$11. u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3) dx$$

$$\begin{aligned} \int \frac{x^2+2x+3}{x^3+3x^2+9x} dx &= \frac{1}{3} \int \frac{3(x^2+2x+3)}{x^3+3x^2+9x} dx \\ &= \frac{1}{3} \ln|x^3+3x^2+9x| + C \end{aligned}$$

$$\begin{aligned} 13. \int \frac{x^2-3x+2}{x+1} dx &= \int \left(x - 4 + \frac{6}{x+1} \right) dx \\ &= \frac{x^2}{2} - 4x + 6 \ln|x+1| + C \end{aligned}$$

$$\begin{aligned} 15. \int \frac{x^3-3x^2+5}{x-3} dx &= \int \left(x^2 + \frac{5}{x-3} \right) dx \\ &= \frac{x^3}{3} + 5 \ln|x-3| + C \end{aligned}$$

$$\begin{aligned} 17. \int \frac{x^4+x-4}{x^2+2} dx &= \int \left(x^2 - 2 + \frac{x}{x^2+2} \right) dx \\ &= \frac{x^3}{3} - 2x + \frac{1}{2} \ln(x^2+2) + C \end{aligned}$$

$$19. u = \ln x, du = \frac{1}{x} dx$$

$$\int \frac{(\ln x)^2}{x} dx = \frac{1}{3} (\ln x)^3 + C$$

$$21. u = x + 1, du = dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{x+1}} dx &= \int (x+1)^{-1/2} dx \\ &= 2(x+1)^{1/2} + C \\ &= 2\sqrt{x+1} + C \end{aligned}$$

$$\begin{aligned} 23. \int \frac{2x}{(x-1)^2} dx &= \int \frac{2x-2+2}{(x-1)^2} dx \\ &= \int \frac{2(x-1)}{(x-1)^2} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \ln|x-1| - \frac{2}{(x-1)} + C \end{aligned}$$

$$25. u = 1 + \sqrt{2x}, du = \frac{1}{\sqrt{2x}} dx \Rightarrow (u - 1) du = dx$$

$$\begin{aligned} \int \frac{1}{1 + \sqrt{2x}} dx &= \int \frac{(u - 1)}{u} du = \int \left(u - \frac{1}{u}\right) du \\ &= u - \ln|u| + C_1 \\ &= (1 + \sqrt{2x}) - \ln|1 + \sqrt{2x}| + C_1 \\ &= \sqrt{2x} - \ln(1 + \sqrt{2x}) + C \end{aligned}$$

where $C = C_1 + 1$.

$$27. u = \sqrt{x} - 3, du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2(u + 3) du = dx$$

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{x} - 3} dx &= 2 \int \frac{(u + 3)^2}{u} du = 2 \int \frac{u^2 + 6u + 9}{u} du = 2 \int \left(u + 6 + \frac{9}{u}\right) du \\ &= 2 \left[\frac{u^2}{2} + 6u + 9 \ln|u| \right] + C_1 = u^2 + 12u + 18 \ln|u| + C_1 \\ &= (\sqrt{x} - 3)^2 + 12(\sqrt{x} - 3) + 18 \ln|\sqrt{x} - 3| + C_1 \\ &= x + 6\sqrt{x} + 18 \ln|\sqrt{x} - 3| + C \text{ where } C = C_1 - 27. \end{aligned}$$

$$29. \int \frac{\cos \theta}{\sin \theta} d\theta = \ln|\sin \theta| + C$$

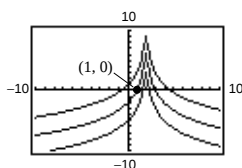
$(u = \sin \theta, du = \cos \theta d\theta)$

$$\begin{aligned} 31. \int \csc 2x dx &= \frac{1}{2} \int (\csc 2x)(2) dx \\ &= -\frac{1}{2} \ln|\csc 2x + \cot 2x| + C \end{aligned}$$

$$33. \int \frac{\cos t}{1 + \sin t} dt = \ln|1 + \sin t| + C$$

$$35. \int \frac{\sec x \tan x}{\sec x - 1} dx = \ln|\sec x - 1| + C$$

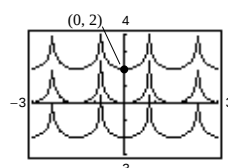
$$\begin{aligned} 37. y &= \int \frac{3}{2 - x} dx \\ &= -3 \int \frac{1}{x - 2} dx \\ &= -3 \ln|x - 2| + C \end{aligned}$$



$$(1, 0): 0 = -3 \ln|1 - 2| + C \Rightarrow C = 0$$

$$y = -3 \ln|x - 2|$$

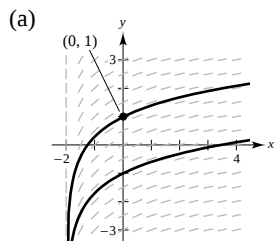
$$\begin{aligned} 39. s &= \int \tan(2\theta) d\theta \\ &= \frac{1}{2} \int \tan(2\theta)(2 d\theta) \\ &= -\frac{1}{2} \ln|\cos 2\theta| + C \end{aligned}$$



$$(0, 2): 2 = -\frac{1}{2} \ln|\cos(0)| + C \Rightarrow C = 2$$

$$s = -\frac{1}{2} \ln|\cos 2\theta| + 2$$

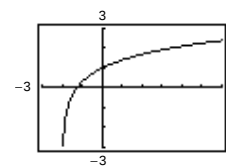
$$41. \frac{dy}{dx} = \frac{1}{x + 2}, (0, 1)$$



$$(b) \quad y = \int \frac{1}{x + 2} dx = \ln|x + 2| + C$$

$$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$$

$$\text{Hence, } y = \ln|x + 2| + 1 - \ln 2 = \ln\left|\frac{x + 2}{2}\right| + 1.$$



$$43. \int_0^4 \frac{5}{3x+1} dx = \left[\frac{5}{3} \ln|3x+1| \right]_0^4$$

$$= \frac{5}{3} \ln 13 \approx 4.275$$

$$45. u = 1 + \ln x, du = \frac{1}{x} dx$$

$$\int_1^e \frac{(1 + \ln x)^2}{x} dx = \left[\frac{1}{3} (1 + \ln x)^3 \right]_1^e = \frac{7}{3}$$

$$47. \int_0^2 \frac{x^2 - 2}{x + 1} dx = \int_0^2 \left(x - 1 - \frac{1}{x + 1} \right) dx$$

$$= \left[\frac{1}{2} x^2 - x - \ln|x + 1| \right]_0^2 = -\ln 3$$

$$49. \int_1^2 \frac{1 - \cos \theta}{\theta - \sin \theta} d\theta = \left[\ln|\theta - \sin \theta| \right]_1^2$$

$$= \ln \left| \frac{2 - \sin 2}{1 - \sin 1} \right| \approx 1.929$$

$$51. -\ln|\cos x| + C = \ln \left| \frac{1}{\cos x} \right| + C = \ln|\sec x| + C$$

$$53. \ln|\sec x + \tan x| + C = \ln \left| \frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)} \right| + C = \ln \left| \frac{\sec^2 x - \tan^2 x}{\sec x - \tan x} \right| + C$$

$$= \ln \left| \frac{1}{\sec x - \tan x} \right| + C = -\ln|\sec x - \tan x| + C$$

$$55. \int \frac{1}{1 + \sqrt{x}} dx = 2(1 + \sqrt{x}) - 2 \ln(1 + \sqrt{x}) + C_1$$

$$= 2[\sqrt{x} - \ln(1 + \sqrt{x})] + C \text{ where } C = C_1 + 2.$$

$$57. \int \cos(1 - x) dx = -\sin(1 - x) + C$$

$$59. \int_{\pi/4}^{\pi/2} (\csc x - \sin x) dx = \left[-\ln|\csc x + \cot x| + \cos x \right]_{\pi/4}^{\pi/2} = \ln(\sqrt{2} + 1) - \frac{\sqrt{2}}{2} \approx 0.174$$

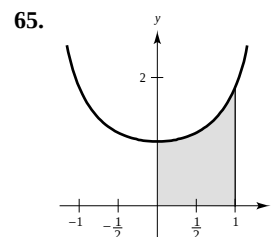
Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.

$$61. F(x) = \int_1^x \frac{1}{t} dt$$

$$F'(x) = \frac{1}{x}$$

$$63. F(x) = \int_x^{3x} \frac{1}{t} dt = \int_1^{3x} \frac{1}{t} dt - \int_1^x \frac{1}{t} dt$$

$$F'(x) = \frac{3}{3x} - \frac{1}{x} = 0$$

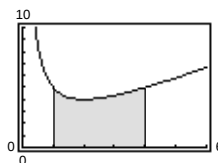


$A \approx 1.25$
Matches (d)

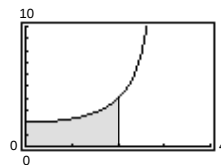
$$67. A = \int_1^4 \frac{x^2 + 4}{x} dx = \int_1^4 \left(x + \frac{4}{x} \right) dx$$

$$= \left[\frac{x^2}{2} + 4 \ln x \right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2}$$

$$= \frac{15}{2} + 8 \ln 2 \approx 13.045 \text{ square units}$$



$$\begin{aligned}
 69. \int_0^2 2 \sec \frac{\pi x}{6} dx &= \frac{12}{\pi} \int_0^2 \sec \left(\frac{\pi x}{6} \right) \frac{\pi}{6} dx \\
 &= \left[\frac{12}{\pi} \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\
 &= \frac{12}{\pi} \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \frac{12}{\pi} \ln |1 + 0| \\
 &= \frac{12}{\pi} \ln(2 + \sqrt{3}) \approx 5.03041
 \end{aligned}$$



71. Power Rule

73. Substitution: ($u = x^2 + 4$)
and Log Rule

75. Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

$$\begin{aligned}
 77. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{8}{x^2} dx = 4 \int_2^4 x^{-2} dx \\
 &= \left[-4 \frac{1}{x} \right]_2^4 \\
 &= -4 \left(\frac{1}{4} - \frac{1}{2} \right) = 1
 \end{aligned}$$

$$\begin{aligned}
 79. \text{Average value} &= \frac{1}{e-1} \int_1^e \frac{\ln x}{x} dx = \frac{1}{e-1} \left[\frac{(\ln x)^2}{2} \right]_1^e \\
 &= \frac{1}{e-1} \left(\frac{1}{2} \right) \\
 &= \frac{1}{2e-2} \approx 0.291
 \end{aligned}$$

$$81. P(t) = \int \frac{3000}{1+0.25t} dt = (3000)(4) \int \frac{0.25}{1+0.25t} dt = 12,000 \ln|1+0.25t| + C$$

$$P(0) = 12,000 \ln|1+0.25(0)| + C = 1000$$

$$C = 1000$$

$$P(t) = 12,000 \ln|1+0.25t| + 1000 = 1000[12 \ln|1+0.25t| + 1]$$

$$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$$

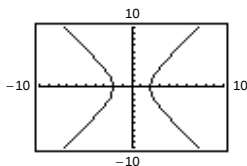
$$83. \frac{1}{50-40} \int_{40}^{50} \frac{90,000}{400+3x} dx = \left[3000 \ln|400+3x| \right]_{40}^{50} \approx \$168.27$$

$$85. (a) 2x^2 - y^2 = 8$$

$$y^2 = 2x^2 - 8$$

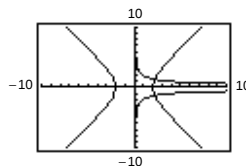
$$y_1 = \sqrt{2x^2 - 8}$$

$$y_2 = -\sqrt{2x^2 - 8}$$



$$(b) y^2 = e^{-\int (1/x) dx} = e^{-\ln x + C} = e^{\ln(1/x)} (e^C) = \frac{1}{x} k$$

$$\text{Let } k = 4 \text{ and graph } y^2 = \frac{4}{x} \quad \left(\begin{array}{l} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x} \end{array} \right)$$



$$(c) \text{ In part (a), } 2x^2 - y^2 = 8$$

$$4x - 2yy' = 0$$

$$y' = \frac{2x}{y}$$

$$\text{In part (b), } y^2 = \frac{4}{x} = 4x^{-1}$$

$$2yy' = \frac{-4}{x^2}$$

$$y' = \frac{-2}{yx^2} = \frac{-2y}{y^2 x^2} = \frac{-2y}{4x} = \frac{-y}{2x}$$

Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

87. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

89. True

$$\begin{aligned}\int \frac{1}{x} dx &= \ln|x| + C_1 \\ &= \ln|x| + \ln|C| = \ln|Cx|, C \neq 0\end{aligned}$$

Section 5.3 Inverse Functions

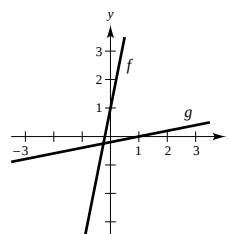
1. (a) $f(x) = 5x + 1$

$$g(x) = \frac{x-1}{5}$$

$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$

$$g(f(x)) = g(5x + 1) = \frac{(5x + 1) - 1}{5} = x$$

(b)



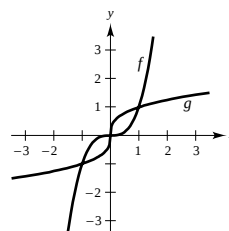
3. (a) $f(x) = x^3$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



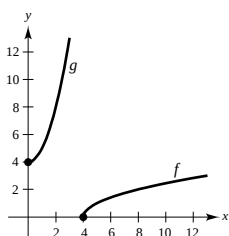
5. (a) $f(x) = \sqrt{x-4}$

$$g(x) = x^2 + 4, x \geq 0$$

$$\begin{aligned}f(g(x)) &= f(x^2 + 4) \\ &= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x\end{aligned}$$

$$\begin{aligned}g(f(x)) &= g(\sqrt{x-4}) \\ &= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x\end{aligned}$$

(b)



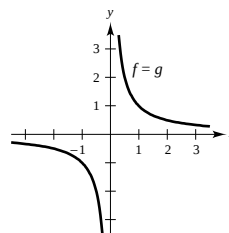
7. (a) $f(x) = \frac{1}{x}$

$$g(x) = \frac{1}{x}$$

$$f(g(x)) = \frac{1}{1/x} = x$$

$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

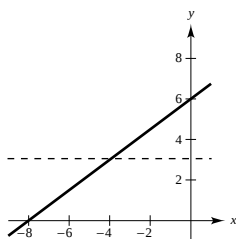


9. Matches (c)

11. Matches (a)

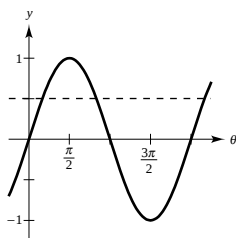
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



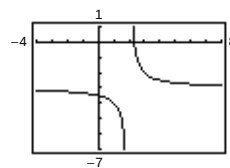
15. $f(\theta) = \sin \theta$

Not one-to-one; does not have an inverse



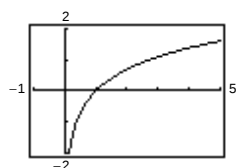
17. $h(s) = \frac{1}{s-2} - 3$

One-to-one; has an inverse



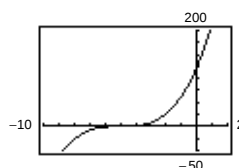
19. $f(x) = \ln x$

One-to-one; has an inverse



21. $g(x) = (x + 5)^3$

One-to-one; has an inverse



23. $f(x) = (x + a)^3 + b$

$$f'(x) = 3(x + a)^2 \geq 0 \text{ for all } x.$$

 f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

25. $f(x) = \frac{x^4}{4} - 2x^2$

$$f'(x) = x^3 - 4x = 0 \text{ when } x = 0, 2, -2.$$

 f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

27. $f(x) = 2 - x - x^3$

$$f'(x) = -1 - 3x^2 < 0 \text{ for all } x.$$

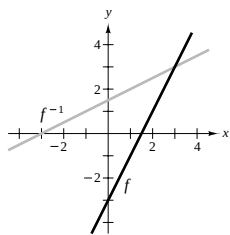
 f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y+3}{2}$$

$$y = \frac{x+3}{2}$$

$$f^{-1}(x) = \frac{x+3}{2}$$

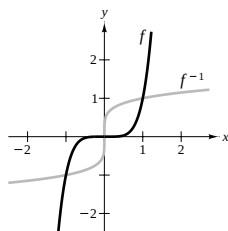


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

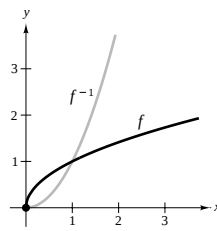


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

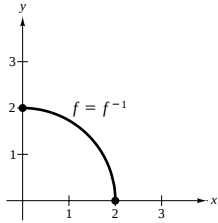


35. $f(x) = \sqrt{4 - x^2} = y, \quad 0 \leq x \leq 2$

$$x = \sqrt{4 - y^2}$$

$$y = \sqrt{4 - x^2}$$

$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

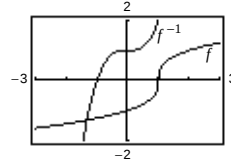


37. $f(x) = \sqrt[3]{x - 1} = y$

$$x = y^3 + 1$$

$$y = x^3 + 1$$

$$f^{-1}(x) = x^3 + 1$$



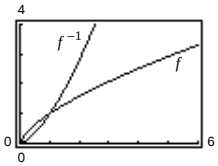
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \quad x \geq 0$

$$x = y^{3/2}$$

$$y = x^{3/2}$$

$$f^{-1}(x) = x^{3/2}, \quad x \geq 0$$



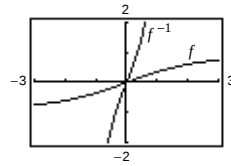
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \frac{x}{\sqrt{x^2 + 7}} = y$

$$x = \frac{\sqrt{7}y}{\sqrt{1 - y^2}}$$

$$y = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}$$

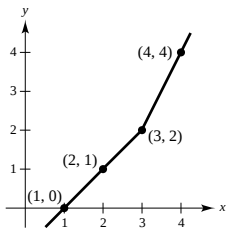
$$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

 43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Let
- x
- be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is
- $50 - x$
- . The total cost is

$$y = 1.25x + 1.60(50 - x)$$

$$= -0.35x + 80 \quad 0 \leq x \leq 50.$$

- (b) We find the inverse of the original function:

$$y = -0.35x + 80$$

$$0.35x = 80 - y$$

$$x = \frac{100}{35}(80 - y)$$

$$\text{Inverse: } y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x).$$

x represents cost and y represents pounds.

- (c) Domain of inverse is
- $62.5 \leq x \leq 80$
- .

- (d) If
- $x = 73$
- in the inverse function,

$$y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20 \text{ pounds.}$$

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$$f'(x) = 2(x - 4) > 0 \text{ on } (4, \infty)$$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$$f'(x) = -\sin x < 0 \text{ on } (0, \pi)$$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \frac{x}{x^2 - 4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} (1 - \sqrt{1 + 16x^2})/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

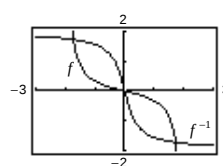
49. $f(x) = \frac{4}{x^2}$ on $(0, \infty)$

$$f'(x) = -\frac{8}{x^3} < 0 \text{ on } (0, \infty)$$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

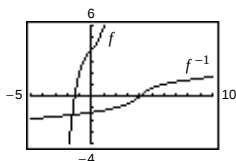
Domain: all x

Range: $-2 < y < 2$



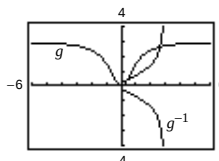
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

59. $f(x) = \sqrt{x - 2}$, Domain: $x \geq 2$

$$f'(x) = \frac{1}{2\sqrt{x - 2}} > 0 \text{ for } x > 2.$$

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2$$

$$x = y^2 + 2$$

$$y = \sqrt{x - 2}$$

$$f^{-1}(x) = x^2 + 2, x \geq 0$$

61. $f(x) = |x - 2|, x \leq 2$

$$= -(x - 2)$$

$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$

$$2 - y = x$$

$$f^{-1}(x) = 2 - x, x \geq 0$$

63. $f(x) = (x - 3)^2$ is one-to-one for $x \geq 3$.

$$(x - 3)^2 = y$$

$$x - 3 = \sqrt{y}$$

$$x = \sqrt{y} + 3$$

$$y = \sqrt{x - 3}$$

$$f^{-1}(x) = \sqrt{x} + 3, x \geq 0$$

(Answer is not unique)

65. $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$

$$x = y - 3$$

$$y = x + 3$$

$$f^{-1}(x) = x - 3, x \geq 0$$

(Answer is not unique)

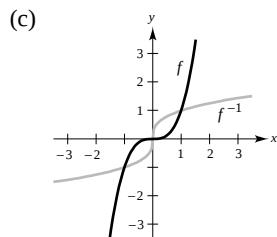
67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V .

$$\begin{aligned} 71. \quad f(x) &= x^3 + 2x - 1, \quad f(1) = 2 = a \\ f'(x) &= 3x^2 + 2 \\ (f^{-1})'(2) &= \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5} \end{aligned}$$

$$\begin{aligned} 75. \quad f(x) &= x^3 - \frac{4}{x}, \quad f(2) = 6 = a \\ f'(x) &= 3x^2 + \frac{4}{x^2} \\ (f^{-1})'(6) &= \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13} \end{aligned}$$

77. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

- (b) Range f = Range $f^{-1} = (-\infty, \infty)$



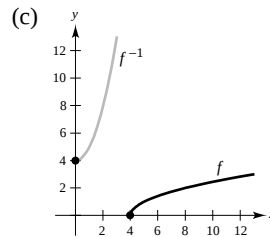
$$\begin{aligned} (d) \quad f(x) &= x^3, \quad \left(\frac{1}{2}, \frac{1}{8}\right) \\ f'(x) &= 3x^2 \\ f'\left(\frac{1}{2}\right) &= \frac{3}{4} \\ f^{-1}(x) &= \sqrt[3]{x}, \quad \left(\frac{1}{8}, \frac{1}{2}\right) \\ (f^{-1})'(x) &= \frac{1}{3\sqrt[3]{x^2}} \\ (f^{-1})'\left(\frac{1}{8}\right) &= \frac{4}{3} \end{aligned}$$

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

$$\begin{aligned} 73. \quad f(x) &= \sin x, \quad f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a \\ f'(x) &= \cos x \\ (f^{-1})'\left(\frac{1}{2}\right) &= \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)} \\ &= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3} \end{aligned}$$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

- (b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$



$$\begin{aligned} (d) \quad f(x) &= \sqrt{x-4}, \quad (5, 1) \\ f'(x) &= \frac{1}{2\sqrt{x-4}} \\ f'(5) &= \frac{1}{2} \\ f^{-1}(x) &= x^2 + 4, \quad (1, 5) \\ (f^{-1})'(x) &= 2x \\ (f^{-1})'(1) &= 2 \end{aligned}$$

81. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2 \frac{dy}{dx} - 14y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}. \text{ At } (-4, 1), \frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}.$$

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}.$$

In Exercises 83 and 85, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x + 3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

In Exercises 87 and 89, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

87. $(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$$= g^{-1}(x - 4)$$

$$= \frac{(x - 4) + 5}{2}$$

$$= \frac{x + 1}{2}$$

89. $(f \circ g)(x) = f(g(x))$

$$= f(2x - 5)$$

$$= (2x - 5) + 4$$

$$= 2x - 1$$

$$\text{Hence, } (f \circ g)^{-1}(x) = \frac{x + 1}{2}$$

$$(\text{Note: } (f \circ g)^{-1} = g^{-1} \circ f^{-1})$$

91. Answers will vary. See page 335 and Example 3.

93. $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.95. f is not one-to-one because many different x -values yield the same y -value.

$$\text{Example: } f(0) = f(\pi) = 0$$

Not continuous at $\frac{(2n-1)\pi}{2}$, where n is an integer97. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$$(f \circ g)(x) = y$$

$$f(g(x)) = y$$

$$g(x) = f^{-1}(y)$$

$$x = g^{-1}(f^{-1}(y))$$

$$= (g^{-1} \circ f^{-1})(y)$$

Since f and g are one-to-one functions,
 $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a) . Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

101. False

$$\text{Let } f(x) = x^2.$$

105. Not true

$$\text{Let } f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1 - x, & 1 < x \leq 2 \end{cases}$$

 f is one-to-one, but not strictly monotonic.

103. True

$$107. \quad f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}, \quad f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

3. $\ln 2 = 0.6931$

$e^{0.6931} \approx 2$

5. $e^{\ln x} = x$

$x = 4$

7. $e^x = 12$

$x = \ln 12 \approx 2.485$

9. $9 - 2e^x = 7$

$2e^x = 2$

$e^x = 1$

$x = 0$

11. $50e^{-x} = 30$

$e^{-x} = \frac{3}{5}$

$-x = \ln\left(\frac{3}{5}\right)$

$x = \ln\left(\frac{5}{3}\right) \approx 0.511$

13. $\ln x = 2$

$x = e^2 \approx 7.3891$

15. $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

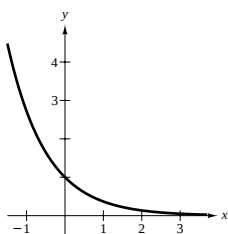
17. $\ln\sqrt{x+2} = 1$

$\sqrt{x+2} = e^1 = e$

$x + 2 = e^2$

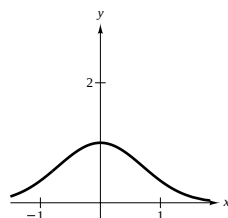
$x = e^2 - 2 \approx 5.389$

19. $y = e^{-x}$

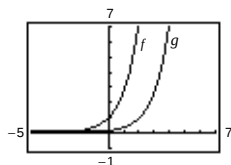


21. $y = e^{-x^2}$

Symmetric with respect to the y-axis

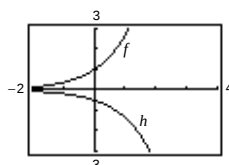
 Horizontal asymptote: $y = 0$


23. (a)



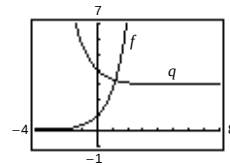
Horizontal shift 2 units to the right

(b)



A reflection in the x -axis and a vertical shrink

(c)



Vertical shift 3 units upward and a reflection in the y -axis

25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

27. $y = C(1 - e^{-ax})$

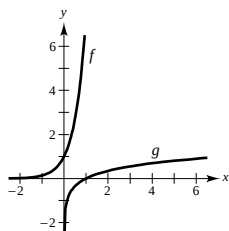
Vertical shift C units

Reflection in both the x - and y -axes

Matches (a)

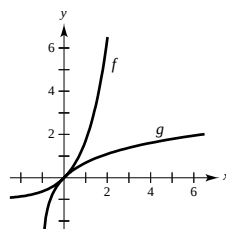
29. $f(x) = e^{2x}$

$$g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

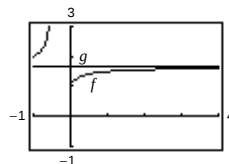


31. $f(x) = e^x - 1$

$$g(x) = \ln(x + 1)$$



33.



As $x \rightarrow \infty$, the graph of f approaches the graph of g .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

35. $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000} \approx 2.718280469$

$$e \approx 2.718281828$$

$$e > \left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$$

37. (a) $y = e^{3x}$

$$y' = 3e^{3x}$$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$$y' = -3e^{-3x}$$

At $(0, 1)$, $y' = -3$.

39. $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

41. $f(x) = e^{-2x+x^2}$

$$\frac{dy}{dx} = 2(x - 1)e^{-2x+x^2}$$

43. $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

45. $g(t) = (e^{-t} + e^t)^3$

$$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$$

47. $y = \ln e^{x^2} = x^2$

$$\frac{dy}{dx} = 2x$$

49. $y = \ln(1 + e^{2x})$

$$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$$

$$51. y = \frac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$$

$$\begin{aligned} \frac{dy}{dx} &= -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) \\ &= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2} \end{aligned}$$

$$55. f(x) = e^{-x} \ln x$$

$$f'(x) = e^{-x} \left(\frac{1}{x} \right) - e^{-x} \ln x = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

$$59. xe^y - 10x + 3y = 0$$

$$xe^y \frac{dy}{dx} + e^y - 10 + 3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

$$63. y = e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$y' = e^x (-\sqrt{2} \sin \sqrt{2}x + \sqrt{2} \cos \sqrt{2}x) + e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$= e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$y'' = e^x [-(\sqrt{2} + 2) \sin \sqrt{2}x + (\sqrt{2} - 2) \cos \sqrt{2}x] + e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$= e^x [(-1 - 2\sqrt{2}) \sin \sqrt{2}x + (-1 + 2\sqrt{2}) \cos \sqrt{2}x]$$

$$-2y' + 3y = -2e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x] + 3e^x [\cos \sqrt{2}x + \sin \sqrt{2}x]$$

$$= e^x [(1 - 2\sqrt{2}) \cos \sqrt{2}x + (1 + 2\sqrt{2}) \sin \sqrt{2}x] = -y''$$

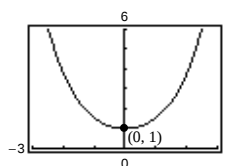
Therefore, $-2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0$.

$$65. f(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{e^x + e^{-x}}{2} > 0$$

Relative minimum: $(0, 1)$



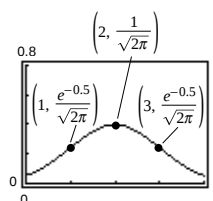
$$67. g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$$

$$g'(x) = \frac{-1}{\sqrt{2\pi}} (x - 2) e^{-(x-2)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}} (x - 1)(x - 3) e^{-(x-2)^2/2}$$

Relative maximum: $\left(2, \frac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



$$53. y = x^2 e^x - 2x e^x + 2e^x = e^x (x^2 - 2x + 2)$$

$$\frac{dy}{dx} = e^x (2x - 2) + e^x (x^2 - 2x + 2) = x^2 e^x$$

$$57. y = e^x (\sin x + \cos x)$$

$$\frac{dy}{dx} = e^x (\cos x - \sin x) + (\sin x + \cos x) e^x$$

$$= e^x (2 \cos x) = 2e^x \cos x$$

$$61. f(x) = (3 + 2x)e^{-3x}$$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

69. $f(x) = x^2 e^{-x}$

$$f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x}(2 - x) = 0 \text{ when } x = 0, 2.$$

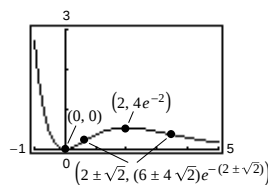
$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2) = 0 \text{ when } x = 2 \pm \sqrt{2}.$$

Relative minimum: $(0, 0)$ Relative maximum: $(2, 4e^{-2})$

$$x = 2 \pm \sqrt{2}$$

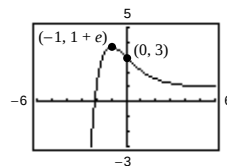
$$y = (2 \pm \sqrt{2})^2 e^{-(2 \pm \sqrt{2})}$$

Points of inflection: $(3.414, 0.384), (0.586, 0.191)$ 

71. $g(t) = 1 + (2 + t)e^{-t}$

$$g'(t) = (1 + t)e^{-t}$$

$$g''(t) = te^{-t}$$

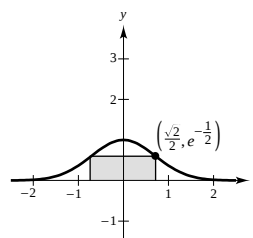
Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$ Point of inflection: $(0, 3)$ 

73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2 e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$



75. $y = \frac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2 \left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right) 2(1 + ae^{-x/b}) \left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b}) \left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right) \left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{La e^{-x/b} [ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

77. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

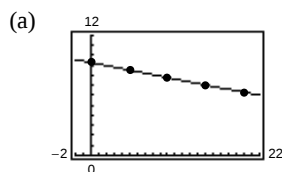
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

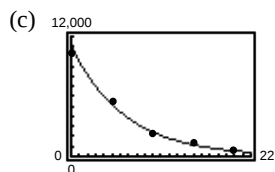
We approximate the root of f to be $x = 0.567$.

 81.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.



83. $f(x) = e^{x/2}, f(0) = 1$

$$f'(x) = \frac{1}{2}e^{x/2}, f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{x/2}, f''(0) = \frac{1}{4}$$

$$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, P_1(0) = 1$$

$$P_1'(x) = \frac{1}{2}, P_1'(0) = \frac{1}{2}$$

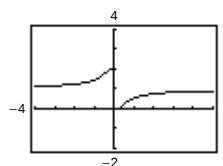
$$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1, P_2(0) = 1$$

$$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, P_2'(0) = \frac{1}{2}$$

$$P_2''(x) = \frac{1}{4}, P_2''(0) = \frac{1}{4}$$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

79. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞ , $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

(b) $\ln P = ah + b$

$$P = e^{ah+b} = e^b e^{ah}$$

$$P = C e^{ah}, C = e^b$$

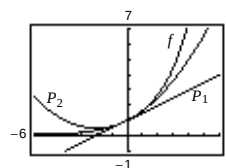
For our data, $a = -0.1499$ and $C = e^{9.3018} = 10,957.7$

$$P = 10,957.7 e^{-0.1499h}$$

(d) $\frac{dP}{dh} = (10,957.71)(-0.1499)e^{-0.1499h}$

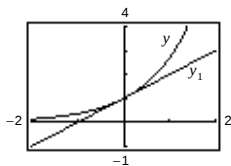
$$= -1642.56 e^{-0.1499h}$$

For $h = 5$, $\frac{dP}{dh} = -776.3$. For $h = 18$, $\frac{dP}{dh} \approx -110.6$.



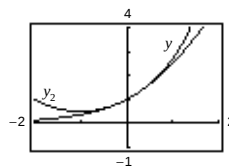
85. (a) $y = e^x$

$$y_1 = 1 + x$$



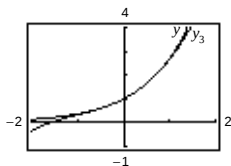
(b) $y = e^x$

$$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$$



(c) $y = e^x$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$



87. Let $u = 5x$, $du = 5 dx$.

$$\int e^{5x} 5dx = e^{5x} + C$$

89. Let $u = -2x$, $du = -2 dx$.

$$\begin{aligned} \int_0^1 e^{-2x} dx &= -\frac{1}{2} \int_0^1 e^{-2x} (-2) dx = \left[-\frac{1}{2} e^{-2x} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-2}) = \frac{e^2 - 1}{2e^2} \end{aligned}$$

91. $\int x e^{-x^2} dx = -\frac{1}{2} \int e^{-x^2} (-2x) dx = -\frac{1}{2} e^{-x^2} + C$

93. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2e^{\sqrt{x}} + C$

95. Let $u = 1 + e^{-x}$, $du = -e^{-x} dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}} dx = -\int \frac{-e^{-x}}{1 + e^{-x}} dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

97. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2} dx$.

$$\begin{aligned} \int_1^3 \frac{e^{3/x}}{x^2} dx &= -\frac{1}{3} \int_1^3 e^{3/x} \left(-\frac{3}{x^2} \right) dx \\ &= \left[-\frac{1}{3} e^{3/x} \right]_1^3 = \frac{e}{3} (e^2 - 1) \end{aligned}$$

99. Let $u = 1 - e^x$, $du = -e^x dx$.

$$\begin{aligned} \int e^x \sqrt{1 - e^x} dx &= -\int (1 - e^x)^{1/2} (-e^x) dx \\ &= -\frac{2}{3} (1 - e^x)^{3/2} + C \end{aligned}$$

101. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x}) dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln|e^x - e^{-x}| + C$$

$$\begin{aligned} 103. \int \frac{5 - e^x}{e^{2x}} dx &= \int 5e^{-2x} dx - \int e^{-x} dx \\ &= -\frac{5}{2} e^{-2x} + e^{-x} + C \end{aligned}$$

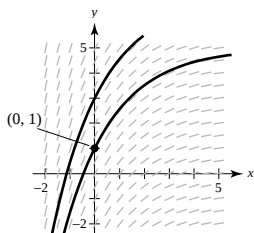
$$\begin{aligned} 105. \int e^{\sin \pi x} \cos \pi x dx &= \frac{1}{\pi} \int e^{\sin \pi x} (\pi \cos \pi x) dx \\ &= \frac{1}{\pi} e^{\sin \pi x} + C \end{aligned}$$

$$\begin{aligned} 107. \int e^{-x} \tan(e^{-x}) dx &= -\int [\tan(e^{-x})] (-e^{-x}) dx \\ &= \ln|\cos(e^{-x})| + C \end{aligned}$$

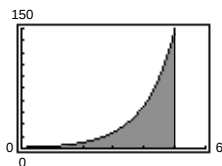
109. Let $u = ax^2$, $du = 2ax \, dx$. (Assume $a \neq 0$)

$$\begin{aligned} y &= \int x e^{ax^2} \, dx \\ &= \frac{1}{2a} \int e^{ax^2} (2ax) \, dx = \frac{1}{2a} e^{ax^2} + C \end{aligned}$$

113. (a)



115. $\int_0^5 e^x \, dx = \left[e^x \right]_0^5 = e^5 - 1 \approx 147.413$



119. (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

123. $\int_0^x e^t \, dt \geq \int_0^x 1 \, dt$

$$\left[e^t \right]_0^x \geq \left[t \right]_0^x$$

$$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x \text{ for } x \geq 0$$

127. Yes. $f(x) = Ce^x$, C a constant.

111. $f'(x) = \int \frac{1}{2}(e^x + e^{-x}) \, dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

$$f'(0) = C_1 = 0$$

$$f(x) = \int \frac{1}{2}(e^x - e^{-x}) \, dx = \frac{1}{2}(e^x + e^{-x}) + C_2$$

$$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$$

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

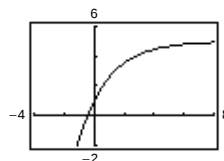
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2} \, dx = -4 \int e^{-x/2} \left(-\frac{1}{2} dx \right)$$

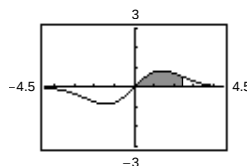
$$= -4e^{-x/2} + C$$

$$(0, 1): 1 = -4e^0 + C = -4 + C \Rightarrow C = 5$$

$$y = -4e^{-x/2} + 5$$



117. $\int_0^{\sqrt{6}} x e^{-x^2/4} \, dx = \left[-2e^{-x^2/4} \right]_0^{\sqrt{6}}$
 $= -2e^{-3/2} + 2 \approx 1.554$



121. $0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2} \, dt$

Graphing Utility: $0.4772 = 47.72\%$

125. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$$\lim_{x \rightarrow -\infty} e^x = 0 \text{ and } \lim_{x \rightarrow \infty} e^x = \infty.$$

129. $e^{-x} > 0 \Rightarrow \int_0^2 e^{-x} \, dx > 0$.

131. $f(x) = \frac{\ln x}{x}$

(a) $f'(x) = \frac{1 - \ln x}{x^2} = 0$ when $x = e$.

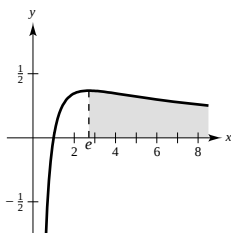
On $(0, e)$, $f'(x) > 0 \Rightarrow f$ is increasing.On (e, ∞) , $f'(x) < 0 \Rightarrow f$ is decreasing.(b) For $e \leq A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.Section 5.5 Bases Other than e and Applications

1. $y = \left(\frac{1}{2}\right)^{t/3}$

At $t_0 = 6$, $y = \left(\frac{1}{2}\right)^{6/3} = \frac{1}{4}$

3. $y = \left(\frac{1}{2}\right)^{t/7}$

At $t_0 = 10$, $y = \left(\frac{1}{2}\right)^{10/7} \approx 0.3715$

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

7. $\log_7 1 = 0$

9. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \frac{1}{3}$

$\log_3 \frac{1}{3} = -1$

11. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

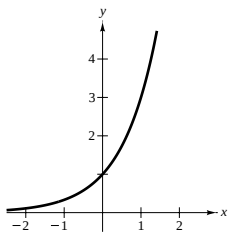
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

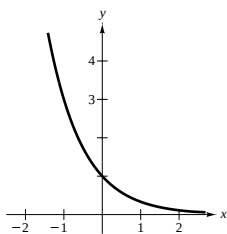
13. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



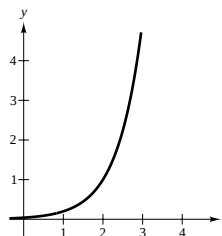
15. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



17. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



19. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$

$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$

$$x = -1$$

23. (a) $x^2 - x = \log_5 25$

$$x^2 - x = \log_5 5^2 = 2$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \text{ OR } x = 2$$

25. $3^{2x} = 75$

$$2x \ln 3 = \ln 75$$

$$x = \frac{1}{2} \frac{\ln 75}{\ln 3} \approx 1.965$$

29. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$

$$t = \frac{1}{12} \frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

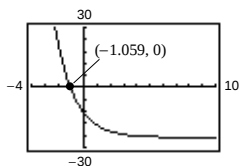
33. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$

$$x = \pm \sqrt{3^{4.5}} \approx \pm 11.845$$

35. $g(x) = 6(2^{1-x}) - 25$

$$\text{Zero: } x \approx -1.059$$



21. (a) $\log_3 x = -1$

$$3^{-1} = x$$

$$x = \frac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$

$$x = \frac{1}{16}$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$

$$3x = 1$$

$$x = \frac{1}{3}$$

27. $2^{3-x} = 625$

$$(3 - x) \ln 2 = \ln 625$$

$$3 - x = \frac{\ln 625}{\ln 2}$$

$$x = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

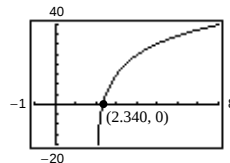
31. $\log_2(x - 1) = 5$

$$x - 1 = 2^5 = 32$$

$$x = 33$$

37. $h(s) = 32 \log_{10}(s - 2) + 15$

$$\text{Zero: } s \approx 2.340$$

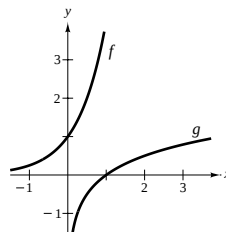


39. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1



41. $f(x) = 4^x$

$f'(x) = (\ln 4) 4^x$

43. $y = 5^{x-2}$

$\frac{dy}{dx} = (\ln 5) 5^{x-2}$

45. $g(t) = t^2 2^t$

$$\begin{aligned} g'(t) &= t^2 (\ln 2) 2^t + (2t) 2^t \\ &= t 2^t (t \ln 2 + 2) \\ &= 2^t t(2 + t \ln 2) \end{aligned}$$

47. $h(\theta) = 2^{-\theta} \cos \pi \theta$

$$\begin{aligned} h'(\theta) &= 2^{-\theta}(-\pi \sin \pi \theta) - (\ln 2) 2^{-\theta} \cos \pi \theta \\ &= -2^{-\theta}[(\ln 2) \cos \pi \theta + \pi \sin \pi \theta] \end{aligned}$$

49. $y = \log_3 x$

$\frac{dy}{dx} = \frac{1}{x \ln 3}$

51. $f(x) = \log_2 \frac{x^2}{x-1}$

$= 2 \log_2 x - \log_2 (x-1)$

$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$

$= \frac{x-2}{(\ln 2)x(x-1)}$

53. $y = \log_5 \sqrt{x^2 - 1} = \frac{1}{2} \log_5 (x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2 - 1) \ln 5} = \frac{x}{(x^2 - 1) \ln 5}$

55. $g(t) = \frac{10 \log_4 t}{t} = \frac{10}{\ln 4} \left(\frac{\ln t}{t} \right)$

$g'(t) = \frac{10}{\ln 4} \left[\frac{t(1/t) - \ln t}{t^2} \right]$

$= \frac{10}{t^2 \ln 4} [1 - \ln t] = \frac{5}{t^2 \ln 2} (1 - \ln t)$

57. $y = x^{2/x}$

$\ln y = \frac{2}{x} \ln x$

$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} \left(\frac{1}{x} \right) + \ln x \left(-\frac{2}{x^2} \right) = \frac{2}{x^2} (1 - \ln x)$

$\frac{dy}{dx} = \frac{2y}{x^2} (1 - \ln x) = 2x^{(2/x)-2} (1 - \ln x)$

59. $y = (x-2)^{x+1}$

$\ln y = (x+1) \ln(x-2)$

$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x+1) \left(\frac{1}{x-2} \right) + \ln(x-2)$

$\frac{dy}{dx} = y \left[\frac{x+1}{x-2} + \ln(x-2) \right]$

$= (x-2)^{x+1} \left[\frac{x+1}{x-2} + \ln(x-2) \right]$

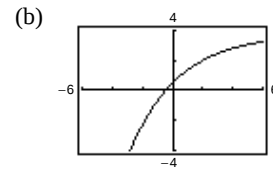
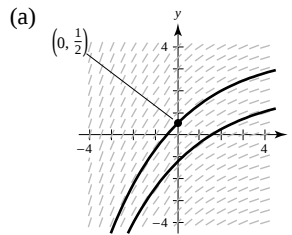
61. $\int 3^x dx = \frac{3^x}{\ln 3} + C$

$$\begin{aligned}
 63. \int_{-1}^2 2^x dx &= \left[\frac{2^x}{\ln 2} \right]_{-1}^2 \\
 &= \frac{1}{\ln 2} \left[4 - \frac{1}{2} \right] \\
 &= \frac{7}{2 \ln 2} = \frac{7}{\ln 4}
 \end{aligned}$$

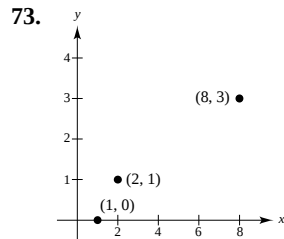
$$\begin{aligned}
 65. \int x 5^{-x^2} dx &= -\frac{1}{2} \int 5^{-x^2} (-2x) dx \\
 &= -\left(\frac{1}{2} \right) \frac{5^{-x^2}}{\ln 5} + C \\
 &= -\frac{1}{2 \ln 5} (5^{-x^2}) + C
 \end{aligned}$$

$$\begin{aligned}
 67. \int \frac{3^{2x}}{1 + 3^{2x}} dx, u = 1 + 3^{2x}, du = 2(\ln 3)3^{2x} dx \\
 \frac{1}{2 \ln 3} \int \frac{(2 \ln 3)3^{2x}}{1 + 3^{2x}} dx = \frac{1}{2 \ln 3} \ln(1 + 3^{2x}) + C
 \end{aligned}$$

$$\begin{aligned}
 69. \frac{dy}{dx} &= 0.4^{x/3}, \left(0, \frac{1}{2}\right) \\
 y &= \int 0.4^{x/3} dx = 3 \int 0.4^{x/3} \left(\frac{1}{3} dx\right) \\
 &= \frac{3}{\ln 0.4} 0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C \\
 y &= 3 \ln 2.5 (0.4)^{x/3} + \frac{1}{2} - 3 \ln 2.5 \\
 &= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}
 \end{aligned}$$



71. Answers will vary. Example: Growth and decay problems.



x	1	2	8
y	0	1	3

- (a) y is an exponential function of x : False
 (b) y is a logarithmic function of x : True; $y = \log_2 x$
 (c) x is an exponential function of y : True, $2^y = x$
 (d) y is a linear function of x : False

$$\begin{aligned}
 75. f(x) &= \log_2 x \Rightarrow f'(x) = \frac{1}{x \ln 2} \\
 g(x) &= x^x \Rightarrow g'(x) = x^x(1 + \ln x)
 \end{aligned}$$

[Note: Let $y = g(x)$. Then: $\ln y = \ln x^x = x \ln x$

$$\begin{aligned}
 \frac{1}{y} y' &= x \cdot \frac{1}{x} + \ln x \\
 y' &= y(1 + \ln x) \\
 y' &= x^x(1 + \ln x) = g'(x).]
 \end{aligned}$$

$$h(x) = x^2 \Rightarrow h'(x) = 2x$$

$$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$$

From greatest to smallest rate of growth:

$$g(x), k(x), h(x), f(x)$$

$$77. C(t) = P(1.05)^t$$

$$(a) C(10) = 24.95(1.05)^{10} \approx \$40.64$$

$$(b) \frac{dC}{dt} = P(\ln 1.05)(1.05)^t$$

$$\text{When } t = 1: \frac{dC}{dt} \approx 0.051P$$

$$\text{When } t = 8: \frac{dC}{dt} \approx 0.072P$$

$$\begin{aligned}
 (c) \frac{dC}{dt} &= (\ln 1.05)[P(1.05)^t] \\
 &= (\ln 1.05)C(t)
 \end{aligned}$$

The constant of proportionality is $\ln 1.05$.

79. $P = \$1000$, $r = 3\frac{1}{2}\% = 0.035$, $t = 10$

$$A = 1000\left(1 + \frac{0.035}{n}\right)^{10n}$$

$$A = 1000e^{(0.035)(10)} = 1419.07$$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

81. $P = \$1000$, $r = 5\% = 0.05$, $t = 30$

$$A = 1000\left(1 + \frac{0.05}{n}\right)^{30n}$$

$$A = 1000e^{(0.05)(30)} = 4481.69$$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

83. $100,000 = Pe^{0.05t} \Rightarrow P = 100,000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,583.53	8208.50

85. $100,000 = P\left(1 + \frac{0.05}{12}\right)^{12t} \Rightarrow P = 100,000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

87. (a) $A = 20,000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32,320.21$

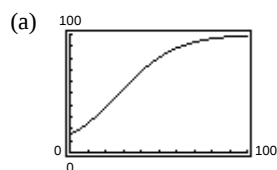
(b) $A = \$30,000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20,000\left(1 + \frac{0.06}{365}\right)^{(365)(4)}$
 $\approx \$12,928.09 + 25,424.48 = \$38,352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right]$
 $\approx \$34,985.11$

Take option (c).

91. $y = \frac{300}{3 + 17e^{-0.0625x}}$



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

89. (a) $\lim_{t \rightarrow \infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \frac{322.27}{t^2} e^{-(48.1)/t}$

$$V'(20) \approx 0.073 \text{ million } \text{ft}^3/\text{yr}$$

$$V'(60) \approx 0.040 \text{ million } \text{ft}^3/\text{yr}$$

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

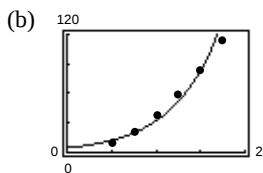
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$$

$$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$$

$$17e^{-0.0625x} - 3 = 0 \Rightarrow x \approx 27.8 \text{ or } 27,800 \text{ egg masses.}$$

93. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$



(c) $B'(d) = 9.0952e^{1.9132d}$

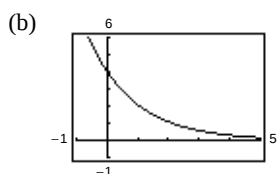
$$B'(0.8) \approx 42.03 \text{ tons/inch}$$

$$B'(1.5) \approx 160.38 \text{ tons/inch}$$

95. (a) $\int_0^4 f(t) dt \approx 5.67$

$$\int_0^4 g(t) dt \approx 5.67$$

$$\int_0^4 h(t) dt \approx 5.67$$



97.
$$P = \int_0^{10} 2000e^{-0.06t} dt$$

$$= \left[\frac{2000}{-0.06} e^{-0.06t} \right]_0^{10}$$

$$\approx \$15,039.61$$

 (c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$$y = C(k^t)$$

When $t = 0, y = 1200 \Rightarrow C = 1200$.

$$y = 1200(k^t)$$

$$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$$

 Let $k = 0.6$.

$$y = 1200(0.6)^t$$

 101. False. e is an irrational number.

103. True.

$$f(g(x)) = 2 + e^{\ln(x-2)}$$

$$= 2 + x - 2 = x$$

$$g(f(x)) = \ln(2 + e^x - 2)$$

$$= \ln e^x = x$$

105. True.

$$\frac{d}{dx}[e^x] = e^x \text{ and } \frac{d}{dx}[e^{-x}] = -e^{-x}$$

$$e^x = e^{-x} \text{ when } x = 0.$$

$$(e^0)(-e^{-0}) = -1$$

$$\begin{aligned}
 107. \quad \frac{dy}{dt} &= \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1 \\
 \frac{dy}{y[(5/4) - y]} &= \frac{8}{25}dt \Rightarrow \frac{4}{5} \int \left(\frac{1}{y} + \frac{1}{(5/4) - y}\right) dy = \int \frac{8}{25} dt \Rightarrow \\
 \ln y - \ln\left(\frac{5}{4} - y\right) &= \frac{2}{5}t + C \\
 \ln\left(\frac{y}{(5/4) - y}\right) &= \frac{2}{5}t + C \\
 \frac{y}{(5/4) - y} &= e^{(2/5)t + C} = C_1 e^{(2/5)t} \\
 y(0) = 1 &\Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{(5/4) - y} \\
 \Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) &= y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y \\
 \Rightarrow y &= \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}
 \end{aligned}$$

Section 5.6 Differential Equations: Growth and Decay

$$1. \quad \frac{dy}{dx} = x + 2$$

$$y = \int (x + 2)dx = \frac{x^2}{2} + 2x + C$$

$$3. \quad \frac{dy}{dx} = y + 2$$

$$\frac{dy}{y + 2} = dx$$

$$\int \frac{1}{y + 2} dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 2$$

$$5. \quad y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy' dx = \int 5x dx$$

$$\int y dy = \int 5x dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

$$7. \quad y' = \sqrt{x}y$$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int \frac{y'}{y} dx = \int \sqrt{x} dx$$

$$\int \frac{dy}{y} = \int \sqrt{x} dx$$

$$\ln y = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2} + C_1}$$

$$= e^{C_1} e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

9. $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$

$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$

$$\int \frac{y'}{y} dx = \int \frac{2x}{1 + x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2} dx$$

$$\ln y = \ln(1 + x^2) + C_1$$

$$\ln y = \ln(1 + x^2) + \ln C$$

$$\ln y = \ln C(1 + x^2)$$

$$y = C(1 + x^2)$$

11. $\frac{dQ}{dt} = \frac{k}{t^2}$

$$\int \frac{dQ}{dt} dt = \int \frac{k}{t^2} dt$$

$$\int dQ = -\frac{k}{t} + C$$

$$Q = -\frac{k}{t} + C$$

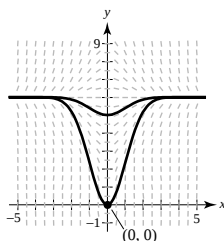
13. $\frac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds} ds = \int k(250 - s) ds$$

$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$

$$N = -\frac{k}{2}(250 - s)^2 + C$$

15. (a)



(b) $\frac{dy}{dx} = x(6 - y), (0, 0)$

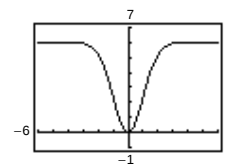
$$\frac{dy}{y - 6} = -x$$

$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0): 0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$$



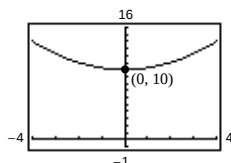
17. $\frac{dy}{dt} = \frac{1}{2}t, (0, 10)$

$$\int dy = \int \frac{1}{2}t dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



19. $\frac{dy}{dt} = -\frac{1}{2}y, (0, 10)$

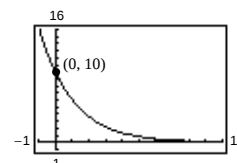
$$\int \frac{dy}{y} = \int -\frac{1}{2} dt$$

$$\ln y = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1} e^{-t/2} = C e^{-t/2}$$

$$10 = C e^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



21. $\frac{dy}{dx} = ky$

$$y = Ce^{kx} \text{ (Theorem 5.16)}$$

$$(0, 4): 4 = Ce^0 = C$$

$$(3, 10): 10 = 4e^{3k} \Rightarrow k = \frac{1}{3} \ln\left(\frac{5}{2}\right)$$

$$\text{When } x = 6, y = 4e^{1/3 \ln(5/2)(6)} = 4e^{\ln(5/2)^2}$$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

23. $\frac{dV}{dt} = kV$

$$V = Ce^{kt} \text{ (Theorem 5.16)}$$

$$(0, 20,000): C = 20,000$$

$$(4, 12,500): 12,500 = 20,000e^{4k} \Rightarrow k = \frac{1}{4} \ln\left(\frac{5}{8}\right)$$

$$\text{When } t = 6, V = 20,000e^{1/4 \ln(5/8)(6)} = 20,000e^{\ln(5/8)^{3/2}}$$

$$= 20,000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

25. $y = Ce^{kt}, \left(0, \frac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2}e^{0.4605t}$$

27. $y = Ce^{kt}, (1, 1), (5, 5)$

$$1 = Ce^k$$

$$5 = Ce^{5k}$$

$$5Ce^k = Ce^{5k}$$

$$5e^k = e^{5k}$$

$$5 = e^{4k}$$

$$k = \frac{\ln 5}{4} \approx 0.4024$$

$$y = Ce^{0.4024t}$$

$$1 = Ce^{0.4024}$$

$$C \approx 0.6687$$

$$y = 0.6687e^{0.4024t}$$

29. A differential equation in x and y is an equation that involves x , y and derivatives of y .

31. $\frac{dy}{dx} = \frac{1}{2}xy$

$$\frac{dy}{dx} > 0 \text{ when } xy > 0. \text{ Quadrants I and III.}$$

33. Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10,000$, $y = 10e^{[\ln(1/2)/1620](10,000)} \approx 0.14$ gram.

35. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10,000)} \Rightarrow C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

37. Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10,000$, $y \approx 1.49$ g.

39. Since $y = Ce^{[\ln(1/2)/24,360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24,360](1000)} \Rightarrow C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10,000$, $y = 2.16e^{[\ln(1/2)/24,360](10,000)} \approx 1.63$ grams.

41. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$

$$k = \frac{-\ln 2}{1620}$$

$$y = y_0e^{-(\ln 2)t/1620}$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$.

Therefore, 95.81% of the present amount still exists.

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \1822.12

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \1833.67

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

49. $500,000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500,000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112,087.09$$

51. $500,000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500,000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30,688.87$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

55. (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.0085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.0085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

(d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

57. $P = Ce^{kt} = Ce^{-0.009t}$

$$P(-1) = 8.2 = Ce^{-0.009(-1)} \Rightarrow C = 8.1265$$

$$P = 8.1265e^{-0.009t}$$

$$P(10) \approx 7.43 \quad \text{or} \quad 7,430,000 \text{ people in 2010}$$

61. If $k < 0$, the population decreases.If $k > 0$, the population increases.

65. (a) $19 = 30(1 - e^{20k})$

$$30e^{20k} = 11$$

$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$

$$N \approx 30(1 - e^{-0.0502t})$$

67. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$$5 = Ce^k$$

$$\lim_{t \rightarrow \infty} Ce^{k/t} = C = 30$$

$$5 = 30e^k$$

$$k = \ln \frac{1}{6} \approx -1.7918$$

$$S \approx 30e^{-1.7918/t}$$

59. $P = Ce^{kt} = Ce^{0.036t}$

$$P(-1) = 4.6 = Ce^{0.036(-1)} \Rightarrow C = 4.7686$$

$$P = 4.7686e^{0.036t}$$

$$P(10) \approx 6.83 \quad \text{or} \quad 6,830,000 \text{ people in 2010}$$

63. $P = Ce^{kx}, (0, 760), (1000, 672.71)$

$$C = 760$$

$$672.71 = 760e^{1000k}$$

$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$

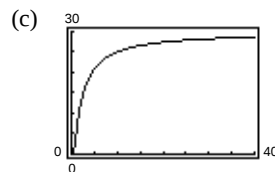
$$P \approx 760e^{-0.000122x}$$

When $x = 3000$, $P \approx 527.06$ mm Hg.

(b) $25 = 30(1 - e^{-0.0502t})$

$$e^{-0.0502t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

69. $A(t) = V(t)e^{-0.10t} = 100,000e^{0.8\sqrt{t}}e^{-0.10t} = 100,000e^{0.8\sqrt{t}-0.10t}$

$$\frac{dA}{dt} = 100,000 \left(\frac{0.4}{\sqrt{t}} - 0.10 \right) e^{0.8\sqrt{t}-0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, $(1998 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \leq x \leq 30$ and $0 \leq y \leq 600,000$.

71. $\beta(I) = 10 \log_{10} \frac{I}{I_0}, I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \frac{10^{-14}}{10^{-16}} = 20 \text{ decibels}$

(b) $\beta(10^{-9}) = 10 \log_{10} \frac{10^{-9}}{10^{-16}} = 70 \text{ decibels}$

(c) $\beta(10^{-6.5}) = 10 \log_{10} \frac{10^{-6.5}}{10^{-16}} = 95 \text{ decibels}$

(d) $\beta(10^{-4}) = 10 \log_{10} \frac{10^{-4}}{10^{-16}} = 120 \text{ decibels}$

73. $R = \frac{\ln I - 0}{\ln 10}, I = e^{R \ln 10} = 10^R$

(a) $8.3 = \frac{\ln I - 0}{\ln 10}$

$$I = 10^{8.3} \approx 199,526,231.5$$

(b) $2R = \frac{\ln I - 0}{\ln 10}$

$$I = e^{2R \ln 10} = e^{2 \ln 10} = (e^{\ln 10})^2 = (10^R)^2$$

Increases by a factor of $e^{2R \ln 10}$ or 10^R .

(c) $\frac{dR}{dI} = \frac{1}{I \ln 10}$

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq \text{constant}$.

77. True

Section 5.7 Differential Equations: Separation of Variables

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$y'' = -C_1 \cos x - C_2 \sin x$

$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$

5. $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x} (\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$

$$= \tan x.$$

In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.

7. $y = 3 \cos x$

$$y^{(4)} = 3 \cos x$$

$$y^{(4)} - 16y = -45 \cos x \neq 0,$$

No.

9. $y = e^{-2x}$

$$y^{(4)} = 16e^{-2x}$$

$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$

Yes.

11. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

$$y^{(4)} = 16C_1 e^{2x} + 16C_2 e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x$$

$$y^{(4)} - 16y = 0,$$

Yes.

In 13–17, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2, y' = 2x$

$$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$$

No.

17. $y = \ln x, y' = \frac{1}{x}$

$$xy' - 2y = x\left(\frac{1}{x}\right) - 2\ln x \neq x^3e^x, \quad \text{No.}$$

19. $y = Ce^{kx}$

$$\frac{dy}{dx} = Cke^{kx}$$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

15. $y = x^2(2 + e^x), y' = x^2(e^x) + 2x(2 + e^x)$

$$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x,$$

Yes.

21. $y^2 = Cx^3$ passes through $(4, 4)$

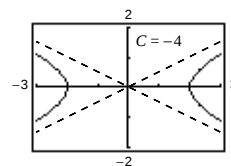
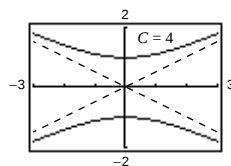
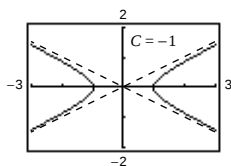
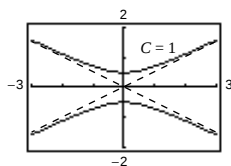
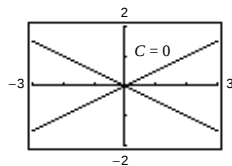
$$16 = C(64) \Rightarrow C = \frac{1}{4}$$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

23. Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines
 $C = \pm 1$, $C = \pm 4$, Hyperbolas



25. Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

27. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$

$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$

$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) +$$

$$9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions: $y\left(\frac{\pi}{6}\right) = 2, y'\left(\frac{\pi}{6}\right) = 1$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right)$$

$$= -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2, y'' = 6C_2x$$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) +$$

$$3(C_1x + C_2x^3) = 0$$

Initial conditions: $y(2) = 0, y'(2) = 4$

$$0 = 2C_1 + 8C_2$$

$$y' = C_1 + 3C_2x^2$$

$$4 = C_1 + 12C_2$$

$$\left. \begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array} \right\} C_2 = \frac{1}{2}, C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

31. $\frac{dy}{dx} = 3x^2$

$$y = \int 3x^2 dx = x^3 + C$$

33. $\frac{dy}{dx} = \frac{x}{1+x^2}$

$$y = \int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$$

$$(u = 1+x^2, du = 2x dx)$$

35. $\frac{dy}{dx} = \frac{x-2}{x} = 1 - \frac{2}{x}$

$$y = \int \left[1 - \frac{2}{x} \right] dx$$

$$= x - 2 \ln|x| + C = x - \ln x^2 + C$$

37. $\frac{dy}{dx} = \sin 2x$

$$y = \int \sin 2x dx = -\frac{1}{2} \cos 2x + C$$

$$(u = 2x, du = 2dx)$$

39. $\frac{dy}{dx} = x\sqrt{x-3}$ Let $u = \sqrt{x-3}$, then $x = u^2 + 3$ and $dx = 2u du$.

$$y = \int x\sqrt{x-3} dx = \int (u^2 + 3)(u)(2u) du$$

$$= 2 \int (u^4 + 3u^2) du = 2 \left(\frac{u^5}{5} + u^3 \right) + C = \frac{2}{5} (x-3)^{5/2} + 2(x-3)^{3/2} + C$$

41. $\frac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$$

$$(u = x^2, du = 2x dx)$$

43. $\frac{dy}{dx} = \frac{x}{y}$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = C$$

45. $\frac{dr}{ds} = 0.05r$

$$\int \frac{dr}{r} = \int 0.05 ds$$

$$\ln |r| = 0.05s + C_1$$

$$r = e^{0.05s + C_1} = Ce^{0.05s}$$

47. $(2+x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2+x} dx$$

$$\ln y = 3 \ln(2+x) + \ln C = \ln C(2+x)^3$$

$$y = C(x+2)^3$$

49. $yy' = \sin x$

$$\int y \, dy = \int \sin x \, dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2 \cos x + C$$

53. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x} dx \quad \left(u = \ln x, du = \frac{dx}{x} \right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

57. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1) dx$$

$$\ln y = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1$, $1 = Ce^{-1/2}$, $C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

61. $\frac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2 \, dv$$

$$\ln u = -\frac{1}{2} \cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1$, $C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

51. $\sqrt{1-4x^2} \frac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned} \int dy &= \int \frac{x}{\sqrt{1-4x^2}} dx \\ &= -\frac{1}{8} \int (1-4x^2)^{-1/2} (-8x \, dx) \end{aligned}$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

55. $yy' - e^x = 0$

$$\int y \, dy = \int e^x dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4$, $16 = 2 + C$, $C = 14$

Particular solution: $y^2 = 2e^x + 14$

59. $y(1+x^2) \frac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2} dy = \frac{x}{1+x^2} dx$$

$$\frac{1}{2} \ln(1+y^2) = \frac{1}{2} \ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1+y^2 = C(1+x^2)$$

$$y(0) = \sqrt{3}: 1+3 = C \Rightarrow C = 4$$

$$1+y^2 = 4(1+x^2)$$

$$y^2 = 3+4x^2$$

63. $dP - kP \, dt = 0$

$$\int \frac{dP}{P} = k \int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0$, $P_0 = Ce^0 = C$

Particular solution: $P = P_0 e^{kt}$

$$65. \quad \frac{dy}{dx} = \frac{-9x}{16y}$$

$$\int 16y \, dy = -\int 9x \, dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

$$\text{Initial condition: } y(1) = 1, \quad 8 = -\frac{9}{2} + C, \quad C = \frac{25}{2}$$

$$\text{Particular solution: } 8y^2 = \frac{-9}{2}x^2 + \frac{25}{2},$$

$$16y^2 + 9x^2 = 25$$

$$69. \quad f(x, y) = x^3 - 4xy^2 + y^3$$

$$f(tx, ty) = t^3 x^3 - 4t x t^2 y^2 + t^3 y^3$$

$$= t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

$$73. \quad f(x, y) = 2 \ln xy$$

$$f(tx, ty) = 2 \ln tx ty$$

$$= 2 \ln t^2 xy = 2(\ln t^2 + \ln xy)$$

Not homogeneous

$$77. \quad y' = \frac{x+y}{2x}, \quad y = vx$$

$$v + x \frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x \frac{dv}{dx} = \frac{1+v}{2} - v$$

$$2 \int \frac{dv}{1-v} = \int \frac{dx}{x}$$

$$-\ln(1-v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1-v^2)} = |Cx|$$

$$\frac{1}{[1-(y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x-y)^2} = |Cx|$$

$$|x| = C(x-y)^2$$

$$67. \quad m = \frac{dy}{dx} = \frac{0-y}{(x+2)-x} = -\frac{y}{2}$$

$$\int \frac{dy}{y} = \int -\frac{1}{2} dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

$$71. \quad f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

$$f(tx, ty) = \frac{t^4 x^2 y^2}{\sqrt{t^2 x^2 + t^2 y^2}} = t^3 \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

$$75. \quad f(x, y) = 2 \ln \frac{x}{y}$$

$$f(tx, ty) = 2 \ln \frac{tx}{ty} = 2 \ln \frac{x}{y}$$

Homogeneous degree 0

$$79. \quad y' = \frac{x-y}{x+y}, \quad y = vx$$

$$v + x \frac{dv}{dx} = \frac{x - xv}{x + xv}$$

$$v \, dx + x \, dv = \frac{1-v}{1+v} dx$$

$$\int \frac{v+1}{v^2+2v-1} dv = -\int \frac{dx}{x}$$

$$\frac{1}{2} \ln|v^2+2v-1| = -\ln|x| + \ln C_1 = \ln \left| \frac{C_1}{x} \right|$$

$$|v^2+2v-1| = \frac{C}{x^2}$$

$$\left| \frac{y^2}{x^2} + 2\frac{y}{x} - 1 \right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

$$81. \quad y' = \frac{xy}{x^2 - y^2}, y = vx$$

$$v + x \frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$

$$v dx + x dv = \frac{v}{1 - v^2} dx$$

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$

$$\frac{-1}{2v^2} = \ln|C_1 x v|$$

$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$

$$y = C e^{-x^2/2y^2}$$

$$85. \quad \left(x \sec \frac{y}{x} + y \right) dx - x dy = 0, y = vx$$

$$(x \sec v + xv) dx - x(v dx + x dv) = 0$$

$$(\sec v + v) dx = v dx + x dv$$

$$\int \cos v dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

$$x = C e^{\sin v}$$

$$= C e^{\sin(y/x)}$$

$$\text{Initial condition: } y(1) = 0, 1 = C e^0 = C$$

$$\text{Particular solution: } x = e^{\sin(y/x)}$$

$$83. \quad x dy - (2xe^{-y/x} + y) dx = 0, y = vx$$

$$x(v dx + x dv) - (2xe^{-v} + vx) dx = 0$$

$$\int e^v dv = \int \frac{2}{x} dx$$

$$e^v = \ln C_1 x^2$$

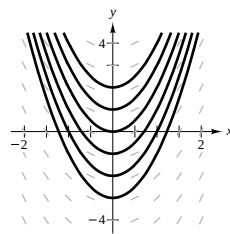
$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

$$\text{Initial condition: } y(1) = 0, 1 = C$$

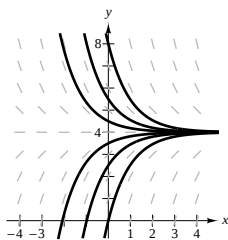
$$\text{Particular solution: } e^{y/x} = 1 + \ln x^2$$

$$87. \quad \frac{dy}{dx} = x$$



$$y = \int x dx = \frac{1}{2} x^2 + C$$

$$89. \quad \frac{dy}{dx} = 4 - y$$



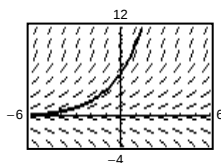
$$\int \frac{dy}{4 - y} = \int dx$$

$$\ln|4 - y| = -x + C_1$$

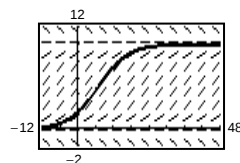
$$4 - y = e^{-x + C_1}$$

$$y = 4 + C e^{-x}$$

$$91. \quad \frac{dy}{dx} = 0.5y, y(0) = 6$$



$$93. \quad \frac{dy}{dx} = 0.02y(10 - y), y(0) = 2$$



95. $\frac{dy}{dt} = ky, y = Ce^{kt}$

 Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

 Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

 When $t = 25$, $y \approx 0.989y_0$, $y = 98.9\%$ of y_0 .

99. $\frac{dy}{dx} = ky(y - 4)$

 The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

101. $\frac{dw}{dt} = k(1200 - w)$

$$\int \frac{dw}{1200 - w} = \int k dt$$

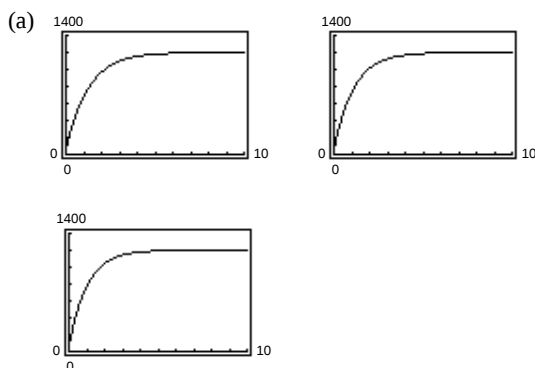
$$\ln(1200 - w) = -kt + C_1$$

$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$


 (b) $k = 0.8$: $t = 1.31$ years

 $k = 0.9$: $t = 1.16$ years

 $k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$$\lim_{t \rightarrow \infty} w = 1200$$

97. $\frac{dy}{dx} = k(y - 4)$

 The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

103. (a) $\frac{dv}{dt} = k(W - v)$

$$\int \frac{dv}{W - v} = \int k dt$$

$$-\ln(W - v) = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

 $W = 20, v = 0$ when $t = 0$, and

 $v = 5$ when $t = 1$.

$$C = 20, k = -\ln(3/4)$$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) \approx 20(1 - e^{-0.2877t})$$

$$\begin{aligned} \text{(b) } s &= \int 20(1 - e^{-0.2877t}) dt \\ &\approx 20[t + 3.4761e^{-0.2877t}] + C \end{aligned}$$

 Since $S(0) = 0$, $C \approx -69.5$ and we have

$$s \approx 20t + 69.5(e^{-0.2877t} - 1).$$

105. Given family (circles):
- $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

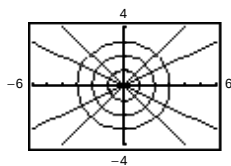
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$



107. Given family (parabolas):
- $x^2 = Cy$

$$2x = Cy'$$

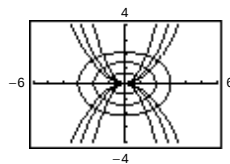
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2 \int y \, dy = - \int x \, dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



109. Given family:
- $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

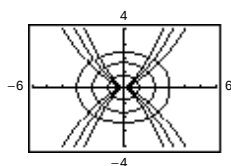
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y} \left(\frac{y^2}{x^3} \right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3 \int y \, dy = -2 \int x \, dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



111. A general solution of order
- n
- has
- n
- arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

- 113.
- $M(x, y)dx + N(x, y)dy = 0$
- , where
- M
- and
- N
- are homogeneous functions of the same degree.

115. False. Consider Example 2.
- $y = x^3$
- is a solution to
- $xy' - 3y = 0$
- , but
- $y = x^3 + 1$
- is not a solution.

117. False

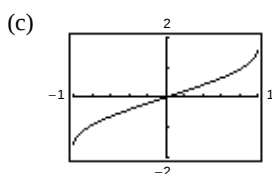
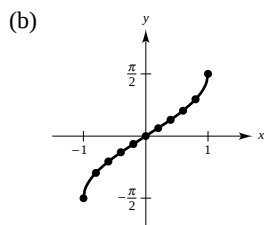
$$\begin{aligned} f(tx, ty) &= t^2x^2 + t^2xy + 2 \\ &\neq t^2f(x, y) \end{aligned}$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571



(d) Symmetric about origin:
 $\arcsin(-x) = -\arcsin x$
 Intercept: (0, 0)

3. False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

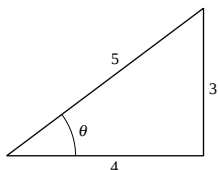
9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

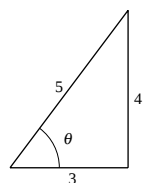
13. $\arccos(-0.8) \approx 2.50$

15. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

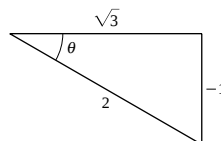
17. (a) $\sin(\arctan \frac{3}{4}) = \frac{3}{5}$



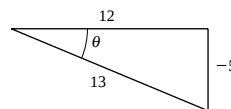
(b) $\sec(\arcsin \frac{4}{5}) = \frac{5}{3}$



19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



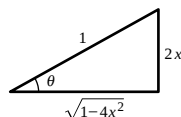
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



21. $y = \cos(\arcsin 2x)$

$$\theta = \arcsin 2x$$

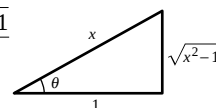
$$y = \cos \theta = \sqrt{1 - 4x^2}$$



23. $y = \sin(\operatorname{arcsec} x)$

$$\theta = \operatorname{arcsec} x, 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$$

$$y = \sin \theta = \frac{\sqrt{x^2 - 1}}{|x|}$$

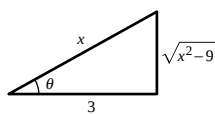


The absolute value bars on x are necessary because of the restriction $0 \leq \theta \leq \pi, \theta \neq \pi/2$, and $\sin \theta$ for this domain must always be nonnegative.

25. $y = \tan\left(\operatorname{arcsec} \frac{x}{3}\right)$

$$\theta = \operatorname{arcsec} \frac{x}{3}$$

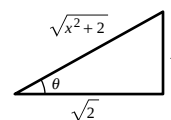
$$y = \tan \theta = \frac{\sqrt{x^2 - 9}}{3}$$



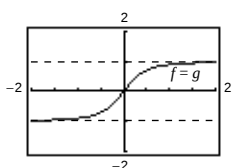
27. $y = \csc\left(\arctan \frac{x}{\sqrt{2}}\right)$

$$\theta = \arctan \frac{x}{\sqrt{2}}$$

$$y = \csc \theta = \frac{\sqrt{x^2 + 2}}{x}$$



29. $\sin(\arctan 2x) = \frac{2x}{\sqrt{1 + 4x^2}}$

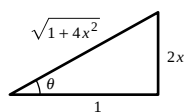


Asymptotes: $y = \pm 1$

$$\arctan 2x = \theta$$

$$\tan \theta = 2x$$

$$\sin \theta = \frac{2x}{\sqrt{1 + 4x^2}}$$



33. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$

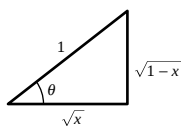
$$\sqrt{2x} = \sin(\arccos \sqrt{x})$$

$$\sqrt{2x} = \sqrt{1 - x}, 0 \leq x \leq 1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$



35. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, |x| \geq 1$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \leq y < 0 \text{ and } 0 < y \leq \frac{\pi}{2},$$

$$\csc y = x \Rightarrow \sin y = 1/x. \text{ Thus, } y = \arcsin(1/x).$$

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x) \tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

37. $f(x) = \arcsin(x - 1)$

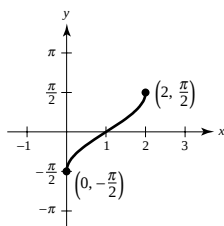
$$x - 1 = \sin y$$

$$x = 1 + \sin y$$

$$\text{Domain: } [0, 2]$$

$$\text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.



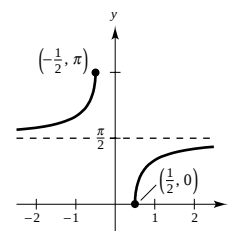
39. $f(x) = \operatorname{arcsec} 2x$

$$2x = \sec y$$

$$x = \frac{1}{2} \sec y$$

$$\text{Domain: } \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$$

$$\text{Range: } \left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$$



41. $f(x) = 2 \arcsin(x - 1)$

$$f'(x) = \frac{2}{\sqrt{1 - (x - 1)^2}} = \frac{2}{\sqrt{2x - x^2}}$$

45. $f(x) = \arctan \frac{x}{a}$

$$f'(x) = \frac{1/a}{1 + (x^2/a^2)} = \frac{a}{a^2 + x^2}$$

49. $h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$

$$h'(t) = \frac{1}{2}(1 - t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1 - t^2}}$$

53. $y = \frac{1}{2} \left(\frac{1}{2} \ln \frac{x+1}{x-1} + \arctan x \right)$

$$= \frac{1}{4} [\ln(x+1) - \ln(x-1)] + \frac{1}{2} \arctan x$$

$$\frac{dy}{dx} = \frac{1}{4} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

57. $y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$

$$\begin{aligned} y' &= 2 \frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x) \\ &= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} \\ &= \frac{16 - (16-x^2) + x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}} \end{aligned}$$

61. $f(x) = \arcsin x, a = \frac{1}{2}$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$

43. $g(x) = 3 \arccos \frac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

47. $g(x) = \frac{\arcsin 3x}{x}$

$$\begin{aligned} g'(x) &= \frac{x(3/\sqrt{1-9x^2}) - \arcsin 3x}{x^2} \\ &= \frac{3x - \sqrt{1-9x^2} \arcsin 3x}{x^2 \sqrt{1-9x^2}} \end{aligned}$$

51. $y = x \arccos x - \sqrt{1-x^2}$

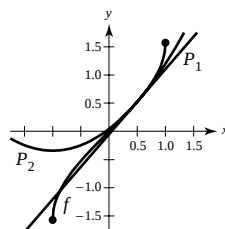
$$\begin{aligned} y' &= \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x) \\ &= \arccos x \end{aligned}$$

55. $y = x \arcsin x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x \left(\frac{1}{\sqrt{1-x^2}} \right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

59. $y = \arctan x + \frac{x}{1+x^2}$

$$\begin{aligned} y' &= \frac{1}{1+x^2} + \frac{(1+x^2) - x(2x)}{(1+x^2)^2} \\ &= \frac{(1+x^2) + (1-x^2)}{(1+x^2)^2} \\ &= \frac{2}{(1+x^2)^2} \end{aligned}$$



$$\begin{aligned}
 63. \quad f(x) &= \operatorname{arcsec} x - x \\
 f'(x) &= \frac{1}{|x|\sqrt{x^2-1}} - 1 \\
 &= 0 \text{ when } |x|\sqrt{x^2-1} = 1. \\
 x^2(x^2-1) &= 1
 \end{aligned}$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm \sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: (1.272, -0.606)

Relative minimum: (-1.272, 3.747)

67. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

$$71. (a) \cot \theta = \frac{x}{5}$$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

$$73. (a) \quad h(t) = -16t^2 + 256$$

$$-16t^2 + 256 = 0 \text{ when } t = 4 \text{ sec.}$$

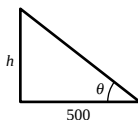
$$(b) \tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$$

$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15,625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.



$$\begin{aligned}
 65. \quad f(x) &= \arctan x - \arctan(x-4) \\
 f'(x) &= \frac{1}{1+x^2} - \frac{1}{1+(x-4)^2} = 0 \\
 1+x^2 &= 1+(x-4)^2 \\
 0 &= -8x + 16 \\
 x &= 2
 \end{aligned}$$

By the First Derivative Test, (2, 2.214) is a relative maximum.

$$69. \quad y = \operatorname{arccot} x, 0 < y < \pi$$

$$x = \cot y$$

$$\tan y = \frac{1}{x}$$

So, graph the function

$$y = \arctan\left(\frac{1}{x}\right) \text{ for } x > 0 \text{ and } y = \arctan\left(\frac{1}{x}\right) + \pi \text{ for } x < 0.$$

$$(b) \frac{d\theta}{dt} = \frac{-\frac{1}{5}}{1 + \left(\frac{x}{5}\right)^2} \frac{dx}{dt} = \frac{-5}{x^2 + 25} \frac{dx}{dt}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 10, \frac{d\theta}{dt} = 16 \text{ rad/hr.}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 3, \frac{d\theta}{dt} \approx 58.824 \text{ rad/hr.}$$

$$75. \tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \frac{x + y}{1 - xy}, xy \neq 1$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan \frac{(1/2) + (1/3)}{1 - [(1/2) \cdot (1/3)]} = \arctan \frac{5/6}{1 - (1/6)} = \arctan \frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

$$77. f(x) = kx + \sin x$$

$$f'(x) = k + \cos x \geq 0 \text{ for } k \geq 1$$

$$f'(x) = k + \cos x \leq 0 \text{ for } k \leq -1$$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

79. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

81. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

$$1. \int \frac{5}{\sqrt{9 - x^2}} dx = 5 \arcsin\left(\frac{x}{3}\right) + C$$

$$3. \text{ Let } u = 3x, du = 3 dx.$$

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}} dx = \frac{1}{3} \int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}} (3) dx = \left[\frac{1}{3} \arcsin(3x) \right]_0^{1/6} = \frac{\pi}{18}$$

$$5. \int \frac{7}{16 + x^2} dx = \frac{7}{4} \arctan\left(\frac{x}{4}\right) + C$$

$$7. \text{ Let } u = 2x, du = 2 dx.$$

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2} dx = \frac{1}{2} \int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2} dx = \left[\frac{1}{2} \arctan(2x) \right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

$$9. \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx = \operatorname{arcsec}|2x| + C$$

$$11. \int \frac{x^3}{x^2 + 1} dx = \int \left[x - \frac{x}{x^2 + 1} \right] dx = \int x dx - \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} x^2 - \frac{1}{2} \ln(x^2 + 1) + C \quad (\text{Use long division.})$$

$$13. \int \frac{1}{\sqrt{1 - (x + 1)^2}} dx = \arcsin(x + 1) + C$$

$$15. \text{ Let } u = t^2, du = 2t dt.$$

$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{1}{\sqrt{1 - (t^2)^2}} (2t) dt = \frac{1}{2} \arcsin(t^2) + C$$

17. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}} dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left[\frac{1}{2} \arcsin^2 x \right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

19. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} \int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_{-1/2}^0 (1-x^2)^{-1/2} (-2x) dx \\ &= \left[-\sqrt{1-x^2} \right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2} \\ &\approx -0.134 \end{aligned}$$

21. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{4 + e^{4x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{4 + (e^{2x})^2} dx = \frac{1}{4} \arctan \frac{e^{2x}}{2} + C$$

23. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned} \int_{\pi/2}^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= -\int_{\pi/2}^{\pi} \frac{-\sin x}{1 + \cos^2 x} dx \\ &= \left[-\arctan(\cos x) \right]_{\pi/2}^{\pi} = \frac{\pi}{4} \end{aligned}$$

25. $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$. $u = \sqrt{x}$, $x = u^2$, $dx = 2u du$

$$\begin{aligned} \int \frac{1}{u\sqrt{1-u^2}} (2u du) &= 2 \int \frac{du}{\sqrt{1-u^2}} = 2 \arcsin u + C \\ &= 2 \arcsin \sqrt{x} + C \end{aligned}$$

27. $\int \frac{x-3}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx - 3 \int \frac{1}{x^2+1} dx$

$$= \frac{1}{2} \ln(x^2+1) - 3 \arctan x + C$$

29. $\int \frac{x+5}{\sqrt{9-(x-3)^2}} dx = \int \frac{(x-3)}{\sqrt{9-(x-3)^2}} dx + \int \frac{8}{\sqrt{9-(x-3)^2}} dx$

$$= -\sqrt{9-(x-3)^2} - 8 \arcsin\left(\frac{x-3}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 8 \arcsin\left(\frac{x}{3}-1\right) + C$$

31. $\int_0^2 \frac{1}{x^2-2x+2} dx = \int_0^2 \frac{1}{1+(x-1)^2} dx = \left[\arctan(x-1) \right]_0^2 = \frac{\pi}{2}$

33. $\int \frac{2x}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{4+(x+3)^2} dx$

$$= \ln|x^2+6x+13| - 3 \arctan\left(\frac{x+3}{2}\right) + C$$

35. $\int \frac{1}{\sqrt{-x^2-4x}} dx = \int \frac{1}{\sqrt{4-(x+2)^2}} dx = \arcsin\left(\frac{x+2}{2}\right) + C$

37. Let $u = -x^2 - 4x$, $du = (-2x - 4) dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}} dx = -\frac{1}{2} \int (-x^2-4x)^{-1/2} (-2x-4) dx = -\sqrt{-x^2-4x} + C$$

39. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}} dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}} dx = -\int_2^3 (4x-x^2)^{-1/2} (4-2x) dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}} dx$

$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right) \right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

41. Let $u = x^2 + 1$, $du = 2x dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2} dx = \frac{1}{2} \int \frac{2x}{(x^2 + 1)^2 + 1} dx = \frac{1}{2} \arctan(x^2 + 1) + C$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u du = e^t dt$, and $\frac{2u du}{u^2 + 3} = dt$.

$$\begin{aligned} \int \sqrt{e^t - 3} dt &= \int \frac{2u^2}{u^2 + 3} du = \int 2 du - \int 6 \frac{1}{u^2 + 3} du \\ &= 2u - 2\sqrt{3} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3} \arctan \sqrt{\frac{e^t - 3}{3}} + C \end{aligned}$$

45. A perfect square trinomial is an expression in x with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x + 3)^2$

47. (a) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$, $u = x$ (b) $\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$, $u = 1-x^2$

- (c) $\int \frac{1}{x\sqrt{1-x^2}} dx$ cannot be evaluated using the basic integration rules.

49. (a) $\int \sqrt{x-1} dx = \frac{2}{3}(x-1)^{3/2} + C$, $u = x-1$

- (b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

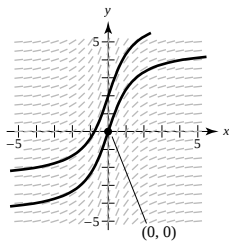
$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (u^2 + 1)(u)(2u) du = 2 \int (u^4 + u^2) du = 2 \left(\frac{u^5}{5} + \frac{u^3}{3} \right) + C \\ &= \frac{2}{15} u^3(3u^2 + 5) + C = \frac{2}{15} (x-1)^{3/2} [3(x-1) + 5] + C = \frac{2}{15} (x-1)^{3/2} (3x+2) + C \end{aligned}$$

- (c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u^2 + 1}{u} (2u) du = 2 \int (u^2 + 1) du = 2 \left(\frac{u^3}{3} + u \right) + C = \frac{2}{3} u(u^2 + 3) + C = \frac{2}{3} \sqrt{x-1} (x+2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

51. (a)

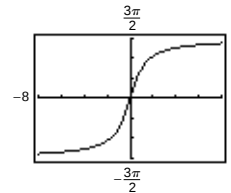


- (b) $\frac{dy}{dx} = \frac{3}{1+x^2}$, $(0, 0)$

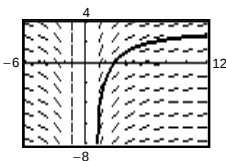
$$y = 3 \int \frac{dx}{1+x^2} = 3 \arctan x + C$$

$$(0, 0): 0 = 3 \arctan(0) + C \Rightarrow C = 0$$

$$y = 3 \arctan x$$



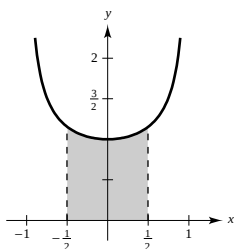
53. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$



$$\begin{aligned} 55. A &= \int_1^3 \frac{1}{x^2 - 2x + 1 + 4} dx = \int_1^3 \frac{1}{(x-1)^2 + 2^2} dx \\ &= \left[\frac{1}{2} \arctan \left(\frac{x-1}{2} \right) \right]_1^3 = \frac{1}{2} \arctan(1) = \frac{\pi}{8} \approx 0.3927 \end{aligned}$$

57. Area $\approx (1)(1) = 1$

Matches (c)



59. (a) $\int_0^1 \frac{4}{1+x^2} dx = \left[4 \arctan x \right]_0^1 = 4 \arctan 1 - 4 \arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4 \int_0^1 \frac{4}{1+x^2} dx \approx 4\left(\frac{1}{36}\right) \left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2} \right] \approx 3.1415918$$

(c) 3.1415927

61. (a) $\frac{d}{dx} \left[\arcsin\left(\frac{u}{a}\right) + C \right] = \frac{1}{\sqrt{1-(u^2/a^2)}} \left(\frac{u'}{a} \right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$.

(b) $\frac{d}{dx} \left[\frac{1}{a} \arctan \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{1+(u/a)^2} \right] = \frac{1}{a^2} \left[\frac{u'}{(a^2+u^2)/a^2} \right] = \frac{u'}{a^2+u^2}$

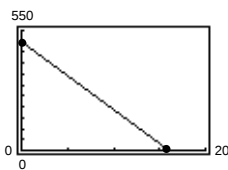
Thus, $\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C$.

(c) Assume $u > 0$.

$$\frac{d}{dx} \left[\frac{1}{a} \operatorname{arccsc} \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}} \right] = \frac{1}{a} \left[\frac{u'}{u\sqrt{u^2-a^2}/a^2} \right] = \frac{u'}{u\sqrt{u^2-a^2}}. \text{ The case } u < 0 \text{ is handled in a similar manner.}$$

Thus, $\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}} dx = \frac{1}{a} \operatorname{arccsc} \frac{|u|}{a} + C$.

63. (a) $v(t) = -32t + 500$



(b) $s(t) = \int v(t) dt = \int (-32t + 500) dt$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height,

$$v(t) = 0.$$

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

$$t = 15.625$$

$$s(15.625) = -16(15.625)^2 + 500(15.625)$$

$$= 3906.25 \text{ ft (Maximum height)}$$

—CONTINUED—

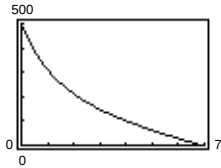
63. —CONTINUED—

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{32 + kv^2} dv &= -\int dt \\
 \frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}} v\right) &= -t + C_1 \\
 \arctan\left(\sqrt{\frac{k}{32}} v\right) &= -\sqrt{32k} t + C \\
 \sqrt{\frac{k}{32}} v &= \tan(C - \sqrt{32k} t) \\
 v &= \sqrt{\frac{32}{k}} \tan(C - \sqrt{32k} t)
 \end{aligned}$$

When $t = 0$, $v = 500$, $C = \arctan(500\sqrt{k/32})$, and we have

$$v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k} t\right].$$

$$\text{(d) When } k = 0.001, v(t) = \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032} t\right].$$



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

$$\text{(e) } h = \int_0^{6.86} \sqrt{32,000} \tan\left[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032} t\right] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

Section 5.10 Hyperbolic Functions

$$1. \text{ (a) } \sinh 3 = \frac{e^3 - e^{-3}}{2} \approx 10.018$$

$$\text{(b) } \tanh(-2) = \frac{\sinh(-2)}{\cosh(-2)} = \frac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$$

$$3. \text{ (a) } \operatorname{csch}(\ln 2) = \frac{2}{e^{\ln 2} - e^{-\ln 2}} = \frac{2}{2 - (1/2)} = \frac{4}{3}$$

$$\begin{aligned}
 \text{(b) } \coth(\ln 5) &= \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} \\
 &= \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}
 \end{aligned}$$

$$5. \text{ (a) } \cosh^{-1}(2) = \ln(2 + \sqrt{3}) \approx 1.317$$

$$\text{(b) } \operatorname{sech}^{-1}\left(\frac{2}{3}\right) = \ln\left(\frac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$$

$$7. \tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

$$\begin{aligned}
 9. \sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^y + e^{-y}}{2} \right) + \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^y - e^{-y}}{2} \right) \\
 &= \frac{1}{4} [e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}] \\
 &= \frac{1}{4} [2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x+y)
 \end{aligned}$$

$$\begin{aligned}
 11. 3 \sinh x + 4 \sinh^3 x &= \sinh x (3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2} \right) \left[3 + 4 \left(\frac{e^x - e^{-x}}{2} \right)^2 \right] \\
 &= \left(\frac{e^x - e^{-x}}{2} \right) [3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2} (e^x - e^{-x})(e^{2x} + e^{-2x} + 1) \\
 &= \frac{1}{2} [e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)
 \end{aligned}$$

$$13. \quad \sinh x = \frac{3}{2}$$

$$\cosh^2 x - \left(\frac{3}{2} \right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$15. y = \sinh(1 - x^2)$$

$$y' = -2x \cosh(1 - x^2)$$

$$17. f(x) = \ln(\sinh x)$$

$$f'(x) = \frac{1}{\sinh}(\cosh x) = \coth x$$

$$19. y = \ln\left(\tanh \frac{x}{2}\right)$$

$$\begin{aligned}
 y' &= \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)} \\
 &= \frac{1}{\sinh x} = \operatorname{csch} x
 \end{aligned}$$

$$21. h(x) = \frac{1}{4} \sinh(2x) - \frac{x}{2}$$

$$h'(x) = \frac{1}{2} \cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

$$23. f(t) = \arctan(\sinh t)$$

$$\begin{aligned}
 f'(t) &= \frac{1}{1 + \sinh^2 t} (\cosh t) \\
 &= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t
 \end{aligned}$$

$$25. \text{ Let } y = g(x).$$

$$y = x^{\cosh x}$$

$$\ln y = \cosh x \ln x$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{\cosh x}{x} + \sinh x \ln x$$

$$\frac{dy}{dx} = \frac{y}{x} [\cosh x + x(\sinh x) \ln x]$$

$$= \frac{x^{\cosh x}}{x} [\cosh x + x(\sinh x) \ln x]$$

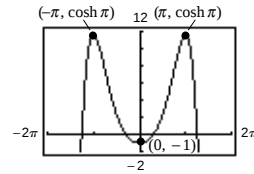
27. $y = (\cosh x - \sinh x)^2$

$$\begin{aligned} y' &= 2(\cosh x - \sinh x)(\sinh x - \cosh x) \\ &= -2(\cosh x - \sinh x)^2 = -2e^{-2x} \end{aligned}$$

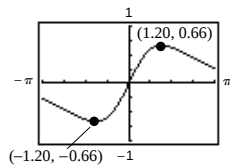
29. $f(x) = \sin x \sinh x - \cos x \cosh x, -4 \leq x \leq 4$

$$\begin{aligned} f'(x) &= \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x \\ &= 2 \sin x \cosh x = 0 \text{ when } x = 0, \pm \pi. \end{aligned}$$

 Relative maxima: $(\pm \pi, \cosh \pi)$

 Relative minimum: $(0, -1)$


31. $g(x) = x \operatorname{sech} x = \frac{x}{\cosh x}$


 Relative maximum: $(1.20, 0.66)$

 Relative minimum: $(-1.20, -0.66)$

33. $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

 Therefore, $y''' - y' = 0$.

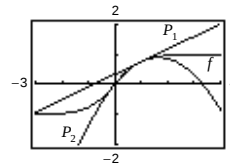
35. $f(x) = \tanh x$ $f(1) = \tanh(1) \approx 0.7616$

$f'(x) = \operatorname{sech}^2 x$ $f'(1) = \frac{1}{\cosh^2(1)} \approx 0.4200$

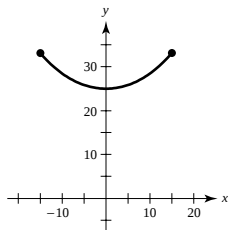
$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x$ $f''(1) \approx -0.6397$

$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$

$P_2(x) = 0.7616 + 0.42(x - 1) - \frac{0.6397}{2}(x - 1)^2$



37. (a) $y = 10 + 15 \cosh \frac{x}{15}, -15 \leq x \leq 15$



(b) At $x = \pm 15, y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0, y = 10 + 15 \cosh(0) = 25$.

(c) $y' = \sinh \frac{x}{15}$. At $x = 15, y' = \sinh(1) \approx 1.175$

39. Let $u = 1 - 2x, du = -2 dx$.

$$\begin{aligned} \int \sinh(1 - 2x) dx &= -\frac{1}{2} \int \sinh u du \\ &= -\frac{1}{2} \cosh u + C \\ &= -\frac{1}{2} \cosh(1 - 2x) + C \end{aligned}$$

41. Let $u = \cosh(x - 1), du = \sinh(x - 1) dx$.

$$\int \cosh^2(x - 1) \sinh(x - 1) dx = \frac{1}{3} \cosh^3(x - 1) + C$$

43. Let $u = \sinh x$, $du = \cosh x \, dx$.

$$\int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x \, dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2} dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2} \right) x dx = -\coth \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2} dx = -\int \operatorname{csch} \frac{1}{x} \coth \frac{1}{x} \left(-\frac{1}{x^2} \right) dx = \operatorname{csch} \frac{1}{x} + C$$

49. $\int_0^4 \frac{1}{25 - x^2} dx = \left[\frac{1}{10} \ln \left| \frac{5+x}{5-x} \right| \right]_0^4 = \frac{1}{10} \ln 9 = \frac{1}{5} \ln 3$

51. Let $u = 2x$, $du = 2 \, dx$.

$$\int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}} (2) dx = \left[\arcsin(2x) \right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

53. Let $u = x^2$, $du = 2x \, dx$.

$$\int \frac{x}{x^4 + 1} dx = \frac{1}{2} \int \frac{2x}{(x^2)^2 + 1} dx = \frac{1}{2} \arctan(x^2) + C$$

55. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

57. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}} (\sec^2 x) = |\sec x|$$

59. $y = \coth^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x} (2 \cos 2x) = 2 \sec 2x$$

61. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x \left(\frac{2}{\sqrt{1 + 4x^2}} \right) + 2 \sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2 \sinh^{-1}(2x)$$

63. See page 395.

65. $y = a \operatorname{sech}^{-1} \left(\frac{x}{a} \right) - \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

67. $\int \frac{1}{\sqrt{1 + e^{2x}}} dx = \int \frac{e^x}{e^x \sqrt{1 + (e^x)^2}} dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln \left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x} \right) + C$

69. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1+x}} dx = 2 \int \frac{1}{\sqrt{1 + (\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh^{-1} \sqrt{x} + C = 2 \ln(\sqrt{x} + \sqrt{1+x}) + C$$

71. $\int \frac{-1}{4x - x^2} dx = \int \frac{1}{(x-2)^2 - 4} dx = \frac{1}{4} \ln \left| \frac{(x-2) - 2}{(x-2) + 2} \right| = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C$

$$\begin{aligned}
 73. \int \frac{1}{1-4x-2x^2} dx &= \int \frac{1}{3-2(x+1)^2} dx = \frac{-1}{\sqrt{2}} \int \frac{\sqrt{2}}{[\sqrt{2}(x+1)]^2 - (\sqrt{3})^2} dx \\
 &= \frac{-1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) - \sqrt{3}}{\sqrt{2}(x+1) + \sqrt{3}} \right| + C = \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) + \sqrt{3}}{\sqrt{2}(x+1) - \sqrt{3}} \right| + C
 \end{aligned}$$

75. Let $u = 4x - 1$, $du = 4 dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}} dx = \frac{1}{4} \int \frac{4}{\sqrt{81-(4x-1)^2}} dx = \frac{1}{4} \arcsin\left(\frac{4x-1}{9}\right) + C$$

$$\begin{aligned}
 77. y &= \int \frac{x^3 - 21x}{5 + 4x - x^2} dx = \int \left(-x - 4 + \frac{20}{5 + 4x - x^2}\right) dx = \int (-x - 4) dx + 20 \int \frac{1}{3^2 - (x-2)^2} dx \\
 &= -\frac{x^2}{2} - 4x + \frac{20}{6} \ln \left| \frac{(x-2) + 3}{(x-2) - 3} \right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3} \ln \left| \frac{x+1}{x-5} \right| + C = -\frac{x^2}{2} - 4x - \frac{10}{3} \ln \left| \frac{x-5}{x+1} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 79. A &= 2 \int_0^4 \operatorname{sech} \frac{x}{2} dx \\
 &= 2 \int_0^4 \frac{2}{e^{x/2} + e^{-x/2}} dx \\
 &= 4 \int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1} dx \\
 &= \left[8 \arctan(e^{x/2}) \right]_0^4 \\
 &= 8 \arctan(e^2) - 2\pi \approx 5.207
 \end{aligned}$$

$$\begin{aligned}
 81. A &= \int_0^2 \frac{5x}{\sqrt{x^4+1}} dx \\
 &= \frac{5}{2} \int_0^2 \frac{2x}{\sqrt{(x^2)^2+1}} dx \\
 &= \left[\frac{5}{2} \ln(x^2 + \sqrt{x^4+1}) \right]_0^2 \\
 &= \frac{5}{2} \ln(4 + \sqrt{17}) \approx 5.237
 \end{aligned}$$

$$\begin{aligned}
 83. \int \frac{3k}{16} dt &= \int \frac{1}{x^2 - 12x + 32} dx \\
 \frac{3kt}{16} &= \int \frac{1}{(x-6)^2 - 4} dx = \frac{1}{2(2)} \ln \left| \frac{(x-6) - 2}{(x-6) + 2} \right| + C = \frac{1}{4} \ln \left| \frac{x-8}{x-4} \right| + C
 \end{aligned}$$

When $x = 0$: $t = 0$

$$C = -\frac{1}{4} \ln(2)$$

When $x = 1$: $t = 10$

$$\frac{30k}{16} = \frac{1}{4} \ln \left| \frac{-7}{-3} \right| - \frac{1}{4} \ln(2) = \frac{1}{4} \ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15} \ln\left(\frac{7}{6}\right)$$

When $t = 20$: $\left(\frac{3}{16}\right)\left(\frac{2}{15}\right) \ln\left(\frac{7}{6}\right)(20) = \frac{1}{4} \ln \frac{x-8}{2x-8}$

$$\ln\left(\frac{7}{6}\right)^2 = \ln \frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

85. As k increases, the time required for the object to reach the ground increases.

$$87. y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$89. y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$91. y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

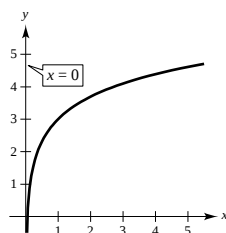
$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward

Vertical asymptote: $x = 0$



$$3. \ln \sqrt[5]{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$$

$$5. \ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$$

$$7. \ln \sqrt{x + 1} = 2$$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

$$13. y = \frac{1}{b^2} \left[\ln(a + bx) + \frac{a}{a + bx} \right]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2} \right] = \frac{x}{(a + bx)^2}$$

$$15. y = -\frac{1}{a} \ln \left(\frac{a + bx}{x} \right) = -\frac{1}{a} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(\frac{b}{a + bx} - \frac{1}{x} \right) = \frac{1}{x(a + bx)}$$

$$17. u = 7x - 2, du = 7dx$$

$$\int \frac{1}{7x - 2} dx = \frac{1}{7} \int \frac{1}{7x - 2} (7) dx = \frac{1}{7} \ln |7x - 2| + C$$

$$\begin{aligned}
 19. \int \frac{\sin x}{1 + \cos x} dx &= - \int \frac{-\sin x}{1 + \cos x} dx \\
 &= -\ln|1 + \cos x| + C
 \end{aligned}$$

$$23. \int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta| \right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

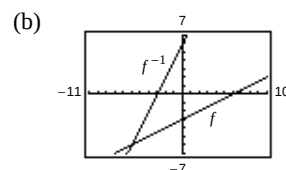
$$\begin{aligned}
 25. (a) \quad f(x) &= \frac{1}{2}x - 3 \\
 y &= \frac{1}{2}x - 3 \\
 2(y + 3) &= x \\
 2(x + 3) &= y \\
 f^{-1}(x) &= 2x + 6
 \end{aligned}$$

$$\begin{aligned}
 27. (a) \quad f(x) &= \sqrt{x+1} \\
 y &= \sqrt{x+1} \\
 y^2 - 1 &= x \\
 x^2 - 1 &= y \\
 f^{-1}(x) &= x^2 - 1, x \geq 0
 \end{aligned}$$

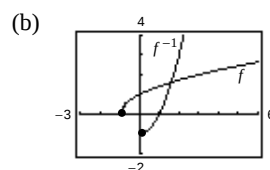
$$\begin{aligned}
 29. (a) \quad f(x) &= \sqrt[3]{x+1} \\
 y &= \sqrt[3]{x+1} \\
 y^3 - 1 &= x \\
 x^3 - 1 &= y \\
 f^{-1}(x) &= x^3 - 1
 \end{aligned}$$

$$\begin{aligned}
 31. \quad f(x) &= x^3 + 2 \\
 f^{-1}(x) &= (x - 2)^{1/3} \\
 (f^{-1})'(x) &= \frac{1}{3}(x - 2)^{-2/3} \\
 (f^{-1})'(-1) &= \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\
 &= \frac{1}{3^{5/3}} \approx 0.160
 \end{aligned}$$

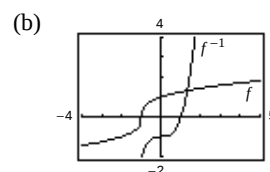
$$21. \int_1^4 \frac{x+1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x| \right]_1^4 = 3 + \ln 4$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x \\
 f(f^{-1}(x)) &= f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt{x+1}) = \sqrt{(x^2 - 1)^2} - 1 = x \\
 f(f^{-1}(x)) &= f(x^2 - 1) = \sqrt{(x^2 - 1) + 1} \\
 &= \sqrt{x^2} = x \text{ for } x \geq 0.
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt[3]{x+1}) = (\sqrt[3]{x+1})^3 - 1 = x \\
 f(f^{-1}(x)) &= f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x
 \end{aligned}$$

$$\begin{aligned}
 33. \quad f(x) &= \tan x \\
 f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\
 f'(x) &= \sec^2 x \\
 f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\
 (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4}
 \end{aligned}$$

35. (a) $f(x) = \ln \sqrt{x}$

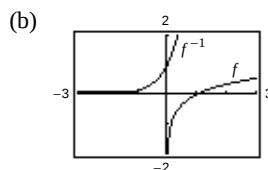
$$y = \ln \sqrt{x}$$

$$e^y = \sqrt{x}$$

$$e^{2y} = x$$

$$e^{2x} = y$$

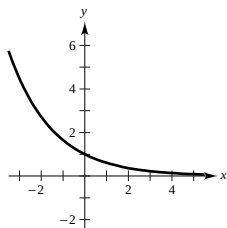
$$f^{-1}(x) = e^{2x}$$



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$$f'(x) = -2x$$

41. $g(t) = t^2 e^t$

$$g'(t) = t^2 e^t + 2te^t = te^t(t + 2)$$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x dx$.

$$\int x e^{-3x^2} dx = -\frac{1}{6} \int e^{-3x^2} (-6x) dx = -\frac{1}{6} e^{-3x^2} + C$$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x} dx = \int (e^{3x} - e^x + e^{-x}) dx$

$$= \frac{1}{3} e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53. $\int x e^{1-x^2} dx = -\frac{1}{2} \int e^{1-x^2} (-2x) dx$

$$= -\frac{1}{2} e^{1-x^2} + C$$

55. Let $u = e^x - 1$, $du = e^x dx$.

$$\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + C$$

57. $y = e^x(a \cos 3x + b \sin 3x)$

$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$

$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

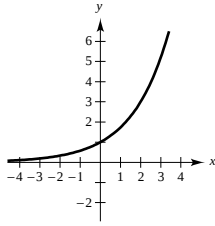
$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$

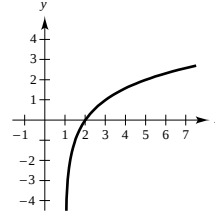
$$y'' - 2y' + 10y = e^x[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x = 0$$

$$59. \text{Area} = \int_0^4 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$$

$$61. y = 3^{3/2}$$



$$63. y = \log_2(x - 1)$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$67. y = x^{2x+1}$$

$$\ln y = (2x + 1) \ln x$$

$$\frac{y'}{y} = \frac{2x + 1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x + 1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x + 1}{x} + 2 \ln x \right)$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \frac{1}{2} \frac{-1}{(1-x) \ln 3} = \frac{1}{2(x-1) \ln 3}$$

$$71. \int (x+1) 5^{(x+1)^2} dx = \frac{1}{2} \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$73. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a) a^x$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$(d) y = a^a$$

$$y' = 0$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$75. 10,000 = P e^{(0.07)(15)}$$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

$$77. P(h) = 30e^{kh}$$

$$P(18,000) = 30e^{18,000k} = 15$$

$$k = \frac{\ln(1/2)}{18,000} = \frac{-\ln 2}{18,000}$$

$$P(h) = 30e^{-(h \ln 2)/18,000}$$

$$P(35,000) = 30e^{-(35,000 \ln 2)/18,000} \approx 7.79 \text{ inches}$$

$$79. P = Ce^{0.015t}$$

$$2C = Ce^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

$$81. \frac{dy}{dx} = \frac{x^2 + 3}{x}$$

$$\int dy = \int \left(x + \frac{3}{x} \right) dx$$

$$y = \frac{x^2}{2} + 3 \ln|x| + C$$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2+C_1} = y$$

$$y = Ce^{x^2}$$

85. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2) dx - 2xy dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$(x^2 + v^2x^2) dx - 2x(vx)(x dv + v dx) = 0$$

$$(x^2 + v^2x^2 - 2x^3v) dx - 2x^3v dv = 0$$

$$(x^2 - x^2v^2) dx = 2x^3v dv$$

$$(1 - v^2) dx = 2x dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2} dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x + C_2x^3) \\ &= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0 \end{aligned}$$

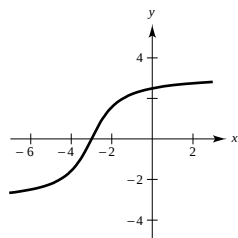
$$x = 2, y = 0: 0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$$

$$x = 2, y' = 4: 4 = C_1 + 12C_2$$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

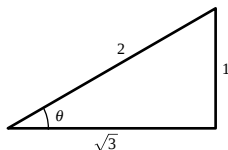
89. $f(x) = 2 \arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x dx$.

$$\int \frac{x}{16+x^2} dx = \frac{1}{2} \int \frac{1}{16+x^2} (2x) dx = \frac{1}{2} \ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\int \frac{\arctan(x/2)}{4+x^2} dx = \frac{1}{2} \int \left(\arctan \frac{x}{2}\right) \left(\frac{2}{4+x^2}\right) dx = \frac{1}{4} \left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}} dt$
 $\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$

109. $y = 2x - \cosh \sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right)$$

111. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$

85. As k increases, the time required for the object to reach the ground increases.

$$87. y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$89. y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$91. y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

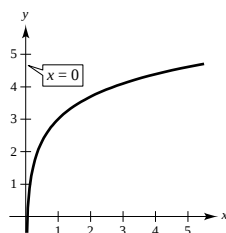
$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward

Vertical asymptote: $x = 0$



$$3. \ln \sqrt[5]{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$$

$$5. \ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$$

$$7. \ln \sqrt{x + 1} = 2$$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

$$13. y = \frac{1}{b^2} \left[\ln(a + bx) + \frac{a}{a + bx} \right]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2} \right] = \frac{x}{(a + bx)^2}$$

$$15. y = -\frac{1}{a} \ln \left(\frac{a + bx}{x} \right) = -\frac{1}{a} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(\frac{b}{a + bx} - \frac{1}{x} \right) = \frac{1}{x(a + bx)}$$

$$17. u = 7x - 2, du = 7dx$$

$$\int \frac{1}{7x - 2} dx = \frac{1}{7} \int \frac{1}{7x - 2} (7) dx = \frac{1}{7} \ln |7x - 2| + C$$

$$\begin{aligned}
 19. \int \frac{\sin x}{1 + \cos x} dx &= -\int \frac{-\sin x}{1 + \cos x} dx \\
 &= -\ln|1 + \cos x| + C
 \end{aligned}$$

$$23. \int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta| \right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

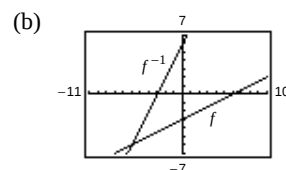
$$\begin{aligned}
 25. (a) \quad f(x) &= \frac{1}{2}x - 3 \\
 y &= \frac{1}{2}x - 3 \\
 2(y + 3) &= x \\
 2(x + 3) &= y \\
 f^{-1}(x) &= 2x + 6
 \end{aligned}$$

$$\begin{aligned}
 27. (a) \quad f(x) &= \sqrt{x+1} \\
 y &= \sqrt{x+1} \\
 y^2 - 1 &= x \\
 x^2 - 1 &= y \\
 f^{-1}(x) &= x^2 - 1, x \geq 0
 \end{aligned}$$

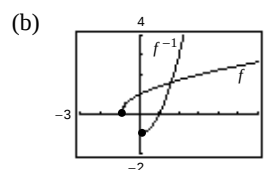
$$\begin{aligned}
 29. (a) \quad f(x) &= \sqrt[3]{x+1} \\
 y &= \sqrt[3]{x+1} \\
 y^3 - 1 &= x \\
 x^3 - 1 &= y \\
 f^{-1}(x) &= x^3 - 1
 \end{aligned}$$

$$\begin{aligned}
 31. \quad f(x) &= x^3 + 2 \\
 f^{-1}(x) &= (x - 2)^{1/3} \\
 (f^{-1})'(x) &= \frac{1}{3}(x - 2)^{-2/3} \\
 (f^{-1})'(-1) &= \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}} \\
 &= \frac{1}{3^{5/3}} \approx 0.160
 \end{aligned}$$

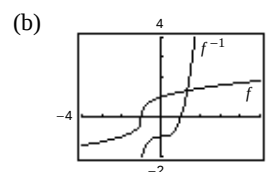
$$21. \int_1^4 \frac{x+1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x| \right]_1^4 = 3 + \ln 4$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x \\
 f(f^{-1}(x)) &= f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt{x+1}) = \sqrt{(x^2 - 1)^2} - 1 = x \\
 f(f^{-1}(x)) &= f(x^2 - 1) = \sqrt{(x^2 - 1) + 1} \\
 &= \sqrt{x^2} = x \text{ for } x \geq 0.
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad f^{-1}(f(x)) &= f^{-1}(\sqrt[3]{x+1}) = (\sqrt[3]{x+1})^3 - 1 = x \\
 f(f^{-1}(x)) &= f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x
 \end{aligned}$$

$$\begin{aligned}
 33. \quad f(x) &= \tan x \\
 f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{3} \\
 f'(x) &= \sec^2 x \\
 f'\left(\frac{\pi}{6}\right) &= \frac{4}{3} \\
 (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) &= \frac{1}{f'(\pi/6)} = \frac{3}{4}
 \end{aligned}$$

35. (a) $f(x) = \ln \sqrt{x}$

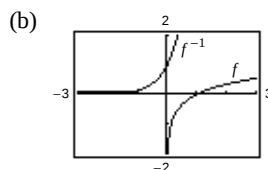
$$y = \ln \sqrt{x}$$

$$e^y = \sqrt{x}$$

$$e^{2y} = x$$

$$e^{2x} = y$$

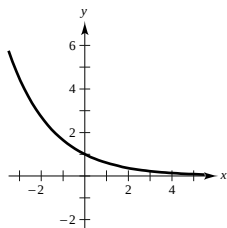
$$f^{-1}(x) = e^{2x}$$



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$$f'(x) = -2x$$

41. $g(t) = t^2 e^t$

$$g'(t) = t^2 e^t + 2te^t = te^t(t + 2)$$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2 - x)}{e^x}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x dx$.

$$\int x e^{-3x^2} dx = -\frac{1}{6} \int e^{-3x^2} (-6x) dx = -\frac{1}{6} e^{-3x^2} + C$$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x} dx = \int (e^{3x} - e^x + e^{-x}) dx$

$$= \frac{1}{3} e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53. $\int x e^{1-x^2} dx = -\frac{1}{2} \int e^{1-x^2} (-2x) dx$

$$= -\frac{1}{2} e^{1-x^2} + C$$

55. Let $u = e^x - 1$, $du = e^x dx$.

$$\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + C$$

57. $y = e^x(a \cos 3x + b \sin 3x)$

$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$

$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

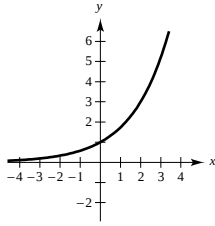
$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$

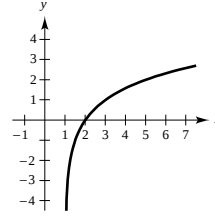
$$y'' - 2y' + 10y = e^x[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x = 0$$

$$59. \text{Area} = \int_0^4 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$$

$$61. y = 3^{3/2}$$



$$63. y = \log_2(x - 1)$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$67. y = x^{2x+1}$$

$$\ln y = (2x + 1) \ln x$$

$$\frac{y'}{y} = \frac{2x + 1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x + 1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x + 1}{x} + 2 \ln x \right)$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \frac{1}{2} \frac{-1}{(1-x) \ln 3} = \frac{1}{2(x-1) \ln 3}$$

$$71. \int (x+1) 5^{(x+1)^2} dx = \frac{1}{2} \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$73. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a) a^x$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$(d) y = a^a$$

$$y' = 0$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$75. 10,000 = P e^{(0.07)(15)}$$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

$$77. P(h) = 30e^{kh}$$

$$P(18,000) = 30e^{18,000k} = 15$$

$$k = \frac{\ln(1/2)}{18,000} = \frac{-\ln 2}{18,000}$$

$$P(h) = 30e^{-(h \ln 2)/18,000}$$

$$P(35,000) = 30e^{-(35,000 \ln 2)/18,000} \approx 7.79 \text{ inches}$$

$$79. P = Ce^{0.015t}$$

$$2C = Ce^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

$$81. \frac{dy}{dx} = \frac{x^2 + 3}{x}$$

$$\int dy = \int \left(x + \frac{3}{x} \right) dx$$

$$y = \frac{x^2}{2} + 3 \ln|x| + C$$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2+C_1} = y$$

$$y = Ce^{x^2}$$

85. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2) dx - 2xy dy = 0$$

Let $y = vx$, $dy = x dv + v dx$.

$$(x^2 + v^2x^2) dx - 2x(vx)(x dv + v dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v) dx - 2x^3v dv = 0$$

$$(x^2 - x^2v^2) dx = 2x^3v dv$$

$$(1 - v^2) dx = 2x dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2} dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x + C_2x^3) \\ &= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0 \end{aligned}$$

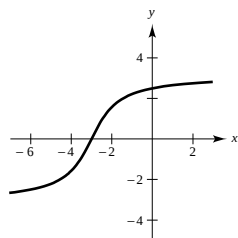
$$x = 2, y = 0: 0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$$

$$x = 2, y' = 4: 4 = C_1 + 12C_2$$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

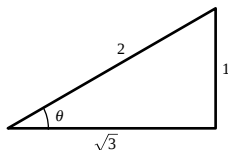
89. $f(x) = 2 \arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x dx$.

$$\int \frac{x}{16+x^2} dx = \frac{1}{2} \int \frac{1}{16+x^2} (2x) dx = \frac{1}{2} \ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\int \frac{\arctan(x/2)}{4+x^2} dx = \frac{1}{2} \int \left(\arctan \frac{x}{2}\right) \left(\frac{2}{4+x^2}\right) dx = \frac{1}{4} \left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}} dt$
 $\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$

109. $y = 2x - \cosh \sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right)$$

111. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$

Problem Solving for Chapter 5

1. $\tan \theta_1 = \frac{3}{x}$

$$\tan \theta_2 = \frac{6}{10-x}$$

Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10-x}\right)$$

$$f'(x) = \frac{1}{1 + \frac{9}{x^2}} \left(\frac{-3}{x^2} \right) + \frac{1}{1 + \frac{36}{(10-x)^2}} \left(\frac{6}{(10-x)^2} \right) = 0$$

$$\frac{3}{x^2 + 9} = \frac{6}{(10-x)^2 + 36}$$

$$(10-x)^2 + 36 = 2(x^2 + 9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

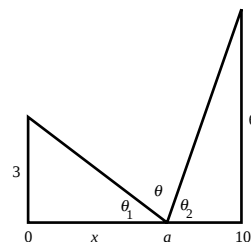
$$a = -10 + \sqrt{218} \approx 4.7648 \quad f(a) \approx 1.4153$$

$$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263 \quad \text{or} \quad 98.9^\circ$$

Endpoints: $a = 0$: $\theta \approx 1.0304$

$$a = 10$$
: $\theta \approx 1.2793$

Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.



3. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \Rightarrow \ln x = \frac{\pi}{2} + 2k\pi$.

Two values are $x = e^{\pi/2}, e^{(\pi/2)+2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \Rightarrow \ln x = \frac{3\pi}{2} + 2k\pi$.

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

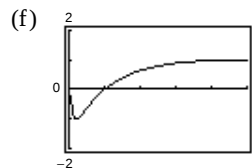
(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.

(e) $f'(x) = \frac{1}{x} \cos(\ln x)$

$$f'(x) = 0 \Rightarrow \cos(\ln x) = 0 \Rightarrow \ln x = \frac{\pi}{2} + k\pi \Rightarrow$$

$$x = e^{\pi/2} \text{ on } [1, 10]$$

$$\left. \begin{array}{l} f(e^{\pi/2}) = 1 \\ f(1) = 0 \\ f(10) \approx 0.7440 \end{array} \right\} \text{Maximum is 1 at } x = e^{\pi/2} \approx 4.8105$$



$\lim_{x \rightarrow 0^+} f(x)$ seems to be $-\frac{1}{2}$. (This is incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \dots$

we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \dots$

we have $f(x) = -1$.

That is, as $x \rightarrow 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus $\lim_{x \rightarrow 0^+} \sin(\ln x)$ does not exist.

You can verify this by graphing $f(x)$ on small intervals close to the origin.

$$5. (a) \frac{\text{Area sector}}{\text{Area circle}} = \frac{t}{2\pi} \Rightarrow \text{Area sector} = \frac{t}{2\pi}(\pi) = \frac{t}{2}$$

$$(b) \text{Area } AOP = \frac{1}{2}(\text{base})(\text{height}) - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$$

$$A(t) = \frac{1}{2} \cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1} \sinh t$$

$$= \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sinh^2 t$$

$$= \frac{1}{2}[\cosh^2 t - \sinh^2 t] = \frac{1}{2}$$

$$A(t) = \frac{1}{2}t + C. \text{ But, } A(0) = C = 0 \Rightarrow C = 0$$

$$\text{Thus, } A(t) = \frac{1}{2}t \text{ or } t = 2A(t).$$

$$7. y = \ln x$$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$$y = \frac{1}{a}x + b - 1 \quad \text{Tangent line}$$

$$\text{If } x = 0, c = b - 1. \text{ Thus, } b - c = b - (b - 1) = 1.$$

$$9. \text{ Let } u = 1 + \sqrt{x}, \sqrt{x} = u - 1, x = u^2 - 2u + 1,$$

$$dx = (2u - 2)du.$$

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x} dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)} du$$

$$= \int_2^3 \frac{2(u - 1)}{u^2 - u} du$$

$$= \int_2^3 \frac{2}{u} du$$

$$= \left[2 \ln u \right]_2^3$$

$$= 2 \ln 3 - 2 \ln 2 = 2 \ln \left(\frac{3}{2} \right)$$

$$\approx 0.8109$$

$$11. (a) \frac{dy}{dt} = y^{1.01}$$

$$\int y^{-1.01} dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$$y(0) = 1: 1 = \frac{1}{C^{100}} \Rightarrow C = 1$$

$$\text{Hence, } y = \frac{1}{(1 - 0.01t)^{100}}.$$

$$\text{For } T = 100, \lim_{t \rightarrow T^-} y = \infty.$$

$$(b) \int y^{-(1+\varepsilon)} dy = \int k dt$$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0} \right)^\varepsilon$$

$$\text{Hence, } y = \frac{1}{\left(\frac{1}{y_0^\varepsilon} - \varepsilon kt \right)^{1/\varepsilon}}.$$

$$\text{For } t \rightarrow \frac{1}{y_0^\varepsilon \varepsilon k}, y \rightarrow \infty.$$

13. Since $\frac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20} dy = \int k dt$$

$$\ln(y - 20) = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$.

Thus, $y = 52e^{\ln(7/13)t} + 20$.

When $t = 5$, $y = 52e^{5 \ln(7/13)} + 20 \approx 22.35^\circ$.

15. (a) $\frac{dS}{dt} = k_1 S(L - S)$

$S = \frac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-Cke^{-kt})$$

$$= \frac{LCke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right) \frac{L}{1 + Ce^{-kt}} \cdot \frac{CLe^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right) \frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

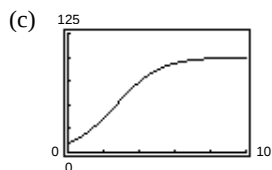
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$. And,

$S = 20$ when $t = 1 \Rightarrow k = -\ln(4/9)$.

Particular Solution. $S = \frac{100}{1 + 9e^{\ln(4/9)t}}$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$



(b) $\frac{dS}{dt} = \ln\left(\frac{4}{9}\right) S(100 - S)$

$$\frac{d^2S}{dt^2} = \ln\left(\frac{4}{9}\right) \left[S\left(-\frac{dS}{dt}\right) + (100 - S) \frac{dS}{dt} \right]$$

$$= \ln\left(\frac{4}{9}\right) (100 - 2S) \frac{dS}{dt}$$

$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

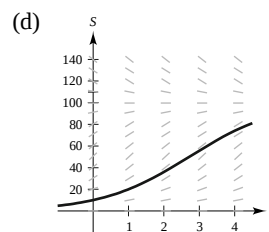
Choosing $S = 50$, we have:

$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$$t \approx 2.7 \text{ months}$$



(e) Sales will decrease toward the line $S = L$.

PART II

CHAPTER 6

Applications of Integration

Section 6.1	Area of a Region Between Two Curves	264
Section 6.2	Volume: The Disk Method	271
Section 6.3	Volume: The Shell Method	278
Section 6.4	Arc Length and Surfaces of Revolution	282
Section 6.5	Work	287
Section 6.6	Moments, Centers of Mass, and Centroids	290
Section 6.7	Fluid Pressure and Fluid Force	297
	Review Exercises	299
	Problem Solving	305

CHAPTER 6

Applications of Integration

Section 6.1 Area of a Region Between Two Curves

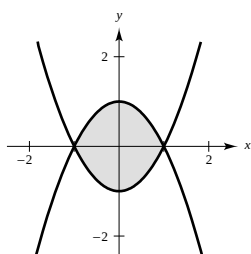
Solutions to Even-Numbered Exercises

$$2. A = \int_{-2}^2 [(2x + 5) - (x^2 + 2x + 1)] dx = \int_{-2}^2 (-x^2 + 4) dx$$

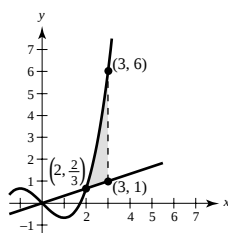
$$4. A = \int_0^1 (x^2 - x^3) dx$$

$$6. A = 2 \int_0^1 [(x - 1)^3 - (x - 1)] dx$$

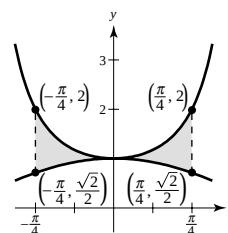
$$8. \int_{-1}^1 [(1 - x^2) - (x^2 - 1)] dx$$



$$10. \int_2^3 \left[\left(\frac{x^3}{3} - x \right) - \frac{x}{3} \right] dx$$



$$12. \int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x) dx$$

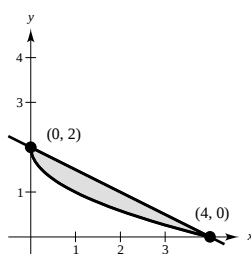


$$14. f(x) = 2 - \frac{1}{2}x$$

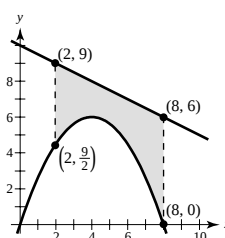
$$g(x) = 2 - \sqrt{x}$$

$$A \approx 1$$

Matches (a)



$$\begin{aligned} 16. A &= \int_2^8 \left[\left(10 - \frac{1}{2}x \right) - \left(-\frac{3}{8}x(x - 8) \right) \right] dx \\ &= \int_2^8 \left(\frac{3}{8}x^2 - \frac{7}{2}x + 10 \right) dx \\ &= \left[\frac{x^3}{8} - \frac{7x^2}{4} + 10x \right]_2^8 \\ &= (64 - 112 + 80) - (1 - 7 + 20) = 18 \end{aligned}$$



18. The points of intersection are given by

$$-x^2 + 4x + 1 = x + 1$$

$$-x^2 + 3x = 0$$

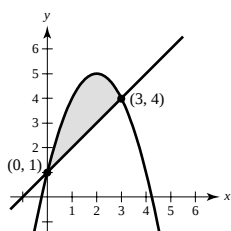
$$x^2 = 3x \text{ when } x = 0, 3$$

$$A = \int_0^3 [(-x^2 + 4x + 1) - (x + 1)] dx$$

$$= \int_0^3 (-x^2 + 3x) dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3$$

$$= -9 + \frac{27}{2} = \frac{9}{2}$$



20. The points of intersection are given by:

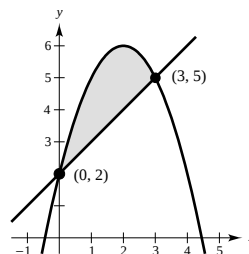
$$-x^2 + 4x + 2 = x + 2$$

$$x(3 - x) = 0 \text{ when } x = 0, 3$$

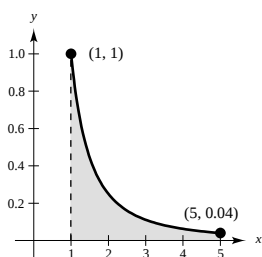
$$A = \int_0^3 [f(x) - g(x)] dx$$

$$= \int_0^3 [(-x^2 + 4x + 2) - (x + 2)] dx$$

$$= \int_0^3 (-x^2 + 3x) dx = \left[-\frac{x^3}{3} + \frac{3}{2}x^2 \right]_0^3 = \frac{9}{2}$$



22. $A = \int_1^5 \left(\frac{1}{x^2} - 0 \right) dx = \left[-\frac{1}{x} \right]_1^5 = \frac{4}{5}$



24. The points of intersection are given by

$$\sqrt[3]{x-1} = x-1$$

$$x-1 = (x-1)^3 = x^3 - 3x^2 + 3x - 1$$

$$x^3 - 3x^2 + 2x = 0$$

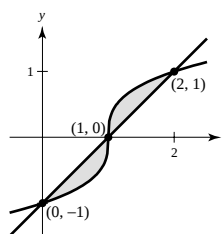
$$x(x^2 - 3x + 2) = 0$$

$$x(x-2)(x-1) = 0 \Rightarrow x = 0, 1, 2$$

$$A = 2 \int_0^1 [(x-1) - \sqrt[3]{x-1}] dx$$

$$= 2 \left[\frac{x^2}{2} - x - \frac{3}{4}(x-1)^{4/3} \right]_0^1$$

$$= 2 \left[\left(\frac{1}{2} - 1 - 0 \right) - \left(-\frac{3}{4} \right) \right] = \frac{1}{2}$$

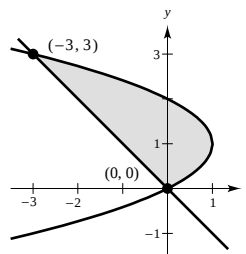


26. The points of intersection are given by:

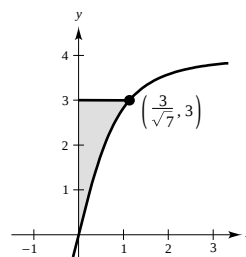
$$2y - y^2 = -y$$

$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$

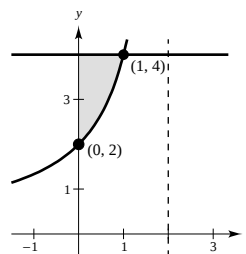
$$\begin{aligned} A &= \int_0^3 [f(y) - g(y)] dy \\ &= \int_0^3 [(2y - y^2) - (-y)] dy \\ &= \int_0^3 (3y - y^2) dy = \left[\frac{3}{2}y^2 - \frac{1}{3}y^3 \right]_0^3 = \frac{9}{2} \end{aligned}$$



$$\begin{aligned} 28. A &= \int_0^3 [f(y) - g(y)] dy \\ &= \int_0^3 \left[\frac{y}{\sqrt{16 - y^2}} - 0 \right] dy \\ &= -\frac{1}{2} \int_0^3 (16 - y^2)^{-1/2} (-2y) dy \\ &= \left[-\sqrt{16 - y^2} \right]_0^3 = 4 - \sqrt{7} \approx 1.354 \end{aligned}$$



$$\begin{aligned} 30. A &= \int_0^1 \left(4 - \frac{4}{2 - x} \right) dx \\ &= \left[4x + 4 \ln |2 - x| \right]_0^1 \\ &= 4 - 4 \ln 2 \\ &\approx 1.227 \end{aligned}$$



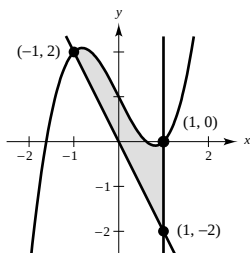
32. The point of intersection is given by:

$$x^3 - 2x + 1 = -2x$$

$$x^3 + 1 = 0 \quad \text{when } x = -1$$

$$\begin{aligned} A &= \int_{-1}^1 [f(x) - g(x)] dx \\ &= \int_{-1}^1 [(x^3 - 2x + 1) - (-2x)] dx \\ &= \int_{-1}^1 (x^3 + 1) dx = \left[\frac{x^4}{4} + x \right]_{-1}^1 = 2 \end{aligned}$$

Numerical Approximation: 2.0



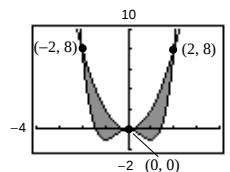
34. The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$

$$x^2(x^2 - 4) = 0 \quad \text{when } x = 0, \pm 2$$

$$\begin{aligned} A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\ &= 2 \int_0^2 (4x^2 - x^4) dx \\ &= 2 \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = \frac{128}{15} \end{aligned}$$

Numerical Approximation: 8.533



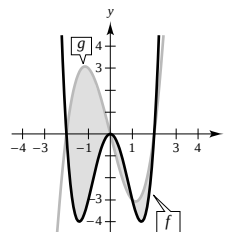
36. $f(x) = x^4 - 4x^2$, $g(x) = x^3 - 4x$

The points of intersection are given by:

$$x^4 - 4x^2 = x^3 - 4x$$

$$x^4 - x^3 - 4x^2 + 4x = 0$$

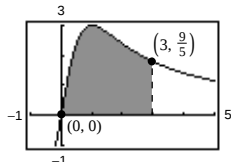
$$x(x-1)(x+2)(x-2) = 0 \quad \text{when } x = -2, 0, 1, 2$$



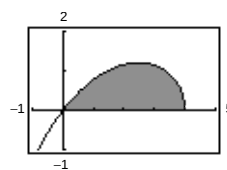
$$\begin{aligned} A &= \int_{-2}^0 [(x^3 - 4x) - (x^4 - 4x^2)] dx + \int_0^1 [(x^4 - 4x^2) - (x^3 - 4x)] dx + \int_1^2 [(x^3 - 4x) - (x^4 - 4x^2)] dx \\ &= \frac{248}{30} + \frac{37}{60} + \frac{53}{60} = \frac{293}{30} \end{aligned}$$

 Numerical Approximation: $8.267 + 0.617 + 0.883 \approx 9.767$

38.
$$\begin{aligned} A &= \int_0^3 \left[\frac{6x}{x^2 + 1} - 0 \right] dx \\ &= \left[3 \ln(x^2 + 1) \right]_0^3 \\ &= 3 \ln 10 \\ &\approx 6.908 \end{aligned}$$

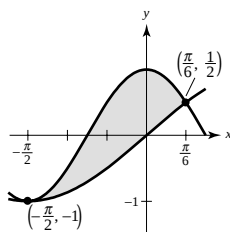


40.
$$A = \int_0^4 x \sqrt{\frac{4-x}{4+x}} dx \approx 3.434$$

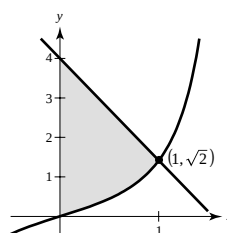


Numerical Approximation: 6.908

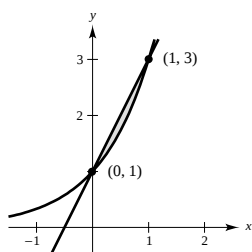
42.
$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x) dx \\ &= \left[\frac{1}{2} \sin 2x + \cos x \right]_{-\pi/2}^{\pi/6} \\ &= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} \right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299 \end{aligned}$$



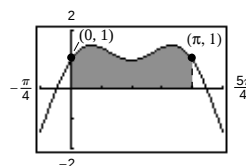
44.
$$\begin{aligned} A &= \int_0^1 \left[(\sqrt{2} - 4)x + 4 - \sec \frac{\pi x}{4} \tan \frac{\pi x}{4} \right] dx \\ &= \left[\frac{\sqrt{2} - 4}{2} x^2 + 4x - \frac{4}{\pi} \sec \frac{\pi x}{4} \right]_0^1 \\ &= \left(\frac{\sqrt{2} - 4}{2} + 4 - \frac{4}{\pi} \sqrt{2} \right) - \left(-\frac{4}{\pi} \right) \\ &= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi} (1 - \sqrt{2}) \approx 2.1797 \end{aligned}$$


 46. From the graph we see that f and g intersect twice at $x = 0$ and $x = 1$.

$$\begin{aligned} A &= \int_0^1 [g(x) - f(x)] dx \\ &= \int_0^1 [(2x + 1) - 3^x] dx \\ &= \left[x^2 + x - \frac{1}{\ln 3} (3^x) \right]_0^1 \\ &= 2 \left(1 - \frac{1}{\ln 3} \right) \approx 0.180 \end{aligned}$$

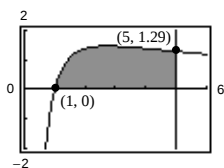


48.
$$\begin{aligned} A &= \int_0^{\pi} [(2 \sin x + \cos 2x) - 0] dx \\ &= \left[-2 \cos x + \frac{1}{2} \sin 2x \right]_0^{\pi} = 4 \end{aligned}$$



$$50. A = \int_1^5 \left[\frac{4 \ln x}{x} - 0 \right] dx$$

$$= \left[2(\ln x)^2 \right]_1^5 = 2(\ln 5)^2 \approx 5.181$$

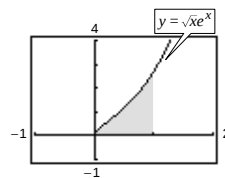


$$52. (a) y = \sqrt{x} e^x, y = 0, x = 0, x = 1$$

$$(b) A = \int_0^1 \sqrt{x} e^x dx.$$

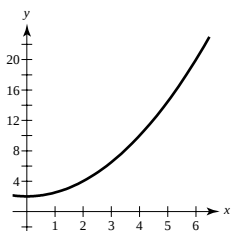
No, it cannot be evaluated by hand.

$$(c) 1.2556$$

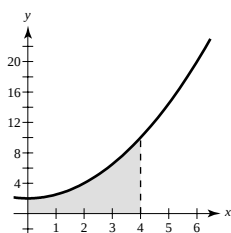


$$54. F(x) = \int_0^x \left(\frac{1}{2} t^2 + 2 \right) dt = \left[\frac{1}{6} t^3 + 2t \right]_0^x = \frac{x^3}{6} + 2x$$

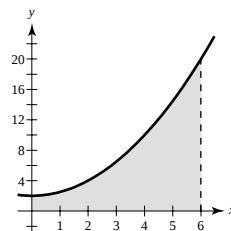
$$(a) F(0) = 0$$



$$(b) F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$$

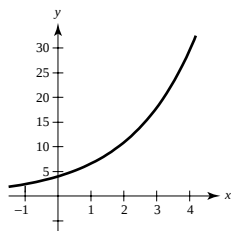


$$(c) F(6) = 36 + 12 = 48$$

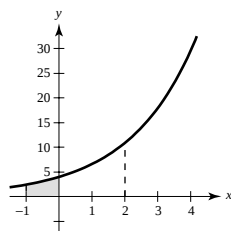


$$56. F(y) = \int_{-1}^y 4e^{x/2} dx = \left[8e^{x/2} \right]_{-1}^y = 8e^{y/2} - 8e^{-1/2}$$

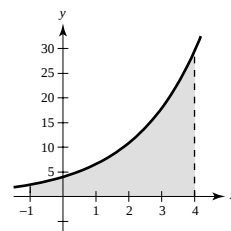
$$(a) F(-1) = 0$$



$$(b) F(0) = 8 - 8e^{-1/2} \approx 3.1478$$



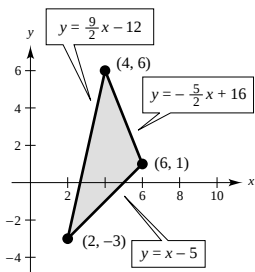
$$(c) F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$$



$$58. A = \int_2^4 \left[\left(\frac{9}{2}x - 12 \right) - (x - 5) \right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16 \right) - (x - 5) \right] dx$$

$$= \int_2^4 \left(\frac{7}{2}x - 7 \right) dx + \int_4^6 \left(-\frac{7}{2}x + 21 \right) dx$$

$$= \left[\frac{7}{4}x^2 - 7x \right]_2^4 + \left[-\frac{7}{4}x^2 + 21x \right]_4^6 = 7 + 7 = 14$$



60. $f(x) = \frac{1}{x^2 + 1}$

$$f'(x) = -\frac{2x}{(x^2 + 1)^2}$$

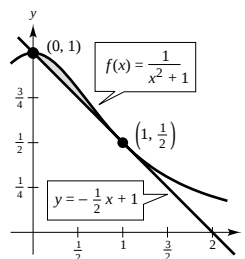
At $\left(1, \frac{1}{2}\right)$, $f'(1) = -\frac{1}{2}$.

Tangent line:

$$y - \frac{1}{2} = -\frac{1}{2}(x - 1) \text{ or } y = -\frac{1}{2}x + 1$$

The tangent line intersects $f(x) = \frac{1}{x^2 + 1}$ at $x = 0$.

$$A = \int_0^1 \left[\frac{1}{x^2 + 1} - \left(-\frac{1}{2}x + 1 \right) \right] dx = \left[\arctan x + \frac{x^2}{4} - x \right]_0^1 = \frac{\pi - 3}{4} \approx 0.0354$$



62. Answers will vary. See page 417.

64. $x^3 \geq x$ on $[-1, 0]$

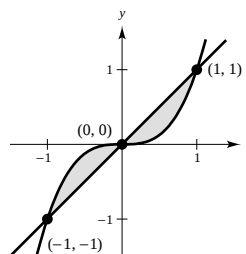
$x^3 \leq x$ on $[0, 1]$

Both functions symmetric to origin

$$\int_{-1}^0 (x^3 - x) dx = -\int_0^1 (x^3 - x) dx.$$

Thus, $\int_{-1}^1 (x^3 - x) dx = 0$.

$$A = 2 \int_0^1 (x - x^3) dx = 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2}$$



66. Proposal 2 is better, since the cumulative deficit (the area under the curve) is less.

68. $A = 2 \int_0^9 (9 - x) dx = 2 \left[9x - \frac{x^2}{2} \right]_0^9 = 81$

$$2 \int_0^{9-b} [(9 - x) - b] dx = \frac{81}{2}$$

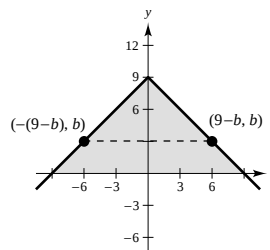
$$2 \int_0^{9-b} [(9 - b) - x] dx = \frac{81}{2}$$

$$2 \left[(9 - b)x - \frac{x^2}{2} \right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

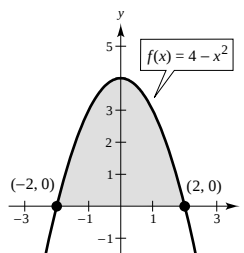
$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



70. $\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (4 - x_i^2) \Delta x$

where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ is the same as

$$\int_{-2}^2 (4 - x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}.$$



$$\begin{aligned} 72. \int_0^5 [(7.21 + 0.26t + 0.02t^2) - (7.21 + 0.1t + 0.01t^2)] dt &= \int_0^5 (0.01t^2 + 0.16t) dt \\ &= \left[\frac{0.01t^3}{3} + \frac{0.16t^2}{2} \right]_0^5 \\ &= \frac{29}{12} \text{ billion} \approx \$2.417 \text{ billion} \end{aligned}$$

74. 5% : $P_1 = 893,000 e^{(0.05)t}$

$3\frac{1}{2}\%$: $P_2 = 893,000 e^{(0.035)t}$

Difference in profits over 5 years:

$$\begin{aligned} \int_0^5 [893,000e^{0.05t} - 893,000e^{0.035t}] dt &= 893,000 \left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035} \right]_0^5 \\ &\approx 893,000[(25.6805 - 34.0356) - (20 - 28.5714)] \\ &\approx 893,000(0.2163) \approx \$193,156 \end{aligned}$$

Note: Using a graphing utility you obtain \$193,183.

76. The curves intersect at the point where the slope of y_2 equals that of y_1 , 1.

$$y_2 = 0.08x^2 + k \Rightarrow y'_2 = 0.16x = 1 \Rightarrow x = \frac{1}{.16} = 6.25$$

(a) The value of k is given by

$$y_1 = y_2$$

$$6.25 = (0.08)(6.25)^2 + k$$

$$k = 3.125.$$

$$\begin{aligned} \text{(b) Area} &= 2 \int_0^{6.25} (y_2 - y_1) dx \\ &= 2 \int_0^{6.25} (0.08x^2 + 3.125 - x) dx \\ &= 2 \left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2} \right]_0^{6.25} \\ &= 2(6.510417) \approx 13.02083 \end{aligned}$$

78. (a) $A \approx 6.031 - 2 \left[\pi \left(\frac{1}{16} \right)^2 \right] - 2 \left[\pi \left(\frac{1}{8} \right)^2 \right] \approx 5.908$

(b) $V = 2A \approx 2(5.908) \approx 11.816 \text{ m}^3$

(c) $5000V \approx 5000(11.816) = 59,082 \text{ pounds}$

80. True

Section 6.2 Volume: The Disk Method

$$2. V = \pi \int_0^2 (4 - x^2)^2 dx = \pi \int_0^2 (x^4 - 8x^2 + 16) dx = \pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2 = \frac{256\pi}{15}$$

$$4. V = \pi \int_0^3 (\sqrt{9 - x^2})^2 dx = \pi \int_0^3 (9 - x^2) dx \\ = \pi \left[9x - \frac{x^3}{3} \right]_0^3 = 18\pi$$

$$6. \begin{aligned} 2 &= 4 - \frac{x^2}{4} & V &= \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[\left(4 - \frac{x^2}{4} \right)^2 - (2)^2 \right] dx \\ 8 &= 16 - x^2 & &= 2\pi \int_0^{2\sqrt{2}} \left[\frac{x^4}{16} - 2x^2 + 12 \right] dx \\ x^2 &= 8 & &= 2\pi \left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x \right]_0^{2\sqrt{2}} \\ x &= \pm 2\sqrt{2} & &= 2\pi \left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2} \right] \\ & & &= \frac{448\sqrt{2}}{15}\pi \approx 132.69 \end{aligned}$$

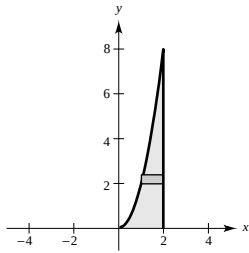
$$8. y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2} \\ V = \pi \int_0^4 (\sqrt{16 - y^2})^2 dy = \pi \int_0^4 (16 - y^2) dy \\ = \pi \left[16y - \frac{y^3}{3} \right]_0^4 = \frac{128\pi}{3}$$

$$10. V = \pi \int_1^4 (-y^2 + 4y)^2 dy = \pi \int_1^4 (y^4 - 8y^3 + 16y^2) dy \\ = \pi \left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3} \right]_1^4 \\ = \frac{459\pi}{15} = \frac{153\pi}{5}$$

$$12. y = 2x^2, y = 0, x = 2$$

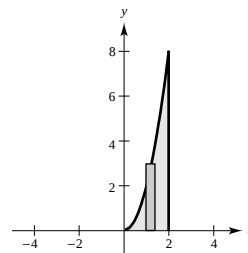
$$(a) R(y) = 2, r(y) = \sqrt{y/2}$$

$$V = \pi \int_0^8 \left(4 - \frac{y}{2} \right) dy = \pi \left[4y - \frac{y^2}{4} \right]_0^8 = 16\pi$$



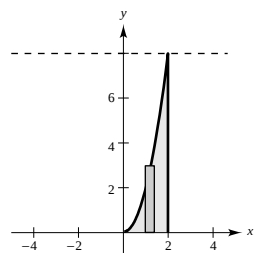
$$(b) R(x) = 2x^2, r(x) = 0$$

$$V = \pi \int_0^2 4x^4 dx = \pi \left[\frac{4x^5}{5} \right]_0^2 = \frac{128\pi}{5}$$



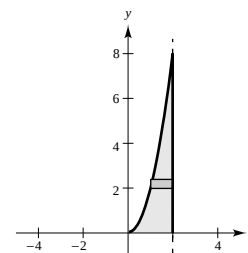
$$(c) R(x) = 8, r(x) = 8 - 2x^2$$

$$V = \pi \int_0^2 [64 - (64 - 32x^2 + 4x^4)] dx \\ = \pi \int_0^2 (32x^2 - 4x^4) dx = 4\pi \int_0^2 (8x^2 - x^4) dx \\ = 4\pi \left[\frac{8}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{896\pi}{15}$$



$$(d) R(y) = 2 - \sqrt{y/2}, r(y) = 0$$

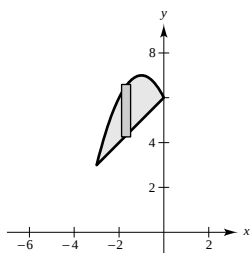
$$V = \pi \int_0^8 \left(2 - \sqrt{\frac{y}{2}} \right)^2 dy \\ = \pi \int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2} \right) dy \\ = \pi \left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4} \right]_0^8 = \frac{16\pi}{3}$$



14. $y = 6 - 2x - x^2$, $y = x + 6$ intersect at $(-3, 3)$ and $(0, 6)$.

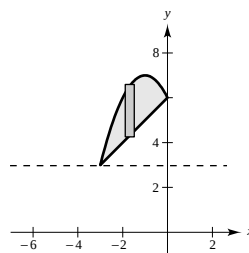
(a) $R(x) = 6 - 2x - x^2$, $r(x) = x + 6$

$$\begin{aligned} V &= \pi \int_{-3}^0 [(6 - 2x - x^2)^2 - (x + 6)^2] dx \\ &= \pi \int_{-3}^0 (x^4 + 4x^3 - 9x^2 - 36x) dx \\ &= \pi \left[\frac{1}{5}x^5 + x^4 - 3x^3 - 18x^2 \right]_{-3}^0 = \frac{243\pi}{5} \end{aligned}$$



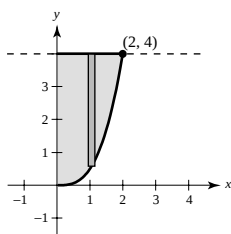
(b) $R(x) = (6 - 2x - x^2) - 3$, $r(x) = (x + 6) - 3$

$$\begin{aligned} V &= \pi \int_{-3}^0 [(3 - 2x - x^2)^2 - (x + 3)^2] dx \\ &= \pi \int_{-3}^0 (x^4 + 4x^3 - 3x^2 - 18x) dx \\ &= \pi \left[\frac{1}{5}x^5 + x^4 - x^3 - 9x^2 \right]_{-3}^0 = \frac{108\pi}{5} \end{aligned}$$



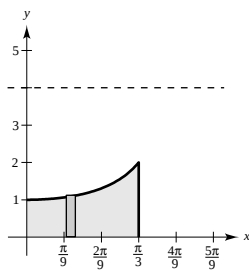
16. $R(x) = 4 - \frac{x^3}{2}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^2 \left(4 - \frac{x^3}{2} \right)^2 dx \\ &= \pi \int_0^2 \left[16 - 4x^3 + \frac{x^6}{4} \right] dx \\ &= \pi \left[16x - x^4 + \frac{x^7}{28} \right]_0^2 \\ &= \pi \left[32 - 16 + \frac{128}{28} \right] = \frac{144}{7}\pi \end{aligned}$$



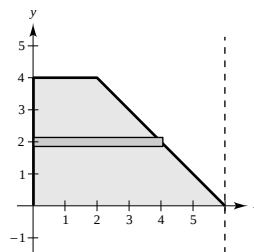
18. $R(x) = 4$, $r(x) = 4 - \sec x$

$$\begin{aligned} V &= \pi \int_0^{\pi/3} [(4)^2 - (4 - \sec x)^2] dx \\ &= \pi \int_0^{\pi/3} (8 \sec x - \sec^2 x) dx \\ &= \pi \left[8 \ln |\sec x + \tan x| - \tan x \right]_0^{\pi/3} \\ &= \pi \left[(8 \ln |2 + \sqrt{3}| - \sqrt{3}) - (8 \ln |1 + 0| - 0) \right] \\ &= \pi [8 \ln(2 + \sqrt{3}) - \sqrt{3}] \approx 27.66 \end{aligned}$$



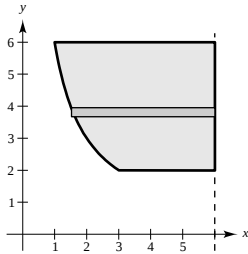
20. $R(y) = 6$, $r(y) = 6 - (6 - y) = y$

$$\begin{aligned} V &= \pi \int_0^4 [(6)^2 - (y)^2] dy \\ &= \pi \left[36y - \frac{y^3}{3} \right]_0^4 = \frac{368\pi}{3} \end{aligned}$$



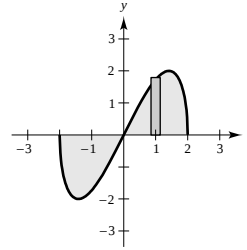
$$22. R(y) = 6 - \frac{6}{y}, \quad r(y) = 0$$

$$\begin{aligned} V &= \pi \int_2^6 \left(6 - \frac{6}{y}\right)^2 dy \\ &= 36\pi \int_2^6 \left(1 - \frac{2}{y} + \frac{1}{y^2}\right) dy \\ &= 36\pi \left[y - 2 \ln|y| - \frac{1}{y} \right]_2^6 \\ &= 36\pi \left[\left(\frac{35}{6} - 2 \ln 6\right) - \left(\frac{3}{2} - 2 \ln 2\right) \right] \\ &= 36\pi \left(\frac{13}{3} + 2 \ln \frac{1}{3} \right) = 12\pi(13 - 6 \ln 3) \approx 241.59 \end{aligned}$$



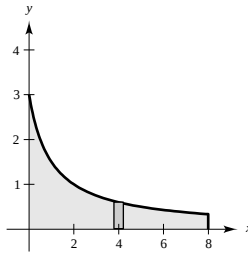
$$24. R(x) = x\sqrt{4-x^2}, \quad r(x) = 0$$

$$\begin{aligned} V &= 2\pi \int_0^2 [x\sqrt{4-x^2}]^2 dx \\ &= 2\pi \int_0^2 (4x^2 - x^4) dx \\ &= 2\pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \\ &= \frac{128\pi}{15} \end{aligned}$$



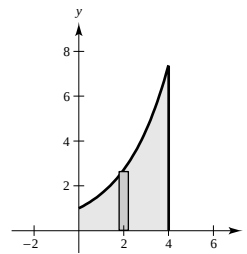
$$26. R(x) = \frac{3}{x+1}, \quad r(x) = 0$$

$$\begin{aligned} V &= \pi \int_0^8 \left(\frac{3}{x+1}\right)^2 dx \\ &= 9\pi \int_0^8 (x+1)^{-2} dx \\ &= 9\pi \left[-\frac{1}{x+1} \right]_0^8 = 8\pi \end{aligned}$$

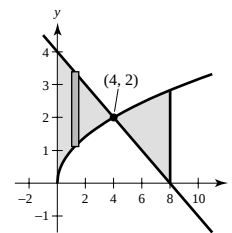


$$28. R(x) = e^{x/2}, \quad r(x) = 0$$

$$\begin{aligned} V &= \pi \int_0^4 (e^{x/2})^2 dx \\ &= \pi \int_0^4 e^x dx \\ &= \left[\pi e^x \right]_0^4 \\ &= \pi(e^4 - 1) \approx 168.38 \end{aligned}$$



$$\begin{aligned} 30. V &= \pi \int_0^4 \left[\left(4 - \frac{1}{2}x\right)^2 - (\sqrt{x})^2 \right] dx + \pi \int_4^8 \left[(\sqrt{x})^2 - \left(4 - \frac{1}{2}x\right)^2 \right] dx \\ &= \pi \int_0^4 \left(\frac{x^2}{4} - 5x + 16 \right) dx + \pi \int_4^8 \left(-\frac{x^2}{4} + 5x - 16 \right) dx \\ &= \pi \left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x \right]_0^4 + \pi \left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x \right]_4^8 \\ &= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi \end{aligned}$$



32. $y = 9 - x^2$, $y = 0$, $x = 2$, $x = 3$

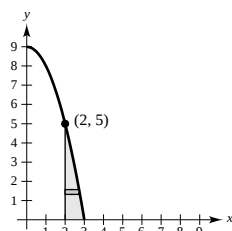
$$x = \sqrt{9 - y}$$

$$V = \pi \int_0^5 [(\sqrt{9 - y})^2 - 2^2] dy$$

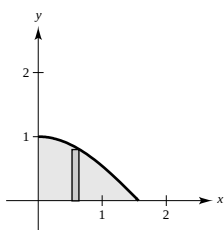
$$= \pi \int_0^5 (5 - y) dy$$

$$= \pi \left[5y - \frac{y^2}{2} \right]_0^5$$

$$= \pi \left(25 - \frac{25}{2} \right) = \frac{25\pi}{2}$$



34. $V = \pi \int_0^{\pi/2} [\cos x]^2 dx \approx 2.4674$

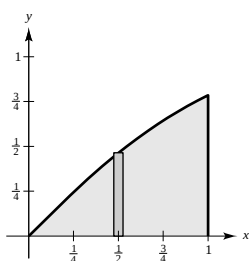


36. $V = \pi \int_1^3 [\ln x]^2 dx \approx 3.2332$

38. $V = \pi \int_0^5 [2 \arctan(0.2x)]^2 dx \approx 15.4115$

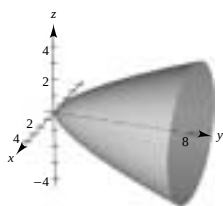
40. $A \approx \frac{3}{4}$

Matches (b)

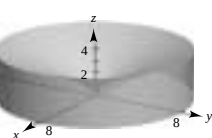


42. $V = \int_a^b A(x) dx$ or $V = \int_c^d A(y) dy$

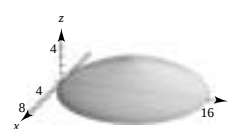
44. (a)



(b)



(c)



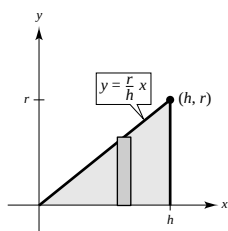
$$a < c < b.$$

46. $R(x) = \frac{r}{h}x$, $r(x) = 0$

$$V = \pi \int_0^h \frac{r^2}{h^2} x^2 dx$$

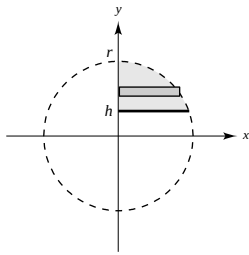
$$= \left[\frac{r^2 \pi}{3h^2} x^3 \right]_0^h$$

$$= \frac{r^2 \pi}{3h^2} h^3 = \frac{1}{3} \pi r^2 h$$

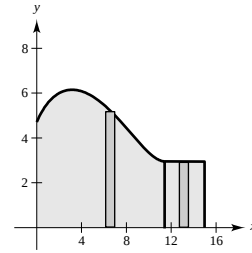


$$48. x = \sqrt{r^2 - y^2}, R(y) = \sqrt{r^2 - y^2}, r(y) = 0$$

$$\begin{aligned} V &= \pi \int_h^r (\sqrt{r^2 - y^2})^2 dy \\ &= \pi \int_h^r (r^2 - y^2) dy \\ &= \pi \left[r^2 y - \frac{y^3}{3} \right]_h^r \\ &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(r^2 h - \frac{h^3}{3} \right) \right] \\ &= \pi \left(\frac{2r^3}{3} - r^2 h + \frac{h^3}{3} \right) \\ &= \frac{\pi}{3} (2r^3 - 3r^2 h + h^3) \end{aligned}$$

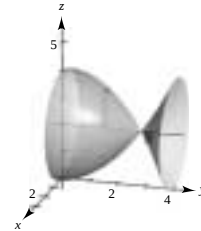


$$\begin{aligned} 52. y &= \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \leq x \leq 11.5 \\ 2.95, & 11.5 < x \leq 15 \end{cases} \\ V &= \pi \int_0^{11.5} (\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2})^2 dx + \pi \int_{11.5}^{15} 2.95^2 dx \\ &= \pi \left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x \right]_0^{11.5} + \pi \left[2.95^2 x \right]_{11.5}^{15} \\ &\approx 1031.9016 \text{ cubic centimeters} \end{aligned}$$



54. (a) First find where $y = b$ intersects the parabola:

$$\begin{aligned} b &= 4 - \frac{x^2}{4} \\ x^2 &= 16 - 4b = 4(4 - b) \\ x &= 2\sqrt{4 - b} \\ V &= \int_0^{2\sqrt{4-b}} \pi \left[4 - \frac{x^2}{4} - b \right]^2 dx + \int_{2\sqrt{4-b}}^4 \pi \left[b - 4 + \frac{x^2}{4} \right]^2 dx \\ &= \int_0^4 \pi \left[4 - \frac{x^2}{4} - b \right]^2 dx \\ &= \pi \int_0^4 \left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16 \right] dx \\ &= \pi \left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x \right]_0^4 \\ &= \pi \left[\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64 \right] = \pi \left[4b^2 - \frac{64}{3}b + \frac{512}{15} \right] \end{aligned}$$



$$50. (a) V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \left[\frac{\pi x^2}{2} \right]_0^4 = 8\pi$$

Let $0 < c < 4$ and set

$$\pi \int_0^c x dx = \left[\frac{\pi x^2}{2} \right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$$c^2 = 8$$

$$c = \sqrt{8} = 2\sqrt{2}$$

Thus, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

(b) Set $\pi \int_0^c x dx = \frac{8\pi}{3}$ (one third of the volume). Then

$$\frac{\pi c^2}{2} = \frac{8\pi}{3}, c^2 = \frac{16}{3}, c = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}.$$

To find the other value, set $\pi \int_0^d x dx = \frac{16\pi}{3}$ (two thirds of the volume). Then

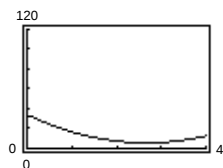
$$\frac{\pi d^2}{2} = \frac{16\pi}{3}, d^2 = \frac{32}{3}, d = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}.$$

The x -values that divide the solid into three parts of equal volume are $x = (4\sqrt{3})/3$ and $x = (4\sqrt{6})/3$.

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54. —CONTINUED—

(b) graph of $V(b) = \pi \left[4b^2 - \frac{64}{3}b + \frac{512}{15} \right]$



Minimum Volume is 17.87 for $b = 2.67$

(c) $V'(b) = \pi \left[8b - \frac{64}{3} \right] = 0 \Rightarrow b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \frac{8}{3}$ is a relative minimum.

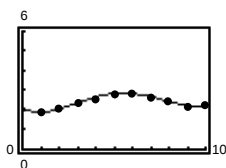
56. (a) $V = \int_0^{10} \pi [f(x)]^2 dx$

Simpson's Rule: $b - a = 10 - 0 = 10$, $n = 10$

$$V \approx \frac{\pi}{3} [(2.1)^2 + 4(1.9)^2 + 2(2.1)^2 + 4(2.35)^2 + 2(2.6)^2 + 4(2.85)^2 + 2(2.9)^2 + 4(2.7)^2 + 2(2.45)^2 + 4(2.2)^2 + (2.3)^2]$$

$$\approx \frac{\pi}{3} [178.405] \approx 186.83 \text{ cm}^3$$

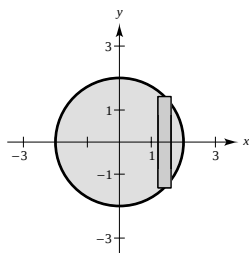
(b) $f(x) = 0.00249x^4 - 0.0529x^3 + 0.3314x^2 - 0.4999x + 2.112$



(c) $V \approx \int_0^{10} \pi f(x)^2 dx \approx 186.35 \text{ cm}^3$

58. $V = \frac{1}{2}(10)(2)(3) = 30 \text{ m}^3$

60.

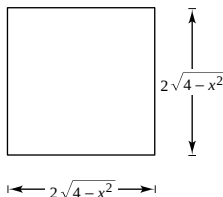


Base of Cross Section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = (2\sqrt{4 - x^2})^2$

$$V = \int_{-2}^2 4(4 - x^2) dx$$

$$= 4 \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{128}{3}$$

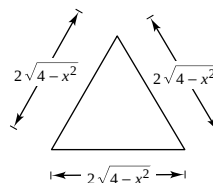


(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{4 - x^2})(\sqrt{3}\sqrt{4 - x^2})$

$$= \sqrt{3}(4 - x^2)$$

$$V = \sqrt{3} \int_{-2}^2 (4 - x^2) dx$$

$$= \sqrt{3} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32\sqrt{3}}{3}$$



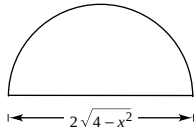
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60. —CONTINUED—

$$(c) A(x) = \frac{1}{2} \pi r^2$$

$$= \frac{\pi}{2} (\sqrt{4-x^2})^2 = \frac{\pi}{2} (4-x^2)$$

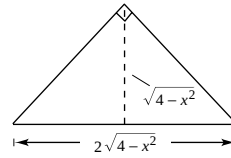
$$V = \frac{\pi}{2} \int_{-2}^2 (4-x^2) dx = \frac{\pi}{2} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{16\pi}{3}$$



$$(d) A(x) = \frac{1}{2}bh$$

$$= \frac{1}{2} (2\sqrt{4-x^2}) (\sqrt{4-x^2}) = 4-x^2$$

$$V = \int_{-2}^2 (4-x^2) dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$$



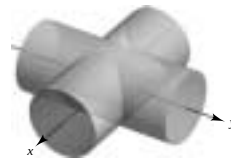
62. The cross sections are squares. By symmetry, we can set up an integral for an eighth of the volume and multiply by 8.

$$A(y) = b^2 = (\sqrt{r^2 - y^2})^2$$

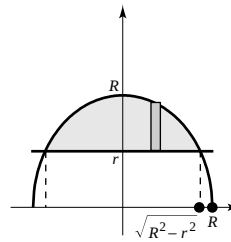
$$V = 8 \int_0^r (r^2 - y^2) dy$$

$$= 8 \left[r^2 y - \frac{1}{3} y^3 \right]_0^r$$

$$= \frac{16}{3} r^3$$



$$\begin{aligned} 64. V &= \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} [(\sqrt{R^2-x^2})^2 - r^2] dx \\ &= 2\pi \int_0^{\sqrt{R^2-r^2}} (R^2 - r^2 - x^2) dx \\ &= 2\pi \left[(R^2 - r^2)x - \frac{x^3}{3} \right]_0^{\sqrt{R^2-r^2}} \\ &= 2\pi \left[(R^2 - r^2)^{3/2} - \frac{(R^2 - r^2)^{3/2}}{3} \right] \\ &= \frac{4}{3} \pi (R^2 - r^2)^{3/2} \end{aligned}$$



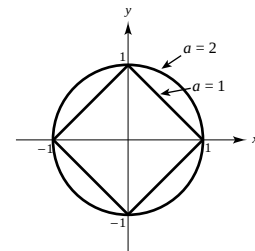
66. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.

- (b) $|y| = (1 - |x|^a)^{1/a}$

$$A = 2 \int_{-1}^1 (1 - |x|^a)^{1/a} dx = 4 \int_0^1 (1 - x^a)^{1/a} dx$$

To approximate the volume of the solid, form n slices, each of whose area is approximated by the integral above. Then sum the volumes of these n slices.



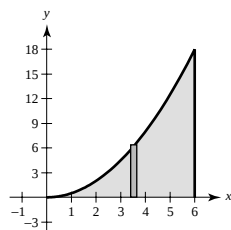
Section 6.3 Volume: The Shell Method

2. $p(x) = x$

$h(x) = 1 - x$

$$V = 2\pi \int_0^1 x(1 - x) dx$$

$$= 2\pi \int_0^1 (x - x^2) dx = 2\pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\pi}{3}$$



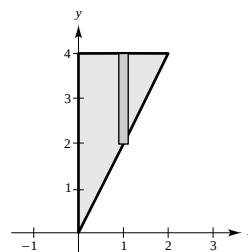
4. $p(x) = x$

$h(x) = 8 - (x^2 + 4) = 4 - x^2$

$$V = 2\pi \int_0^2 x(4 - x^2) dx$$

$$= 2\pi \int_0^2 (4x - x^3) dx$$

$$= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 = 8\pi$$



6. $p(x) = x$

$h(x) = \frac{1}{2}x^2$

$$V = 2\pi \int_0^6 \frac{1}{2}x^3 dx$$

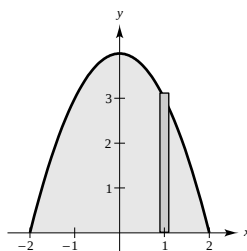
$$= \left[\pi \frac{x^4}{4} \right]_0^6 = 324\pi$$

8. $p(x) = x$

$h(x) = 4 - x^2$

$$V = 2\pi \int_0^2 (4x - x^3) dx$$

$$= 2\pi \left[2x^2 - \frac{1}{4}x^4 \right]_0^2 = 8\pi$$



10. $p(x) = x$

$h(x) = 4 - 2x$

$$V = 2\pi \int_0^2 x(4 - 2x) dx$$

$$= 2\pi \int_0^2 (4x - 2x^2) dx$$

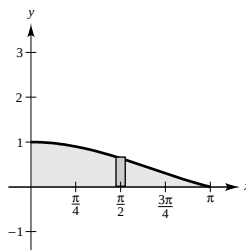
$$= 2\pi \left[2x^2 - \frac{2}{3}x^3 \right]_0^2 = \frac{16\pi}{3}$$

12. $p(x) = x$

$h(x) = \frac{\sin x}{x}$

$$V = 2\pi \int_0^\pi x \left[\frac{\sin x}{x} \right] dx$$

$$= 2\pi \int_0^\pi \sin x dx = \left[-2\pi \cos x \right]_0^\pi = 4\pi$$



14. $p(y) = -y$ ($p(y) \geq 0$ on $[-2, 0]$)

$h(y) = 4 - (2 - y) = 2 + y$

$$V = 2\pi \int_{-2}^0 (-y)(2 + y) dy$$

$$= 2\pi \int_{-2}^0 (-2y - y^2) dy$$

$$= 2\pi \left[-y^2 - \frac{y^3}{3} \right]_{-2}^0 = \frac{8\pi}{3}$$

16. $p(y) = y$

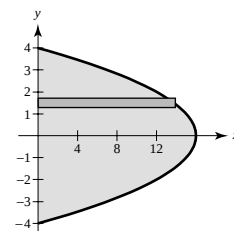
$h(y) = 16 - y^2$

$$V = 2\pi \int_0^4 y(16 - y^2) dy$$

$$= 2\pi \int_0^4 (16y - y^3) dy$$

$$= 2\pi \left[8y^2 - \frac{y^4}{4} \right]_0^4$$

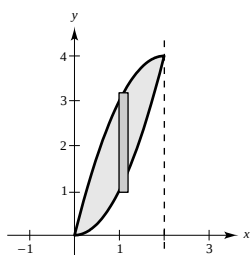
$$= 2\pi[128 - 64] = 128\pi$$



18. $p(x) = 2 - x$

$$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$$

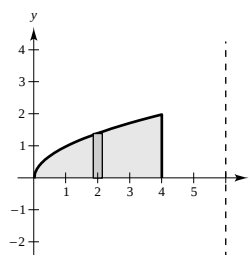
$$\begin{aligned} V &= 2\pi \int_0^2 (2-x)(4x-2x^2) dx \\ &= 2\pi \int_0^2 (8x - 8x^2 + 2x^3) dx \\ &= 2\pi \left[4x^2 - \frac{8}{3}x^3 + \frac{1}{2}x^4 \right]_0^2 = \frac{16\pi}{3} \end{aligned}$$



20. $p(x) = 6 - x$

$$h(x) = \sqrt{x}$$

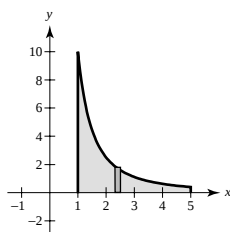
$$\begin{aligned} V &= 2\pi \int_0^4 (6-x)\sqrt{x} dx \\ &= 2\pi \int_0^4 (6x^{1/2} - x^{3/2}) dx \\ &= 2\pi \left[4x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^4 = \frac{192\pi}{5} \end{aligned}$$



22. (a) Disk

$$R(x) = \frac{10}{x^2}, r(x) = 0$$

$$\begin{aligned} V &= \pi \int_1^5 \left(\frac{10}{x^2} \right)^2 dx \\ &= 100\pi \int_1^5 x^{-4} dx \\ &= 100\pi \left[\frac{x^{-3}}{-3} \right]_1^5 \\ &= -\frac{100\pi}{3} \left[\frac{1}{125} - 1 \right] = \frac{496\pi}{15} \end{aligned}$$



(b) Shell

$$R(x) = x, r(x) = 0$$

$$\begin{aligned} V &= 2\pi \int_1^5 x \left(\frac{10}{x^2} \right) dx \\ &= 20\pi \int_1^5 \frac{1}{x} dx \\ &= 20\pi \left[\ln|x| \right]_1^5 = 20\pi \ln 5 \end{aligned}$$

(c) Disk

$$R(x) = 10, r(x) = 10 - \frac{10}{x^2}$$

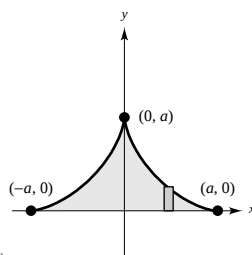
$$\begin{aligned} V &= \pi \int_1^5 \left[10^2 - \left(10 - \frac{10}{x^2} \right)^2 \right] dx \\ &= \pi \left[\frac{100}{3x^3} - \frac{200}{x} \right]_1^5 = \frac{1904\pi}{15} \end{aligned}$$

24. (a) Disk

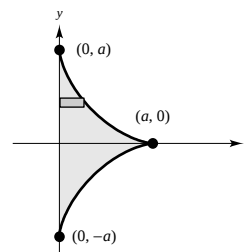
$$R(x) = (a^{2/3} - x^{2/3})^{3/2}$$

$$r(x) = 0$$

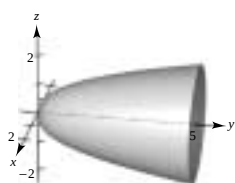
$$\begin{aligned} V &= \pi \int_{-a}^a (a^{2/3} - x^{2/3})^3 dx \\ &= 2\pi \int_0^a (a^2 - 3a^{4/3}x^{2/3} + 3a^{2/3}x^{4/3} - x^2) dx \\ &= 2\pi \left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3 \right]_0^a \\ &= 2\pi \left(a^3 - \frac{9}{5}a^3 + \frac{9}{7}a^3 - \frac{1}{3}a^3 \right) = \frac{32\pi a^3}{105} \end{aligned}$$



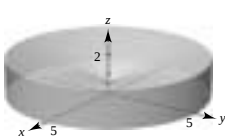
(b) Same as part a by symmetry



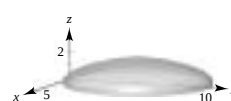
26. (a)



(b)



(c)



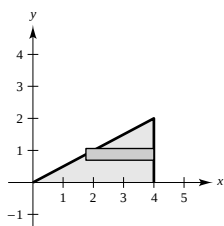
$$a < c < b$$

$$28. 2\pi \int_0^4 x \left(\frac{x}{2} \right) dx$$

represents the volume of the solid generated by revolving the region bounded by $y = x/2$, $y = 0$, and $x = 4$ about the y -axis by using the Shell Method.

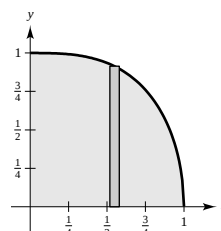
$$\pi \int_0^2 [16 - (2y)^2] dy = \pi \int_0^2 [(4)^2 - (2y)^2] dy$$

represents this same volume by using the Disk Method.



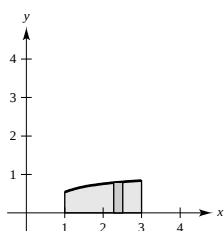
Disk Method

30. (a)



$$(b) V = 2\pi \int_0^1 x \sqrt{1 - x^3} dx \approx 2.3222$$

32. (a)

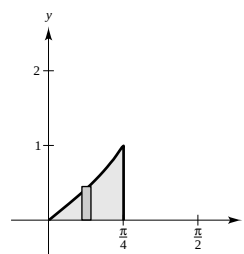


$$(b) V = 2\pi \int_1^3 \frac{2x}{1 + e^{1/x}} dx \approx 19.0162$$

$$34. y = \tan x, y = 0, x = 0, x = \frac{\pi}{4}$$

Volume ≈ 1

Matches (e)



36. Total volume of the hemisphere is $\frac{1}{2}(\frac{4}{3})\pi r^3 = \frac{2}{3}\pi(3)^3 = 18\pi$. By the Shell Method, $p(x) = x$, $h(x) = \sqrt{9 - x^2}$. Find x_0 such that

$$6\pi = 2\pi \int_0^{x_0} x \sqrt{9 - x^2} dx$$

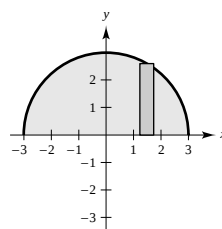
$$6 = - \int_0^{x_0} (9 - x^2)^{1/2} (-2x) dx$$

$$= \left[-\frac{2}{3} (9 - x^2)^{3/2} \right]_0^{x_0} = 18 - \frac{2}{3} (9 - x_0^2)^{3/2}$$

$$(9 - x_0^2)^{3/2} = 18$$

$$x_0 = \sqrt{9 - 18^{2/3}} \approx 1.460.$$

$$\text{Diameter: } 2\sqrt{9 - 18^{2/3}} \approx 2.920$$



$$\begin{aligned}
 38. V &= 4\pi \int_{-r}^r (R-x)\sqrt{r^2-x^2} dx \\
 &= 4\pi R \int_{-r}^r \sqrt{r^2-x^2} dx - 4\pi \int_{-r}^r x\sqrt{r^2-x^2} dx \\
 &= 4\pi R \left(\frac{\pi r^2}{2} \right) + \left[2\pi \left(\frac{2}{3} \right) (r^2-x^2)^{3/2} \right]_{-r}^r \\
 &= 2\pi^2 r^2 R
 \end{aligned}$$

$$\begin{aligned}
 40. (a) \text{ Area region} &= \int_0^b [ab^n - ax^n] dx \\
 &= \left[ab^n x - a \frac{x^{n+1}}{n+1} \right]_0^b \\
 &= ab^{n+1} - a \frac{b^{n+1}}{n+1} \\
 &= ab^{n+1} \left(1 - \frac{1}{n+1} \right) = ab^{n+1} \left(\frac{n}{n+1} \right) \\
 R_1(n) &= \frac{ab^{n+1} \left(\frac{n}{n+1} \right)}{(ab^n)b} = \frac{n}{n+1}
 \end{aligned}$$

$$\begin{aligned}
 (b) \lim_{n \rightarrow \infty} R_1(n) &= \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \\
 \lim_{n \rightarrow \infty} (ab^n)b &= \infty
 \end{aligned}$$

(c) **Disk Method:**

$$\begin{aligned}
 V &= 2\pi \int_0^b x(ab^n - ax^n) dx \\
 &= 2\pi a \int_0^b (xb^n - x^{n+1}) dx \\
 &= 2\pi a \left[\frac{b^n}{2} x^2 - \frac{x^{n+2}}{n+2} \right]_0^b \\
 &= 2\pi a \left[\frac{b^{n+2}}{2} - \frac{b^{n+2}}{n+2} \right] = \pi ab^{n+2} \left(\frac{n}{n+2} \right) \\
 R_2(n) &= \frac{\pi ab^{n+2} \left(\frac{n}{n+2} \right)}{(\pi b^2)(ab^n)} = \left(\frac{n}{n+2} \right)
 \end{aligned}$$

$$(d) \lim_{n \rightarrow \infty} R_2(n) = \lim_{n \rightarrow \infty} \left(\frac{n}{n+2} \right) = 1$$

$$\lim_{n \rightarrow \infty} (\pi b^2)(ab^n) = \infty$$

(e) As $n \rightarrow \infty$, the graph approaches the line $x = 1$.

$$\begin{aligned}
 42. (a) V &= 2\pi \int_0^4 x f(x) dx \\
 &= \frac{2\pi(40)}{3(4)} [0 + 4(10)(45) + 2(20)(40) + 4(30)(20) + 0] \\
 &= \frac{20\pi}{3} [5800] \approx 121,475 \text{ cubic feet}
 \end{aligned}$$

$$(b) \text{ Top line: } y - 50 = \frac{40 - 50}{20 - 0}(x - 0) = -\frac{1}{2}x \Rightarrow y = -\frac{1}{2}x + 50$$

$$\text{Bottom line: } y - 40 = \frac{0 - 40}{40 - 20}(x - 20) = -2(x - 20) \Rightarrow y = -2x + 80$$

$$\begin{aligned}
 V &= 2\pi \int_0^{20} x \left(-\frac{1}{2}x + 50 \right) dx + 2\pi \int_{20}^{40} x(-2x + 80) dx \\
 &= 2\pi \int_0^{20} \left(-\frac{1}{2}x^2 + 50x \right) dx + 2\pi \int_{20}^{40} (-2x^2 + 80x) dx \\
 &= 2\pi \left[-\frac{x^3}{6} + 25x^2 \right]_0^{20} + 2\pi \left[-\frac{2x^3}{3} + 40x^2 \right]_{20}^{40} \\
 &= 2\pi \left[\frac{26,000}{3} \right] + 2\pi \left[\frac{32,000}{3} \right] \\
 &\approx 121,475 \text{ cubic feet}
 \end{aligned}$$

(Note that Simpson's Rule is exact for this problem.)

Section 6.4 Arc Length and Surfaces of Revolution

2. (1, 2), (7, 10)

(a) $d = \sqrt{(7-1)^2 + (10-2)^2} = 10$

(b) $y = \frac{4}{3}x + \frac{2}{3}$

$y' = \frac{4}{3}$

$$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2} dx = \left[\frac{5}{3}x\right]_1^7 = 10$$

6. $y = \frac{3}{2}x^{2/3} + 4$

$y' = x^{-1/3}, [1, 27]$

$$\begin{aligned}
 s &= \int_1^{27} \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx \\
 &= \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx \\
 &= \frac{3}{2} \int_1^{27} \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx \\
 &= \left[\frac{3}{2} \cdot \frac{2}{3} (x^{2/3} + 1)^{3/2}\right]_1^{27} \\
 &= 10^{3/2} - 2^{3/2} \approx 28.794
 \end{aligned}$$

10. $y = \frac{1}{2}(e^x + e^{-x})$

$y' = \frac{1}{2}(e^x - e^{-x}), [0, 2]$

$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, [0, 2]$

$$\begin{aligned}
 s &= \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2} dx \\
 &= \frac{1}{2} \int_0^2 (e^x + e^{-x}) dx \\
 &= \frac{1}{2} \left[e^x - e^{-x}\right]_0^2 = \frac{1}{2} \left(e^2 - \frac{1}{e^2}\right) \approx 3.627
 \end{aligned}$$

4. $y = 2x^{3/2} + 3$

$y' = 3x^{1/2}, [0, 9]$

$$\begin{aligned}
 s &= \int_0^9 \sqrt{1 + 9x} dx \\
 &= \left[\frac{2}{27}(1 + 9x)^{3/2}\right]_0^9 \\
 &= \frac{2}{27}(82^{3/2} - 1) \approx 54.929
 \end{aligned}$$

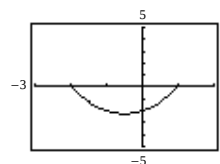
8. $y = \frac{x^5}{10} + \frac{1}{6x^3}$

$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$

$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, [1, 2]$

$$\begin{aligned}
 s &= \int_a^b \sqrt{1 + (y')^2} dx \\
 &= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2} dx \\
 &= \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right) dx \\
 &= \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.246
 \end{aligned}$$

12. (a) $y = x^2 + x - 2, -2 \leq x \leq 1$



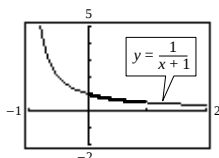
(b) $y' = 2x + 1$

$1 + (y')^2 = 1 + 4x^2 + 4x + 1$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2} dx$$

(c) $L \approx 5.653$

14. (a) $y = \frac{1}{1+x}, 0 \leq x \leq 1$



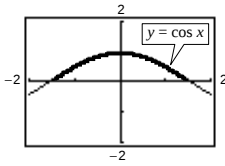
(b) $y' = -\frac{1}{(1+x)^2}$

(c) $L \approx 1.132$

$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$

$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}} dx$$

16. (a) $y = \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



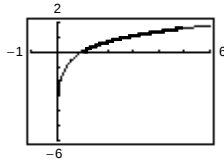
(b) $y' = -\sin x$

(c) 3.820

$$1 + (y')^2 = 1 + \sin^2 x$$

$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} \, dx$$

18. (a) $y = \ln x, 1 \leq x \leq 5$



(b) $y' = \frac{1}{x}$

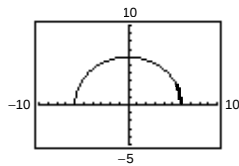
(c) $L \approx 4.367$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}} \, dx$$

20. (a) $x = \sqrt{36 - y^2}, 0 \leq y \leq 3$

$$y = \sqrt{36 - x^2}, 3\sqrt{3} \leq x \leq 6$$



(b) $\frac{dx}{dy} = \frac{1}{2}(36 - y^2)^{-1/2}(-2y)$

(c) $L \approx 3.142 (\pi!)$

$$= \frac{-y}{\sqrt{36 - y^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}} \, dy$$

$$= \int_0^3 \frac{6}{\sqrt{36 - y^2}} \, dy$$

Alternatively, you can convert to a function of x .

$$y = \sqrt{36 - x^2}$$

$$y' = \frac{dy}{dx} = -\frac{x}{\sqrt{36 - x^2}}$$

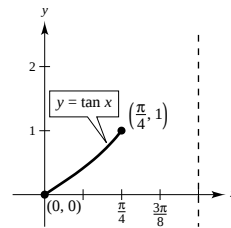
$$L = \int_{3\sqrt{3}}^6 \sqrt{1 + \frac{x^2}{36 - x^2}} \, dx = \int_{3\sqrt{3}}^6 \frac{6}{\sqrt{36 - x^2}} \, dx$$

Although this integral is undefined at $x = 0$, a graphing utility still gives $L \approx 3.142$.

22. $\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2} \, dx$

$$s \approx 1$$

Matches (e)



24. $f(x) = (x^2 - 4)^2, [0, 4]$

(a) $d = \sqrt{(4 - 0)^2 + (144 - 16)^2} \approx 128.062$

(b) $d = \sqrt{(1 - 0)^2 + (9 - 16)^2} + \sqrt{(2 - 1)^2 + (0 - 9)^2} + \sqrt{(3 - 2)^2 + (25 - 0)^2} + \sqrt{(4 - 3)^2 + (144 - 25)^2}$
 ≈ 160.151

(c) $s = \int_0^4 \sqrt{1 + [4x(x^2 - 4)]^2} \, dx \approx 159.087$

(d) 160.287

26. Let $y = \ln x$, $1 \leq x \leq e$, $y' = \frac{1}{x}$ and $L_1 = \int_1^e \sqrt{1 + \frac{1}{x^2}} dx$.

Equivalently, $x = e^y$, $0 \leq y \leq 1$, $\frac{dx}{dy} = e^y$, and $L_2 = \int_0^1 \sqrt{1 + e^{2y}} dy = \int_0^1 \sqrt{1 + e^{2x}} dx$.

Numerically, both integrals yield $L = 2.0035$

28. $y = 31 - 10(e^{x/20} + e^{-x/20})$

$$y' = -\frac{1}{2}(e^{x/20} - e^{-x/20})$$

$$1 + (y')^2 = 1 + \frac{1}{4}(e^{x/10} - 2 + e^{-x/10}) = \left[\frac{1}{2}(e^{x/20} + e^{-x/20}) \right]^2$$

$$\begin{aligned} s &= \int_{-20}^{20} \sqrt{\left[\frac{1}{2}(e^{x/20} + e^{-x/20}) \right]^2} dx \\ &= \frac{1}{2} \int_{-20}^{20} (e^{x/20} + e^{-x/20}) dx = \left[10(e^{x/20} - e^{-x/20}) \right]_{-20}^{20} = 20 \left(e - \frac{1}{e} \right) \approx 47 \text{ ft} \end{aligned}$$

Thus, there are $100(47) = 4700$ square feet of roofing on the barn.

30. $y = 693.8597 - 68.7672 \cosh 0.0100333x$

$$y' = -0.6899619478 \sinh 0.0100333x$$

$$s = \int_{-299.2239}^{299.2239} \sqrt{1 + (-0.6899619478 \sinh 0.0100333x)^2} dx \approx 1480$$

(Use Simpson's Rule with $n = 100$ or a graphing utility.)

32. $y = \sqrt{25 - x^2}$

$$y' = \frac{-x}{\sqrt{25 - x^2}}$$

$$1 + (y')^2 = \frac{25}{25 - x^2}$$

$$\begin{aligned} s &= \int_{-3}^4 \sqrt{\frac{25}{25 - x^2}} dx \\ &= \int_{-3}^4 \frac{5}{\sqrt{25 - x^2}} dx \\ &= \left[5 \arcsin \frac{x}{5} \right]_{-3}^4 \\ &= 5 \left[\arcsin \frac{4}{5} - \arcsin \left(-\frac{3}{5} \right) \right] \approx 7.8540 \end{aligned}$$

$$\frac{1}{4}[2\pi(5)] \approx 7.8540 = s$$

34. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}, [4, 9]$$

$$\begin{aligned} S &= 2\pi \int_4^9 2\sqrt{x} \sqrt{1 + \frac{1}{x}} dx \\ &= 4\pi \int_4^9 \sqrt{x+1} dx \\ &= \frac{8}{3} \pi (x+1)^{3/2} \Big|_4^9 \\ &= \frac{8\pi}{3} (10^{3/2} - 5^{3/2}) \approx 171.258 \end{aligned}$$

36. $y = \frac{x}{2}$

$$y' = \frac{1}{2}$$

$$1 + (y')^2 = \frac{5}{4}, [0, 6]$$

$$\begin{aligned} S &= 2\pi \int_0^6 \frac{x}{2} \sqrt{\frac{5}{4}} dx \\ &= \left[\frac{2\pi\sqrt{5}}{8} x^2 \right]_0^6 = 9\sqrt{5} \pi \end{aligned}$$

40. $y = \ln x$

$$y' = \frac{1}{x}$$

$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, [1, e]$$

$$\begin{aligned} S &= 2\pi \int_1^e x \sqrt{\frac{x^2 + 1}{x^2}} dx \\ &= 2\pi \int_1^e \sqrt{x^2 + 1} dx \approx 22.943 \end{aligned}$$

44. The surface of revolution given by f_1 will be larger. $r(x)$ is larger for f_1 .

46. $y = \sqrt{r^2 - x^2}$

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$

$$\begin{aligned} S &= 2\pi \int_{-r}^r \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}} dx \\ &= 2\pi \int_{-r}^r r dx = \left[2\pi r x \right]_{-r}^r = 4\pi r^2 \end{aligned}$$

38. $y = 9 - x^2, [0, 3]$

$$y' = -2x$$

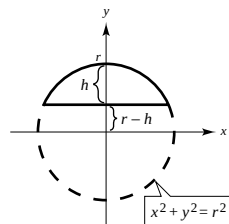
$$\begin{aligned} S &= 2\pi \int_0^3 x \sqrt{1 + 4x^2} dx \\ &= \frac{\pi}{4} \int_0^3 (1 + 4x^2)^{1/2} (8x) dx \\ &= \left[\frac{\pi}{6} (1 + 4x^2)^{3/2} \right]_0^3 \\ &= \frac{\pi}{6} (37^{3/2} - 1) \approx 117.319 \end{aligned}$$

42. The precalculus formula is the distance formula between two points. The representative element is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i} \right)^2} \Delta x_i.$$

48. From Exercise 47 we have:

$$\begin{aligned} S &= 2\pi \int_0^a \frac{rx}{\sqrt{r^2 - x^2}} dx \\ &= -r\pi \int_0^a \frac{-2x dx}{\sqrt{r^2 - x^2}} \\ &= \left[-2r\pi \sqrt{r^2 - x^2} \right]_0^a \\ &= 2r^2\pi - 2r\pi \sqrt{r^2 - a^2} \\ &= 2r\pi(r - \sqrt{r^2 - a^2}) \\ &= 2\pi rh \text{ (where } h \text{ is the height of the zone)} \end{aligned}$$

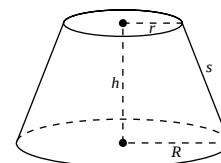


50. (a) We approximate the volume by summing 6 disks of thickness 3 and circumference C_i equal to the average of the given circumferences:

$$\begin{aligned} V &\approx \sum_{i=1}^6 \pi r_i^2(3) = \sum_{i=1}^6 \pi \left(\frac{C_i}{2\pi} \right)^2 (3) = \frac{3}{4\pi} \sum_{i=1}^6 C_i^2 \\ &= \frac{3}{4\pi} \left[\left(\frac{50 + 65.5}{2} \right)^2 + \left(\frac{65.5 + 70}{2} \right)^2 + \left(\frac{70 + 66}{2} \right)^2 + \left(\frac{66 + 58}{2} \right)^2 + \left(\frac{58 + 51}{2} \right)^2 + \left(\frac{51 + 48}{2} \right)^2 \right] \\ &= \frac{3}{4\pi} [57.75^2 + 67.75^2 + 68^2 + 62^2 + 54.5^2 + 49.5^2] \\ &= \frac{3}{4\pi} [21813.625] = 5207.62 \text{ cubic inches} \end{aligned}$$

- (b) The lateral surface area of a frustum of a right circular cone is $\pi s(R + r)$. For the first frustum,

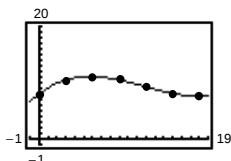
$$\begin{aligned} S_1 &\approx \pi \left[3^2 + \left(\frac{65.5 - 50}{2\pi} \right)^2 \right]^{1/2} \left[\frac{50}{2\pi} + \frac{65.5}{2\pi} \right] \\ &= \left(\frac{50 + 65.5}{2} \right) \left[9 + \left(\frac{65.5 - 50}{2\pi} \right)^2 \right]^{1/2}. \end{aligned}$$



Adding the six frustums together,

$$\begin{aligned} S &\approx \left(\frac{50 + 65.5}{2} \right) \left[9 + \left(\frac{15.5}{2\pi} \right)^2 \right]^{1/2} + \left(\frac{65.5 + 70}{2} \right) \left[9 + \left(\frac{4.5}{2\pi} \right)^2 \right]^{1/2} + \\ &\quad \left(\frac{70 + 66}{2} \right) \left[9 + \left(\frac{4}{2\pi} \right)^2 \right]^{1/2} + \left(\frac{66 + 58}{2} \right) \left[9 + \left(\frac{8}{2\pi} \right)^2 \right]^{1/2} + \\ &\quad \left(\frac{58 + 51}{2} \right) \left[9 + \left(\frac{7}{2\pi} \right)^2 \right]^{1/2} + \left(\frac{51 + 48}{2} \right) \left[9 + \left(\frac{3}{2\pi} \right)^2 \right]^{1/2} \\ &\approx 224.30 + 208.96 + 208.54 + 202.06 + 174.41 + 150.37 \\ &= 1168.64 \end{aligned}$$

- (c) $r = 0.00401y^3 - 0.1416y^2 + 1.232y + 7.943$



- (d) $V = \int_0^{18} \pi r^2 dy \approx 5275.9$ cubic inches

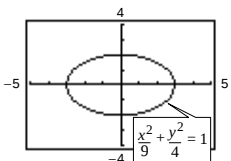
$$\begin{aligned} S &= \int_0^{18} 2\pi r(y) \sqrt{1 + r'(y)^2} dy \\ &\approx 1179.5 \text{ square inches} \end{aligned}$$

52. Individual project, see Exercise 50, 51.

54. (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Ellipse: $y_1 = 2\sqrt{1 - \frac{x^2}{9}}$

$y_2 = -2\sqrt{1 - \frac{x^2}{9}}$



- (b) $y = 2\sqrt{1 - \frac{x^2}{9}}, 0 \leq x \leq 3$

$$\begin{aligned} y' &= 2 \left(\frac{1}{2} \right) \left(1 - \frac{x^2}{9} \right)^{-1/2} \left(-\frac{2x}{9} \right) \\ &= \frac{-2x}{9\sqrt{1 - \frac{x^2}{9}}} = \frac{-2x}{3\sqrt{9 - x^2}} \end{aligned}$$

$$L = \int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}} dx$$

- (c) You cannot evaluate this definite integral, since the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason. Also, the integrand does not have an elementary antiderivative.

56. Essay

Section 6.5 Work

2. $W = Fd = (2800)(4) = 11,200 \text{ ft} \cdot \text{lb}$

4. $W = Fd = [9(2000)]\left[\frac{1}{2}(5280)\right] = 47,520,000 \text{ ft} \cdot \text{lb}$

6. $W = \int_a^b F(x) dx$ is the work done by a force F moving an object along a straight line from $x = a$ to $x = b$.

8. (a) $W = \int_0^9 6 dx = 54 \text{ ft} \cdot \text{lbs}$

(b) $W = \int_0^7 20 dx + \int_7^9 (-10x + 90) dx = 140 + 20$
 $= 160 \text{ ft} \cdot \text{lbs}$

(c) $W = \int_0^9 \frac{1}{27} x^2 dx = \left[\frac{x^3}{81} \right]_0^9 = 9 \text{ ft} \cdot \text{lbs}$

(d) $W = \int_0^9 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^9 = \frac{2}{3}(27) = 18 \text{ ft} \cdot \text{lbs}$

10. $W = \int_0^{10} \frac{5}{4} x dx = \left[\frac{5}{8} x^2 \right]_0^{10}$
 $= 40 \text{ in} \cdot \text{lb} \approx 3.33 \text{ ft} \cdot \text{lb}$

12. $F(x) = kx$

$$800 = k(70) \Rightarrow k = \frac{80}{7}$$

$$W = \int_0^{70} F(x) dx = \int_0^{70} \frac{80}{7} x dx = \left[\frac{40x^2}{7} \right]_0^{70}$$

$$= 28000 \text{ n} \cdot \text{cm} = 280 \text{ Nm}$$

14. $F(x) = kx$

$$15 = k(1) = k$$

$$W = 2 \int_0^4 15x dx = \left[15x^2 \right]_0^4$$

$$= 240 \text{ ft} \cdot \text{lb}$$

16. $W = 7.5 = \int_0^{1/6} kx dx = \left[\frac{kx^2}{2} \right]_0^{1/6} = \frac{k}{72} \Rightarrow k = 540$

$$W = \int_{1/6}^{5/24} 540x dx = 270x^2 \Big|_{1/6}^{5/24} = 4.21875 \text{ ft} \cdot \text{lbs}$$

18. $W = \int_{4000}^h \frac{80,000,000}{x^2} dx = \left[-\frac{80,000,000}{x} \right]_{4000}^h$
 $= \frac{-80,000,000}{h} + 20,000$

$$\lim_{h \rightarrow \infty} W = 20,000 \text{ mi/ton} \approx 2.1 \times 10^{11} \text{ ft} \cdot \text{lb}$$

20. Weight on surface of moon: $\frac{1}{6}(12) = 2 \text{ tons}$

Weight varies inversely as the square of distance from the center of the moon. Therefore,

$$F(x) = \frac{k}{x^2}$$

$$2 = \frac{k}{(1100)^2}$$

$$k = 2.42 \times 10^6$$

$$W = \int_{1100}^{1150} \frac{2.42 \times 10^6}{x^2} dx = \left[\frac{-2.42 \times 10^6}{x} \right]_{1100}^{1150} = 2.42 \times 10^6 \left(\frac{1}{1100} - \frac{1}{1150} \right)$$

$$\approx 95.652 \text{ mi} \cdot \text{ton} \approx 1.01 \times 10^9 \text{ ft} \cdot \text{lb}$$

22. The bottom half had to be pumped a greater distance than the top half.

24. Volume of disk: $4\pi \Delta y$

Weight of disk: $9800(4\pi) \Delta y$

Distance the disk of water is moved: y

$$\begin{aligned} W &= \int_{10}^{12} y(9800)(4\pi) dy = 39,200\pi \left[\frac{y^2}{2} \right]_{10}^{12} \\ &= 39,200\pi(22) \\ &= 862,400\pi \text{ newton-meters} \end{aligned}$$

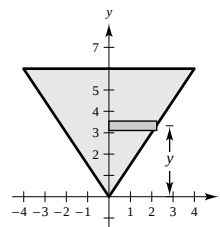
26. Volume of disk: $\pi \left(\frac{2}{3}y \right)^2 \Delta y$

Weight of disk: $62.4\pi \left(\frac{2}{3}y \right)^2 \Delta y$

Distance: y

$$(a) \quad W = \frac{4}{9}(62.4)\pi \int_0^2 y^3 dy = \left[\frac{4}{9}(62.4)\pi \left(\frac{1}{4}y^4 \right) \right]_0^2 \approx 110.9\pi \text{ ft} \cdot \text{lb}$$

$$(b) \quad W = \frac{4}{9}(62.4)\pi \int_4^6 y^3 dy = \left[\frac{4}{9}(62.4)\pi \left(\frac{1}{4}y^4 \right) \right]_4^6 \approx 7210.7\pi \text{ ft} \cdot \text{lb}$$

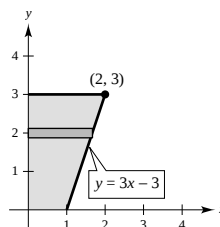


28. Volume of each layer: $\frac{y+3}{3}(3) \Delta y = (y+3) \Delta y$

Weight of each layer: $55.6(y+3) \Delta y$

Distance: $6-y$

$$\begin{aligned} W &= \int_0^3 55.6(6-y)(y+3) dy = 55.6 \int_0^3 (18+3y-y^2) dy \\ &= 55.6 \left[18y + \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 \\ &= 3252.6 \text{ ft} \cdot \text{lb} \end{aligned}$$



30. Volume of layer: $V = 12(2)\sqrt{(25/4) - y^2} \Delta y$

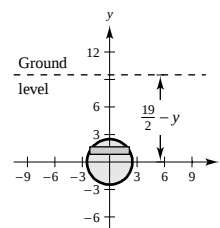
Weight of layer: $W = 42(24)\sqrt{(25/4) - y^2} \Delta y$

Distance: $\frac{19}{2} - y$

$$\begin{aligned} W &= \int_{-2.5}^{2.5} 42(24)\sqrt{\frac{25}{4} - y^2} \left(\frac{19}{2} - y \right) dy \\ &= 1008 \left[\frac{19}{2} \int_{-2.5}^{2.5} \sqrt{\frac{25}{4} - y^2} dy + \int_{-2.5}^{2.5} \sqrt{\frac{25}{4} - y^2} (-y) dy \right] \end{aligned}$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{5}{2}$. Thus, the work is

$$W = 1008 \left(\frac{19}{2} \right) \pi \left(\frac{5}{2} \right)^2 \left(\frac{1}{2} \right) = 29,925\pi \text{ ft} \cdot \text{lb} \approx 94,012.16 \text{ ft} \cdot \text{lb}.$$



32. The lower 10 feet of chain are raised 5 feet with a constant force.

$$W_1 = 3(10)5 = 150 \text{ ft} \cdot \text{lb}$$

The top 5 feet will be raised with variable force.

Weight of section: $3 \Delta y$

Distance: $5 - y$

$$W_2 = 3 \int_0^5 (5 - y) dy = \left[-\frac{3}{2}(5 - y)^2 \right]_0^5 = \frac{75}{2} \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 150 + \frac{75}{2} = \frac{375}{2} \text{ ft} \cdot \text{lb}$$

34. The work required to lift the chain is $337.5 \text{ ft} \cdot \text{lb}$ (from Exercise 31). The work required to lift the 500-pound load is $W = (500)(15) = 7500$. The work required to lift the chain with a 100-pound load attached is

$$W = 337.5 + 7500 = 7837.5 \text{ ft} \cdot \text{lbs}$$

$$36. W = 3 \int_0^6 (12 - 2y) dy = \left[-\frac{3}{4}(12 - 2y)^2 \right]_0^6 = \frac{3}{4}(12)^2 = 108 \text{ ft} \cdot \text{lb}$$

38. Work to pull up the ball: $W_1 = 500(40) = 20,000 \text{ ft} \cdot \text{lb}$

Work to pull up the cable: force is variable

Weight per section: $1 \Delta x$

Distance: $40 - x$

$$W_2 = \int_0^{40} (40 - x) dx = \left[-\frac{1}{2}(40 - x)^2 \right]_0^{40}$$

$$= 800 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 = 20,000 + 800 = 20,800 \text{ ft} \cdot \text{lb}$$

$$40. p = \frac{k}{V}$$

$$2500 = \frac{k}{1} \Rightarrow k = 2500$$

$$W = \int_1^3 \frac{2500}{V} dV = \left[2500 \ln V \right]_1^3$$

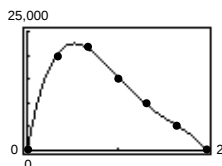
$$= 2500 \ln 3 \approx 2746.53 \text{ ft} \cdot \text{lb}$$

42. (a) $W = FD = (8000\pi)(2) = 16,000\pi \text{ ft} \cdot \text{lbs}$

$$(b) W \approx \frac{2-0}{3(6)} [0 + 4(20,000) + 2(22,000) + 4(15,000) + 2(10,000) + 4(5000) + 0]$$

$$\approx 24,88.889 \text{ ft} \cdot \text{lb}$$

$$(c) F(x) = -16,261.36x^4 + 85,295.45x^3 - 157,738.64x^2 + 104,386.36x - 32.4675$$



- (d) $F(x) = 0$ when $x \approx 0.524$ feet. $F(x)$ is a maximum when $x \approx 0.524$ feet.

$$(e) W = \int_0^2 F(x) dx \approx 25,180.5 \text{ ft} \cdot \text{lbs}$$

$$44. W = \int_0^4 \left(\frac{e^{x^2} - 1}{100} \right) dx \approx 11,494 \text{ ft} \cdot \text{lb}$$

$$46. W = \int_0^2 1000 \sinh x dx \approx 2762.2 \text{ ft} \cdot \text{lb}$$

Section 6.6 Moments, Centers of Mass, and Centroids

$$2. \bar{x} = \frac{7(-3) + 4(-2) + 3(5) + 8(6)}{7 + 4 + 3 + 8} = \frac{17}{11}$$

6. The center of mass is translated k units as well.

$$4. \bar{x} = \frac{12(-6) + 1(-4) + 6(-2) + 3(0) + 11(8)}{12 + 1 + 6 + 3 + 11} = 0$$

$$8. 200x = 550(5 - x) \text{ (Person on left)}$$

$$200x = 2750 - 550x$$

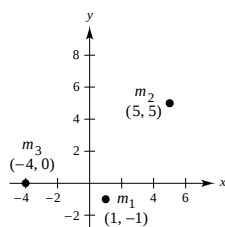
$$750x = 2750$$

$$x = 3\frac{2}{3} \text{ feet}$$

$$10. \bar{x} = \frac{10(1) + 2(5) + 5(-4)}{10 + 2 + 5} = 0$$

$$\bar{y} = \frac{10(-1) + 2(5) + 5(0)}{10 + 2 + 5} = 0$$

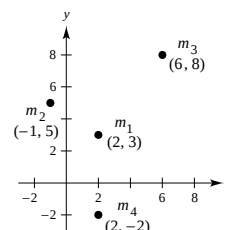
$$(\bar{x}, \bar{y}) = (0, 0)$$



$$12. \bar{x} = \frac{12(2) + 6(-1) + \frac{15}{2}(6) + 15(2)}{12 + 6 + \frac{15}{2} + 15} = \frac{93}{40.5} = \frac{62}{27}$$

$$\bar{y} = \frac{12(3) + 6(5) + \frac{15}{2}(8) + 15(-2)}{12 + 6 + \frac{15}{2} + 15} = \frac{96}{40.5} = \frac{64}{27}$$

$$(\bar{x}, \bar{y}) = \left(\frac{62}{27}, \frac{64}{27}\right)$$



$$14. m = \rho \int_0^2 \frac{1}{2}x^2 dx = \left[\rho \frac{x^3}{6}\right]_0^2 = \frac{4}{3}\rho$$

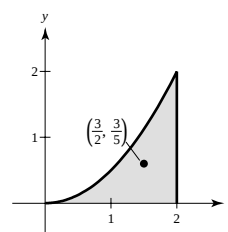
$$M_x = \rho \int_0^2 \frac{1}{2} \left(\frac{1}{2}x^2\right) \left(\frac{1}{2}x^2\right) dx = \frac{\rho}{8} \int_0^2 x^4 dx = \left[\frac{\rho}{40}x^5\right]_0^2 = \frac{32}{40}\rho = \frac{4}{5}\rho$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{4}{5}\rho}{\frac{4}{3}\rho} = \frac{3}{5}$$

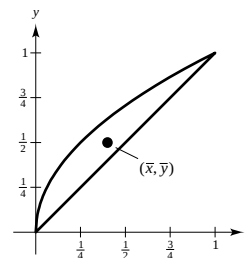
$$M_y = \rho \int_0^2 x \left(\frac{1}{2}x^2\right) dx = \frac{1}{2}\rho \int_0^2 x^3 dx = \left[\frac{\rho}{8}x^4\right]_0^2 = 2\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{2\rho}{\frac{4}{3}\rho} = \frac{3}{2}$$

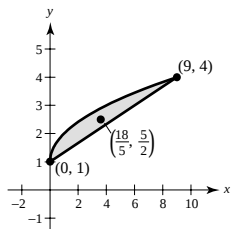
$$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{3}{5}\right)$$



$$\begin{aligned}
 16. \quad m &= \rho \int_0^1 (\sqrt{x} - x) dx = \rho \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} \right]_0^1 = \frac{\rho}{6} \\
 M_x &= \rho \int_0^1 \frac{(\sqrt{x} + x)}{2} (\sqrt{x} - x) dx = \frac{\rho}{2} \int_0^1 (x - x^2) dx = \frac{\rho}{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{\rho}{12} \\
 \bar{y} &= \frac{M_x}{m} = \frac{\rho}{12} \left(\frac{6}{\rho} \right) = \frac{1}{2} \\
 M_y &= \rho \int_0^1 x(\sqrt{x} - x) dx = \rho \int_0^1 (x^{3/2} - x^2) dx = \rho \left[\frac{2}{5} x^{5/2} - \frac{x^3}{3} \right]_0^1 = \frac{\rho}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{\rho}{15} \left(\frac{6}{\rho} \right) = \frac{2}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{2}{5}, \frac{1}{2} \right)
 \end{aligned}$$



$$\begin{aligned}
 18. \quad m &= \rho \int_0^9 \left[(\sqrt{x} + 1) - \left(\frac{1}{3}x + 1 \right) \right] dx = \rho \int_0^9 \left(\sqrt{x} - \frac{1}{3}x \right) dx \\
 &= \rho \left[\frac{2}{3} x^{3/2} - \frac{x^2}{6} \right]_0^9 = \rho \left(18 - \frac{27}{2} \right) = \frac{9}{2} \rho \\
 M_x &= \rho \int_0^9 \frac{\sqrt{x} + 1 + \frac{1}{3}x + 1}{2} \left(\sqrt{x} + 1 - \frac{1}{3}x - 1 \right) dx = \frac{\rho}{2} \int_0^9 \left(\sqrt{x} + \frac{1}{3}x + 2 \right) \left(\sqrt{x} - \frac{1}{3}x \right) dx \\
 &= \frac{\rho}{2} \int_0^9 \left(x - \frac{1}{3}x^{3/2} + \frac{1}{3}x^{3/2} - \frac{1}{9}x^2 + 2\sqrt{x} - \frac{2}{3}x \right) dx = \frac{\rho}{2} \int_0^9 \left(\frac{1}{3}x - \frac{1}{9}x^2 + 2\sqrt{x} \right) dx \\
 &= \frac{\rho}{2} \left[\frac{x^2}{6} - \frac{x^3}{27} + \frac{4}{3}x^{3/2} \right]_0^9 = \frac{\rho}{2} \left[\frac{27}{2} - 27 + 36 \right] = \frac{45}{3} \rho \\
 M_y &= \rho \int_0^9 x \left[\sqrt{x} + 1 - \frac{1}{3}x - 1 \right] dx = \rho \int_0^9 \left(x^{3/2} - \frac{1}{3}x^2 \right) dx = \rho \left[\frac{2}{5} x^{5/2} - \frac{1}{9}x^3 \right]_0^9 \\
 &= \rho \left[\frac{486}{5} - 81 \right] = \frac{81}{5} \rho \\
 \bar{x} &= \frac{M_y}{m} = \frac{\frac{81}{5} \rho}{\frac{9}{2} \rho} = \frac{18}{5}; \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{45}{3} \rho}{\frac{9}{2} \rho} = \frac{5}{2} \\
 (\bar{x}, \bar{y}) &= \left(\frac{18}{5}, \frac{5}{2} \right)
 \end{aligned}$$



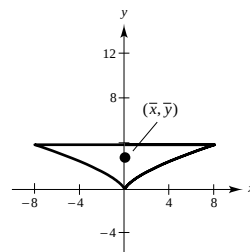
$$20. \quad m = 2\rho \int_0^8 (4 - x^{2/3}) dx = 2\rho \left[4x - \frac{3}{5}x^{5/3} \right]_0^8 = \frac{128\rho}{5}$$

By symmetry, M_y and $\bar{x} = 0$.

$$M_x = 2\rho \int_0^8 \left(\frac{4 + x^{2/3}}{2} \right) (4 - x^{2/3}) dx = \rho \left[16x - \frac{3}{7}x^{7/3} \right]_0^8 = \frac{512\rho}{7}$$

$$\bar{y} = \frac{512\rho}{7} \left(\frac{5}{128\rho} \right) = \frac{20}{7}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{20}{7} \right)$$



$$22. \quad m = \rho \int_0^2 (2y - y^2) dy = \rho \left[y^2 - \frac{y^3}{3} \right]_0^2 = \frac{4\rho}{3}$$

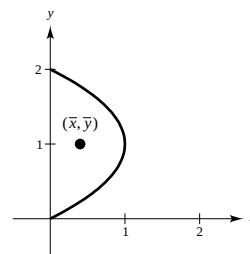
$$M_y = \rho \int_0^2 \left(\frac{2y - y^2}{2} \right) (2y - y^2) dy = \frac{\rho}{2} \left[\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right]_0^2 = \frac{8\rho}{15}$$

$$\bar{x} = \frac{M_y}{m} = \frac{8\rho}{15} \left(\frac{3}{4\rho} \right) = \frac{2}{5}$$

$$M_x = \rho \int_0^2 y(2y - y^2) dy = \rho \left[\frac{2y^3}{3} - \frac{y^4}{4} \right]_0^2 = \frac{4\rho}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{3} \left(\frac{3}{4\rho} \right) = 1$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{5}, 1 \right)$$



$$24. \quad m = \rho \int_{-1}^2 [(y + 2) - y^2] dy = \rho \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \frac{9\rho}{2}$$

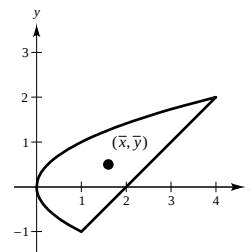
$$\begin{aligned} M_y &= \rho \int_{-1}^2 \frac{[(y + 2) + y^2]}{2} [(y + 2) - y^2] dy \\ &= \frac{\rho}{2} \int_{-1}^2 [(y + 2)^2 - y^4] dy = \frac{\rho}{2} \left[\frac{(y + 2)^3}{3} - \frac{y^5}{5} \right]_{-1}^2 = \frac{36\rho}{5} \end{aligned}$$

$$\bar{x} = \frac{M_y}{m} = \frac{36\rho}{5} \left(\frac{2}{9\rho} \right) = \frac{8}{5}$$

$$\begin{aligned} M_x &= \rho \int_{-1}^2 y[(y + 2) - y^2] dy \\ &= \rho \int_{-1}^2 (2y + y^2 - y^3) dy = \rho \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_{-1}^2 = \frac{9\rho}{4} \end{aligned}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{1}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{1}{2} \right)$$



$$26. \quad A = \int_1^4 \frac{1}{x} dx = \left[\ln|x| \right]_1^4 = \ln 4$$

$$M_x = \frac{1}{2} \int_1^4 \frac{1}{x^2} dx = \left[\frac{1}{2} \left(-\frac{1}{x} \right) \right]_1^4 = \left(-\frac{1}{8} + \frac{1}{2} \right) = \frac{3}{8}$$

$$M_y = \int_1^4 x \left(\frac{1}{x} \right) dx = \left[x \right]_1^4 = 3$$

$$28. \quad A = \int_{-2}^2 -(x^2 - 4) dx = 2 \int_0^2 (4 - x^2) dx = \left[8x - \frac{2x^3}{3} \right]_0^2 = 16 - \frac{16}{3} = \frac{32}{3}$$

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-2}^2 (x^2 - 4)(4 - x^2) dx = -\frac{1}{2} \int_{-2}^2 (x^4 - 8x^2 + 16) dx \\ &= -\frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-2}^2 = -\left[\frac{32}{5} - \frac{64}{3} + 32 \right] = -\frac{256}{15} \end{aligned}$$

$M_y = 0$ by symmetry.

$$30. \quad m = \rho \int_0^4 x e^{-x/2} dx \approx 2.3760\rho$$

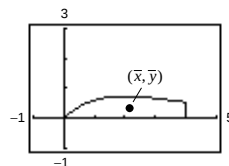
$$M_x = \rho \int_0^4 \left(\frac{x e^{-x/2}}{2} \right) (x e^{-x/2}) dx = \frac{\rho}{2} \int_0^4 x^2 e^{-x} dx \approx 0.7619\rho$$

$$M_y = \rho \int_0^4 x^2 e^{-x/2} dx \approx 5.1732\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 2.2$$

$$\bar{y} = \frac{M_x}{m} \approx 0.3$$

Therefore, the centroid is (2.2, 0.3).

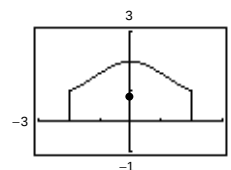


$$32. \quad m = \rho \int_{-2}^2 \frac{8}{x^2 + 4} dx \approx 6.2832\rho$$

$$M_x = \rho \int_{-2}^2 \frac{1}{2} \left(\frac{8}{x^2 + 4} \right) \left(\frac{8}{x^2 + 4} \right) dx = 32\rho \int_{-2}^2 \frac{1}{(x^2 + 4)^2} dx \approx 5.14149\rho$$

$$\bar{y} = \frac{M_x}{m} \approx 0.8$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 0.8).



$$34. \quad A = bh = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \frac{1}{ac} \frac{1}{2} \int_0^c \left[\left(\frac{b}{c}y + a \right)^2 - \left(\frac{b}{c}y \right)^2 \right] dy$$

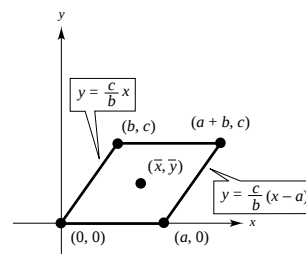
$$= \frac{1}{2ac} \int_0^c \left(\frac{2ab}{c}y + a^2 \right) dy$$

$$= \frac{1}{2ac} \left[\frac{ab}{c}y^2 + a^2y \right]_0^c = \frac{1}{2ac} [abc + a^2c] = \frac{1}{2}(b + a)$$

$$\bar{y} = \frac{1}{ac} \int_0^c y \left[\left(\frac{b}{c}y + a \right) - \left(\frac{b}{c}y \right) \right] dy = \left[\frac{1}{c} \frac{y^2}{2} \right]_0^c = \frac{c}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b+a}{2}, \frac{c}{2} \right)$$

This is the point of intersection of the diagonals.



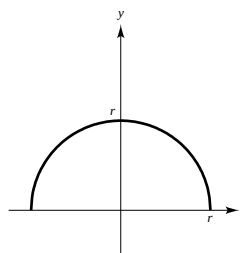
36. $\bar{x} = 0$ by symmetry

$$A = \frac{1}{2} \pi r^2$$

$$\frac{1}{A} = \frac{2}{\pi r^2}$$

$$\begin{aligned} \bar{y} &= \frac{2}{\pi r^2} \frac{1}{2} \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx \\ &= \frac{1}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \frac{1}{\pi r^2} \left[\frac{4r^3}{3} \right] = \frac{4r}{3\pi} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4r}{3\pi} \right)$$



$$38. \quad A = \int_0^1 [1 - (2x - x^2)] dx = \frac{1}{3}$$

$$\frac{1}{A} = 3$$

$$\bar{x} = 3 \int_0^1 x[1 - (2x - x^2)] dx = 3 \int_0^1 [x - 2x^2 + x^3] dx = 3 \left[\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

$$\begin{aligned} \bar{y} &= 3 \int_0^1 \frac{[1 + (2x - x^2)]}{2} [1 - (2x - x^2)] dx = \frac{3}{2} \int_0^1 [1 - (2x - x^2)^2] dx \\ &= \frac{3}{2} \int_0^1 [1 - 4x^2 + 4x^3 - x^4] dx = \frac{3}{2} \left[x - \frac{4}{3}x^3 + x^4 - \frac{x^5}{5} \right]_0^1 = \frac{7}{10} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{7}{10} \right)$$

40. (a) $M_y = 0$ by symmetry

$$M_y = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} x(b - x^{2n}) dx = 0$$

because $bx - x^{2n+1}$ is an odd function.

$$(c) \quad M_x = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{(b + x^{2n})(b - x^{2n})}{2} dx = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} \frac{1}{2} (b^2 - x^{4n}) dx$$

$$= \frac{1}{2} \left(b^2 x - \frac{x^{4n+1}}{4n+1} \right) \Big|_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}}$$

$$= b^2 b^{1/2n} - \frac{b^{(4n+1)/2n}}{4n+1} = \frac{4n}{4n+1} b^{(4n+1)/2n}$$

$$A = \int_{-2\sqrt[n]{b}}^{2\sqrt[n]{b}} (b - x^{2n}) dx = 2 \left[bx - \frac{x^{2n+1}}{2n+1} \right]_0^{2\sqrt[n]{b}}$$

$$= 2 \left[b \cdot b^{1/2n} - \frac{b^{(2n+1)/2n}}{2n+1} \right] = \frac{4n}{2n+1} b^{(2n+1)/2n}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4n b^{(4n+1)/2n} / (4n+1)}{4n b^{(2n+1)/2n} / (2n+1)} = \frac{2n+1}{4n+1} b$$

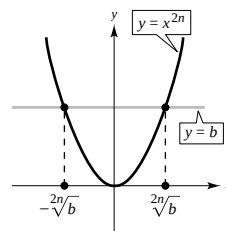
(b) $\bar{y} > \frac{b}{2}$ because there is more area above $y = \frac{b}{2}$ than below.

(d)

n	1	2	3	4
$\bar{y}$	$\frac{3}{5}b$	$\frac{5}{9}b$	$\frac{7}{13}b$	$\frac{9}{17}b$

$$(e) \quad \lim_{n \rightarrow \infty} \bar{y} = \lim_{n \rightarrow \infty} \frac{2n+1}{4n+1} b = \frac{1}{2} b$$

(f) As $n \rightarrow \infty$, the figure gets narrower.

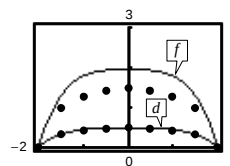


42. Let $f(x)$ be the top curve, given by $l + d$. The bottom curve is $d(x)$.

x	0	0.5	1.0	1.5	2.0
f	2.0	1.93	1.73	1.32	0
d	0.50	0.48	0.43	0.33	0

$$\begin{aligned}
 \text{(a) Area} &= 2 \int_0^2 [f(x) - d(x)] dx \\
 &\approx 2 \frac{2}{3(4)} [1.50 + 4(1.45) + 2(1.30) + 4(.99) + 0] \\
 &= \frac{1}{3} [13.86] = 4.62 \\
 M_x &= \int_{-2}^2 \frac{f(x) + d(x)}{2} (f(x) - d(x)) dx \\
 &= \int_0^2 [f(x)^2 - d(x)^2] dx \\
 &= \frac{2}{3(4)} [3.75 + 4(3.4945) + 2(2.808) + 4(1.6335) + 0] \\
 &= \frac{1}{6} [29.878] = 4.9797 \\
 \bar{y} &= \frac{M_x}{A} = \frac{4.9797}{4.62} = 1.078 \\
 (\bar{x}, \bar{y}) &= (0, 1.078)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } f(x) &= -0.1061x^4 - 0.06126x^2 + 1.9527 \\
 d(x) &= -0.02648x^4 - 0.01497x^2 + .4862 \\
 \text{(c) } \bar{y} &= \frac{M_x}{A} \approx \frac{4.9133}{4.59998} = 1.068 \\
 (\bar{x}, \bar{y}) &= (0, 1.068)
 \end{aligned}$$



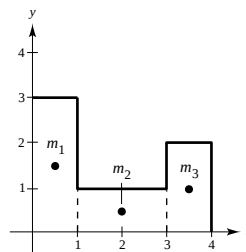
44. Centroids of the given regions: $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(\frac{7}{2}, 1\right)$

$$\text{Area: } A = 3 + 2 + 2 = 7$$

$$\bar{x} = \frac{3(1/2) + 2(2) + 2(7/2)}{7} = \frac{25/2}{7} = \frac{25}{14}$$

$$\bar{y} = \frac{3(3/2) + 2(1/2) + 2(1)}{7} = \frac{15/2}{7} = \frac{15}{14}$$

$$(\bar{x}, \bar{y}) = \left(\frac{25}{14}, \frac{15}{14}\right)$$



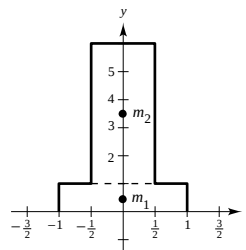
$$46. m_1 = \frac{7}{8}(2) = \frac{7}{4}, P_1 = \left(0, \frac{7}{16}\right)$$

$$m_2 = \frac{7}{8}\left(6 - \frac{7}{8}\right) = \frac{287}{64}, P_2 = \left(0, \frac{55}{16}\right)$$

By symmetry, $\bar{x} = 0$.

$$\bar{y} = \frac{(7/4)(7/16) + (287/64)(55/16)}{(7/4) + (287/64)} = \frac{16,569}{6384} = \frac{5523}{2128}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{5523}{2128}\right) \approx (0, 2.595)$$



48. Centroids of the given regions: (3, 0) and (1, 0)

$$\text{Mass: } 8 + \pi$$

$$\bar{y} = 0$$

$$\bar{x} = \frac{8(1) + \pi(3)}{8 + \pi} = \frac{8 + 3\pi}{8 + \pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8 + 3\pi}{8 + \pi}, 0 \right) \approx (1.56, 0)$$

50. $V = 2\pi rA = 2\pi(3)(4\pi) = 24\pi^2$

52. $A = \int_2^6 2\sqrt{x-2} \, dx = \frac{4}{3}(x-2)^{3/2} \Big|_2^6 = \frac{32}{3}$

$$M_y = \int_2^6 (x)2\sqrt{x-2} \, dx = 2 \int_2^6 x\sqrt{x-2} \, dx$$

$$\text{Let } u = x - 2, x = u + 2, du = dx:$$

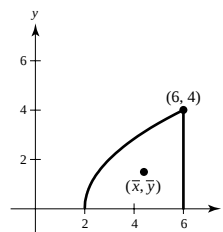
$$M_y = 2 \int_0^4 (u+2)\sqrt{u} \, du = 2 \int_0^4 (u^{3/2} + 2u^{1/2}) \, du = 2 \left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} \right]_0^4$$

$$= 2 \left[\frac{64}{5} + \frac{32}{3} \right] = \frac{704}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{704/15}{32/3} = \frac{22}{5}$$

$$r = \bar{x} = \frac{22}{5}$$

$$V = 2\pi rA = 2\pi \left(\frac{22}{5} \right) \left(\frac{32}{3} \right) = \frac{1408\pi}{15} \approx 294.89$$



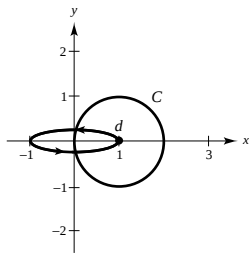
54. A planar lamina is a thin flat plate of constant density. The center of mass
- $(\bar{x}, \bar{y})$
- is the balancing point on the lamina.

56. Let
- R
- be a region in a plane and let
- L
- be a line such that
- L
- does not intersect the interior of
- R
- . If
- r
- is the distance between the centroid of
- R
- and
- L
- , then the volume
- V
- of the solid of revolution formed by revolving
- R
- about
- L
- is

$$V = 2\pi rA$$

where A is the area of R .

58. The centroid of the circle is (1, 0). The distance traveled by the centroid is
- 2π
- . The arc length of the circle is also
- 2π
- . Therefore,
- $S = (2\pi)(2\pi) = 4\pi^2$
- .



Section 6.7 Fluid Pressure and Fluid Force

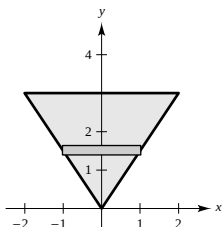
$$2. F = PA = [62.4(5)](16) = 4992 \text{ lb}$$

$$6. h(y) = 3 - y$$

$$L(y) = \frac{4}{3}y$$

$$\begin{aligned} F &= 62.4 \int_0^3 (3 - y) \left(\frac{4}{3}y \right) dy \\ &= \frac{4}{3}(62.4) \int_0^3 (3y - y^2) dy \\ &= \frac{4}{3}(62.4) \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = 374.4 \text{ lb} \end{aligned}$$

Force is one-third that of Exercise 5.

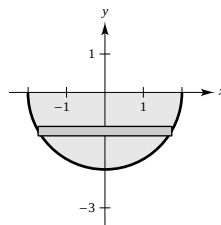


$$\begin{aligned} 4. F &= 62.4(h + 4)(48) - (62.4)(h)(48) \\ &= 62.4(4)(48) = 11,980.8 \text{ lb} \end{aligned}$$

$$8. h(y) = -y$$

$$L(y) = 2\sqrt{4 - y^2}$$

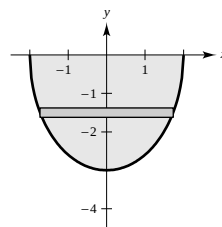
$$\begin{aligned} F &= 62.4 \int_{-2}^0 (-y)(2)\sqrt{4 - y^2} dy \\ &= \left[62.4 \left(\frac{2}{3} \right) (4 - y^2)^{3/2} \right]_{-2}^0 = 332.8 \text{ lb} \end{aligned}$$



$$10. h(y) = -y$$

$$L(y) = \frac{4}{3}\sqrt{9 - y^2}$$

$$\begin{aligned} F &= 62.4 \int_{-3}^0 (-y) \frac{4}{3} \sqrt{9 - y^2} dy \\ &= 62.4 \left(\frac{2}{3} \right) \int_{-3}^0 (9 - y^2)^{1/2} (-2y) dy \\ &= \left[62.4 \left(\frac{4}{9} \right) (9 - y^2)^{3/2} \right]_{-3}^0 = 748.8 \text{ lb} \end{aligned}$$

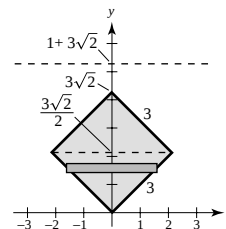


$$12. h(y) = (1 + 3\sqrt{2}) - y$$

$$L_1(y) = 2y \quad (\text{lower part})$$

$$L_2(y) = 2(3\sqrt{2} - y) \quad (\text{upper part})$$

$$\begin{aligned} F &= 2(9800) \left[\int_0^{3\sqrt{2}/2} (1 + 3\sqrt{2} - y)y dy + \int_{3\sqrt{2}/2}^{3\sqrt{2}} (1 + 3\sqrt{2} - y)(3\sqrt{2} - y) dy \right] \\ &= 19,600 \left[\left[\frac{y^2}{2} - 3\sqrt{2}y - \frac{y^3}{3} \right]_0^{3\sqrt{2}/2} + \left[3\sqrt{2}y + 18y + \frac{y^3}{3} - \frac{6\sqrt{2} + 1}{2}y \right]_{3\sqrt{2}/2}^{3\sqrt{2}} \right] \\ &= 19,600 \left[\frac{9(2\sqrt{2} + 1)}{4} + \frac{9(\sqrt{2} + 1)}{4} \right] \\ &= 44,100(3\sqrt{2} + 2) \text{ Newtons} \end{aligned}$$

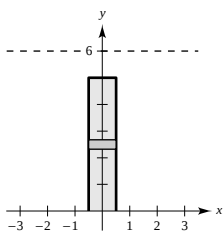


14. $h(y) = 6 - y$

$L(y) = 1$

$$F = 9800 \int_0^5 1(6 - y) dy$$

$$= 9800 \left[6y - \frac{y^2}{2} \right]_0^5 = 171,500 \text{ Newtons}$$



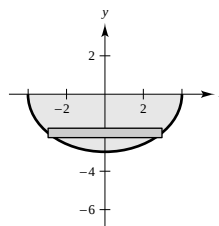
16. $h(y) = -y$

$L(y) = 2\left(\frac{4}{3}\sqrt{9 - y^2}\right)$

$$F = 140.7 \int_{-3}^0 (-y)(2)\left(\frac{4}{3}\sqrt{9 - y^2}\right) dy$$

$$= \frac{(140.7)(4)}{3} \int_{-3}^0 \sqrt{9 - y^2} (-2y) dy$$

$$= \left[\frac{(140.7)(4)}{3} \left(\frac{2}{3} \right) (9 - y^2)^{3/2} \right]_{-3}^0 = 3376.8 \text{ lb}$$



18. $h(y) = -y$

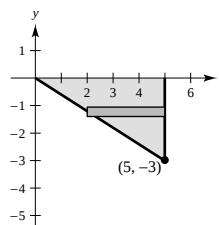
$L(y) = 5 + \frac{5}{3}y$

$$F = 140.7 \int_{-3}^0 (-y) \left(5 + \frac{5}{3}y \right) dy$$

$$= 140.7 \int_{-3}^0 \left(-5y - \frac{5}{3}y^2 \right) dy$$

$$= 140.7 \left[-\frac{5}{2}y^2 - \frac{5}{9}y^3 \right]_{-3}^0$$

$$= 140.7 \left[\frac{45}{2} - 15 \right] = 1055.25 \text{ lb}$$



20. $h(y) = \frac{3}{2} - y$

$L(y) = 2\left(\frac{1}{2}\right)\sqrt{9 - 4y^2}$

$$F = 42 \int_{-3/2}^{3/2} \left(\frac{3}{2} - y \right) \sqrt{9 - 4y^2} dy = 63 \int_{-3/2}^{3/2} \sqrt{9 - 4y^2} dy + \frac{21}{4} \int_{-3/2}^{3/2} \sqrt{9 - 4y^2} (-8y) dy$$

The second integral is zero since it is an odd function and the limits of integration are symmetric to the origin. The first integral is twice the area of a semicircle of radius $\frac{3}{2}$.

$$(\sqrt{9 - 4y^2} = 2\sqrt{(9/4) - y^2})$$

Thus, the force is $63\left(\frac{9}{4}\pi\right) = 141.75\pi \approx 445.32 \text{ lb}$.

22. (a) $F = wk\pi r^2 = (62.4)(7)(\pi 2^2) = 1747.2\pi \text{ lbs}$

(b) $F = wk\pi r^2 = (62.4)(5)(\pi 3^2) = 2808\pi \text{ lbs}$

24. (a) $F = wkhb = (62.4)\left(\frac{11}{2}\right)(3)(5) = 5148 \text{ lbs}$

(b) $F = wkhb = (62.4)\left(\frac{17}{5}\right)(5)(10) = 10,608 \text{ lbs}$

26. From Exercise 21:

$$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98 \text{ lb}$$

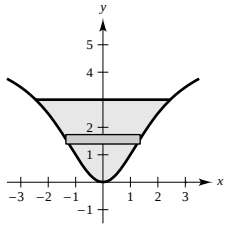
28. $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for x , you obtain

$$x = \sqrt{4y/(5 - y)}.$$

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

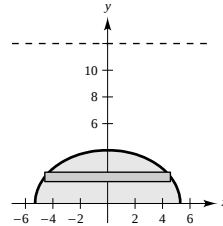
$$\begin{aligned} F &= 62.4(2) \int_0^3 (3 - y) \sqrt{\frac{4y}{5 - y}} dy \\ &= 2(124.8) \int_0^3 (3 - y) \sqrt{\frac{y}{5 - y}} dy \approx 546.265 \text{ lb} \end{aligned}$$



30. $h(y) = 12 - y$

$$L(y) = 2 \frac{\sqrt{7(16 - y^2)}}{2} = \sqrt{7(16 - y^2)}$$

$$\begin{aligned} F &= 62.4 \int_0^4 (12 - y) \sqrt{7(16 - y^2)} dy \\ &= 62.4\sqrt{7} \int_0^4 (12 - y) \sqrt{16 - y^2} dy \approx 21373.7 \text{ lb} \end{aligned}$$

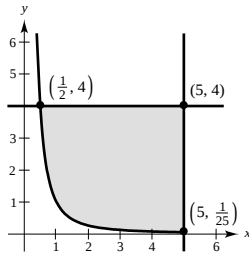


32. Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

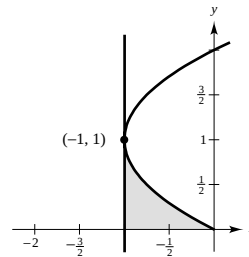
34. The left window experiences the greater fluid force because its centroid is lower.

Review Exercises for Chapter 6

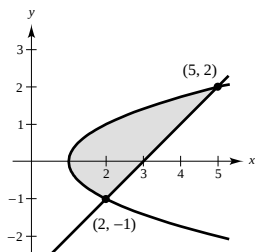
$$\begin{aligned} 2. A &= \int_{1/2}^5 \left(4 - \frac{1}{x^2}\right) dx \\ &= \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5} \end{aligned}$$



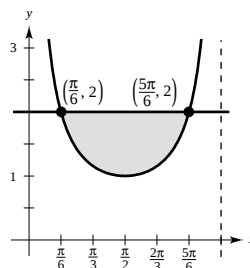
$$\begin{aligned} 4. A &= \int_0^1 [(y^2 - 2y) - (-1)] dy \\ &= \int_0^1 (y^2 - 2y + 1) dy \\ &= \int_0^1 (y - 1)^2 dy \\ &= \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$



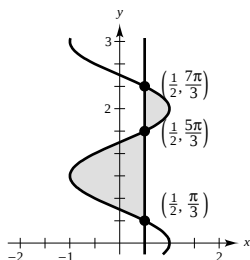
$$\begin{aligned}
 6. A &= \int_{-1}^2 [(y+3) - (y^2+1)] dy \\
 &= \int_{-1}^2 (2+y-y^2) dy \\
 &= \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-1}^2 = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 8. A &= 2 \int_{\pi/6}^{\pi/2} (2 - \csc x) dx \\
 &= 2 \left[2x - \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/2} \\
 &= 2 \left([\pi - 0] - \left[\frac{\pi}{3} - \ln(2 - \sqrt{3}) \right] \right) \\
 &= 2 \left[\frac{2\pi}{3} + \ln(2 - \sqrt{3}) \right] \approx 1.555
 \end{aligned}$$



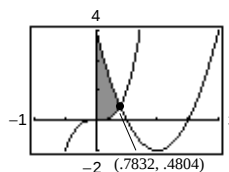
$$\begin{aligned}
 10. A &= \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y \right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2} \right) dy \\
 &= \left[\frac{y}{2} - \sin y \right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2} \right]_{5\pi/3}^{7\pi/3} \\
 &= \frac{\pi}{3} + 2\sqrt{3}
 \end{aligned}$$



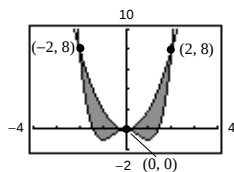
12. Point of intersection is given by:

$$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$$

$$\begin{aligned}
 A &\approx \int_0^{0.783} (3 - 4x + x^2 - x^3) dx \\
 &= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^{0.783} \\
 &\approx 1.189
 \end{aligned}$$



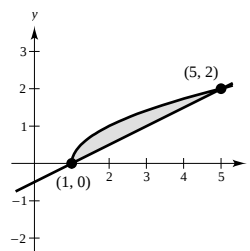
$$\begin{aligned}
 14. A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\
 &= 2 \int_0^2 (4x^2 - x^4) dx \\
 &= 2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{128}{15} \approx 8.5333
 \end{aligned}$$



$$16. y = \sqrt{x-1} \Rightarrow x = y^2 + 1$$

$$y = \frac{x-1}{2} \Rightarrow x = 2y + 1$$

$$\begin{aligned}
 A &= \int_0^2 [(2y+1) - (y^2+1)] dy \\
 &= \int_1^5 \left[\sqrt{x-1} - \frac{x-1}{2} \right] dx \\
 &= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{4}(x-1)^2 \right]_1^5 = \frac{4}{3}
 \end{aligned}$$

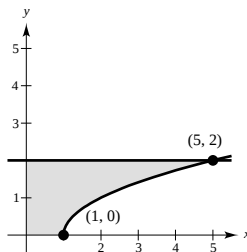


$$18. A = \int_0^1 2 \, dx + \int_1^5 [2 - \sqrt{x-1}] \, dx$$

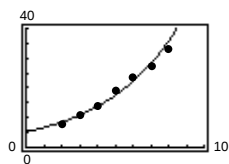
$$x = y^2 + 1$$

$$A = \int_0^2 (y^2 + 1) \, dy$$

$$= \left[\frac{1}{3}y^3 + y \right]_0^2 = \frac{14}{3}$$



$$20. (a) R_1(t) = 5.2834(1.2701)^t = 5.2834 e^{0.2391t}$$

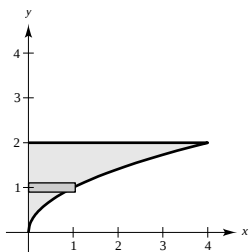


$$(b) R_2(t) = 10 + 5.28 e^{0.2t}$$

$$\text{Difference} = \int_{10}^{15} [R_1(t) - R_2(t)] \, dt \approx 171.25 \text{ billion dollars}$$

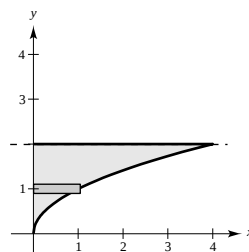
22. (a) Shell

$$V = 2\pi \int_0^2 y^3 \, dy = \left[\frac{\pi}{2} y^4 \right]_0^2 = 8\pi$$



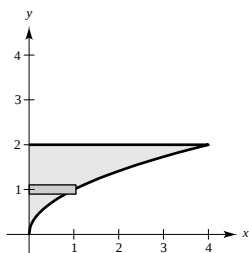
(b) Shell

$$\begin{aligned} V &= 2\pi \int_0^2 (2-y)y^2 \, dy \\ &= 2\pi \int_0^2 (2y^2 - y^3) \, dy \\ &= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 = \frac{8\pi}{3} \end{aligned}$$



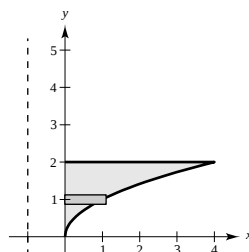
(c) Disk

$$V = \pi \int_0^2 y^4 \, dy = \left[\frac{\pi}{5} y^5 \right]_0^2 = \frac{32\pi}{5}$$



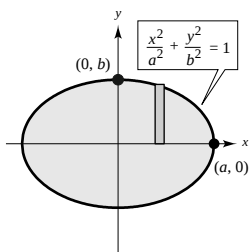
(d) Disk

$$\begin{aligned} V &= \pi \int_0^2 [(y^2 + 1)^2 - 1^2] \, dy \\ &= \pi \int_0^2 (y^4 + 2y^2) \, dy \\ &= \pi \left[\frac{1}{5}y^5 + \frac{2}{3}y^3 \right]_0^2 = \frac{176\pi}{15} \end{aligned}$$



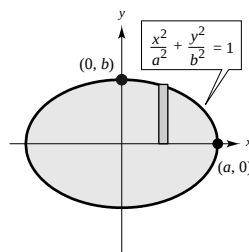
24. (a) Shell

$$\begin{aligned}
 V &= 4\pi \int_0^a (x) \frac{b}{a} \sqrt{a^2 - x^2} dx \\
 &= \frac{-2\pi b}{a} \int_0^a (a^2 - x^2)^{1/2} (-2x) dx \\
 &= \left[\frac{-4\pi b}{3a} (a^2 - x^2)^{3/2} \right]_0^a = \frac{4}{3} \pi a^2 b
 \end{aligned}$$



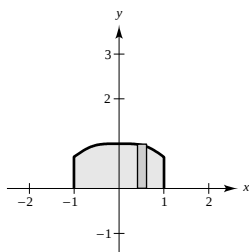
(b) Disk

$$\begin{aligned}
 V &= 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx \\
 &= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
 &= \frac{4}{3} \pi a b^2
 \end{aligned}$$



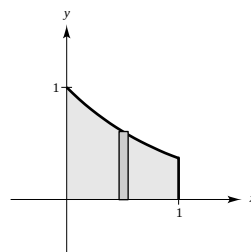
26. Disk

$$\begin{aligned}
 V &= 2\pi \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \right]^2 dx \\
 &= \left[2\pi \arctan x \right]_0^1 \\
 &= 2\pi \left(\frac{\pi}{4} - 0 \right) \\
 &= \frac{\pi^2}{2}
 \end{aligned}$$



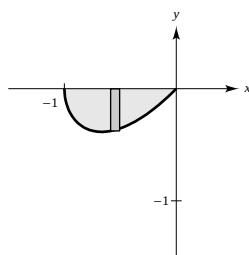
28. Disk

$$\begin{aligned}
 V &= \pi \int_0^1 (e^{-x})^2 dx \\
 &= \pi \int_0^1 e^{-2x} dx = \left[-\frac{\pi}{2} e^{-2x} \right]_0^1 \\
 &= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \left(1 - \frac{1}{e^2} \right)
 \end{aligned}$$



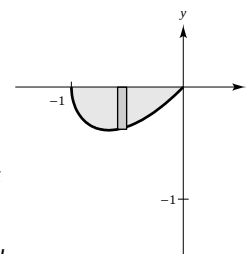
30. (a) Disk

$$\begin{aligned}
 V &= \pi \int_{-1}^0 x^2(x+1) dx \\
 &= \pi \int_{-1}^0 (x^3 + x^2) dx \\
 &= \pi \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 = \frac{\pi}{12}
 \end{aligned}$$



(b) Shell

$$\begin{aligned}
 u &= \sqrt{x+1} \\
 x &= u^2 - 1 \\
 dx &= 2u du \\
 V &= 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx \\
 &= 4\pi \int_0^1 (u^2 - 1)^2 u^2 du \\
 &= 4\pi \int_0^1 (u^6 - 2u^4 + u^2) du \\
 &= 4\pi \left[\frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 \right]_0^1 = \frac{32\pi}{105}
 \end{aligned}$$



$$32. A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{3}\sqrt{a^2 - x^2})$$

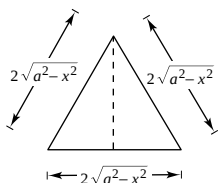
$$= \sqrt{3}(a^2 - x^2)$$

$$V = \sqrt{3} \int_{-a}^a (a^2 - x^2) dx = \sqrt{3} \left[a^2x - \frac{x^3}{3} \right]_{-a}^a$$

$$= \sqrt{3} \left(\frac{4a^3}{3} \right)$$

Since $(4\sqrt{3}a^3)/3 = 10$, we have $a^3 = (5\sqrt{3})/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answers (b).

40. $F = kx$

$$50 = k(9) \Rightarrow k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x dx = \left[\frac{25}{9}x^2 \right]_0^9$$

$$= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

34. $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x} \right]_1^3 = \frac{14}{3}$$

38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^3 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx = 4\pi \int_0^3 \sqrt{x+1} dx$$

$$= 4\pi \left[\left(\frac{2}{3} \right) (x+1)^{3/2} \right]_0^3 = \frac{56\pi}{3}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3/\text{min}$$

$$V = \pi r^2 h = \pi \left(\frac{1}{9} \right) h$$

$$\frac{dV}{dt} = \frac{\pi}{9} \left(\frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = \frac{9}{\pi} \left(\frac{dV}{dt} \right) = \frac{9}{\pi} \left(-\frac{8}{7.481} \right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49 \text{ minutes}$

$(49)(12) = 588 \text{ gallons pumped}$

Volume of water pumped in Exercise 41: 391.7 gallons

$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi \text{ ft} \cdot \text{ton}$

44. (a) Weight of section of cable:
- $4 \Delta x$

Distance: $200 - x$

$$W = 4 \int_0^{200} (200 - x) dx = \left[-2(200 - x)^2 \right]_0^{200} = 80,000 \text{ ft} \cdot \text{lb} = 40 \text{ ft} \cdot \text{ton}$$

- (b) Work to move 300 pounds 200 feet vertically:
- $200(300) = 60,000 \text{ ft} \cdot \text{lb} = 30 \text{ ft} \cdot \text{ton}$

Total work = work for drawing up the cable + work of lifting the load

$$= 40 \text{ ft} \cdot \text{ton} + 30 \text{ ft} \cdot \text{ton} = 70 \text{ ft} \cdot \text{ton}$$

46. $W = \int_a^b F(x) dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \leq x \leq 9 \\ -(4/3)x + 16, & 9 \leq x \leq 12 \end{cases}$$

$$\begin{aligned} W &= \int_0^9 \left(-\frac{2}{9}x + 6 \right) dx + \int_9^{12} \left(-\frac{4}{3}x + 16 \right) dx \\ &= \left[-\frac{1}{9}x^2 + 6x \right]_0^9 + \left[-\frac{2}{3}x^2 + 16x \right]_9^{12} \\ &= (-9 + 54) + (-96 + 192 + 54 - 144) = 51 \text{ ft} \cdot \text{lbs} \end{aligned}$$

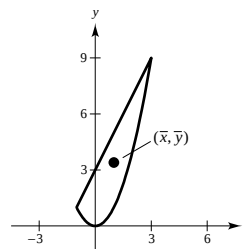
48. $A = \int_{-1}^3 [(2x + 3) - x^2] dx = \left[x^2 + 3x - \frac{1}{3}x^3 \right]_{-1}^3 = \frac{32}{3}$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32} \int_{-1}^3 x(2x + 3 - x^2) dx = \frac{3}{32} \int_{-1}^3 (3x + 2x^2 - x^3) dx = \frac{3}{32} \left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^3 = 1$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{32} \right) \frac{1}{2} \int_{-1}^3 [(2x + 3)^2 - x^4] dx = \frac{3}{64} \int_{-1}^3 (9 + 12x + 4x^2 - x^4) dx \\ &= \frac{3}{64} \left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^3 = \frac{17}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5} \right)$$



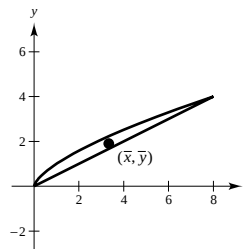
50. $A = \int_0^8 \left(x^{2/3} - \frac{1}{2}x \right) dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2 \right]_0^8 = \frac{16}{5}$

$$\frac{1}{A} = \frac{5}{16}$$

$$\begin{aligned} \bar{x} &= \frac{5}{16} \int_0^8 x \left(x^{2/3} - \frac{1}{2}x \right) dx \\ &= \frac{5}{16} \left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3 \right]_0^8 = \frac{10}{3} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \left(\frac{5}{16} \right) \frac{1}{2} \int_0^8 \left(x^{4/3} - \frac{1}{4}x^2 \right) dx \\ &= \frac{1}{2} \left(\frac{5}{16} \right) \left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3 \right]_0^8 = \frac{40}{21} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21} \right)$$



26. From Exercise 21:

$$F = 64(15)\pi\left(\frac{1}{2}\right)^2 \approx 753.98 \text{ lb}$$

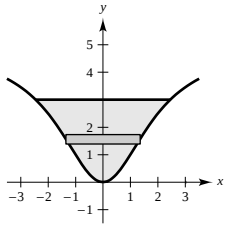
28. $h(y) = 3 - y$

Solving $y = 5x^2/(x^2 + 4)$ for x , you obtain

$$x = \sqrt{4y/(5 - y)}.$$

$$L(y) = 2\sqrt{\frac{4y}{5 - y}}$$

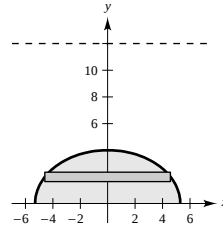
$$\begin{aligned} F &= 62.4(2) \int_0^3 (3 - y) \sqrt{\frac{4y}{5 - y}} dy \\ &= 2(124.8) \int_0^3 (3 - y) \sqrt{\frac{y}{5 - y}} dy \approx 546.265 \text{ lb} \end{aligned}$$



30. $h(y) = 12 - y$

$$L(y) = 2 \frac{\sqrt{7(16 - y^2)}}{2} = \sqrt{7(16 - y^2)}$$

$$\begin{aligned} F &= 62.4 \int_0^4 (12 - y) \sqrt{7(16 - y^2)} dy \\ &= 62.4\sqrt{7} \int_0^4 (12 - y) \sqrt{16 - y^2} dy \approx 21373.7 \text{ lb} \end{aligned}$$

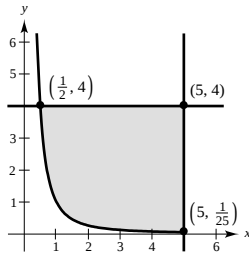


32. Fluid pressure is the force per unit of area exerted by a fluid over the surface of a body.

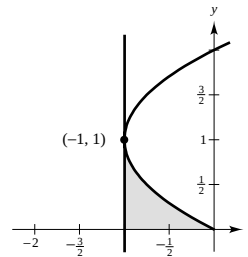
34. The left window experiences the greater fluid force because its centroid is lower.

Review Exercises for Chapter 6

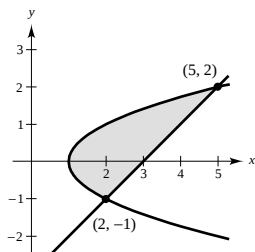
$$\begin{aligned} 2. A &= \int_{1/2}^5 \left(4 - \frac{1}{x^2}\right) dx \\ &= \left[4x + \frac{1}{x}\right]_{1/2}^5 = \frac{81}{5} \end{aligned}$$



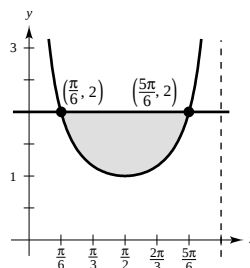
$$\begin{aligned} 4. A &= \int_0^1 [(y^2 - 2y) - (-1)] dy \\ &= \int_0^1 (y^2 - 2y + 1) dy \\ &= \int_0^1 (y - 1)^2 dy \\ &= \left[\frac{(y - 1)^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$



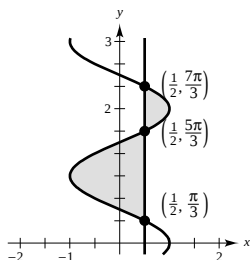
$$\begin{aligned}
 6. A &= \int_{-1}^2 [(y+3) - (y^2+1)] dy \\
 &= \int_{-1}^2 (2+y-y^2) dy \\
 &= \left[2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{-1}^2 = \frac{9}{2}
 \end{aligned}$$



$$\begin{aligned}
 8. A &= 2 \int_{\pi/6}^{\pi/2} (2 - \csc x) dx \\
 &= 2 \left[2x - \ln |\csc x - \cot x| \right]_{\pi/6}^{\pi/2} \\
 &= 2 \left([\pi - 0] - \left[\frac{\pi}{3} - \ln(2 - \sqrt{3}) \right] \right) \\
 &= 2 \left[\frac{2\pi}{3} + \ln(2 - \sqrt{3}) \right] \approx 1.555
 \end{aligned}$$



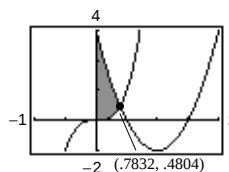
$$\begin{aligned}
 10. A &= \int_{\pi/3}^{5\pi/3} \left(\frac{1}{2} - \cos y \right) dy + \int_{5\pi/3}^{7\pi/3} \left(\cos y - \frac{1}{2} \right) dy \\
 &= \left[\frac{y}{2} - \sin y \right]_{\pi/3}^{5\pi/3} + \left[\sin y - \frac{y}{2} \right]_{5\pi/3}^{7\pi/3} \\
 &= \frac{\pi}{3} + 2\sqrt{3}
 \end{aligned}$$



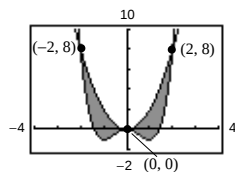
12. Point of intersection is given by:

$$x^3 - x^2 + 4x - 3 = 0 \Rightarrow x \approx 0.783.$$

$$\begin{aligned}
 A &\approx \int_0^{0.783} (3 - 4x + x^2 - x^3) dx \\
 &= \left[3x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^{0.783} \\
 &\approx 1.189
 \end{aligned}$$



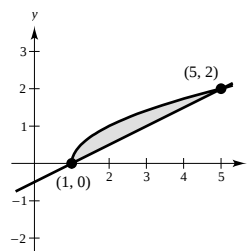
$$\begin{aligned}
 14. A &= 2 \int_0^2 [2x^2 - (x^4 - 2x^2)] dx \\
 &= 2 \int_0^2 (4x^2 - x^4) dx \\
 &= 2 \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2 = \frac{128}{15} \approx 8.5333
 \end{aligned}$$



$$16. y = \sqrt{x-1} \Rightarrow x = y^2 + 1$$

$$y = \frac{x-1}{2} \Rightarrow x = 2y + 1$$

$$\begin{aligned}
 A &= \int_0^2 [(2y+1) - (y^2+1)] dy \\
 &= \int_1^5 \left[\sqrt{x-1} - \frac{x-1}{2} \right] dx \\
 &= \left[\frac{2}{3}(x-1)^{3/2} - \frac{1}{4}(x-1)^2 \right]_1^5 = \frac{4}{3}
 \end{aligned}$$

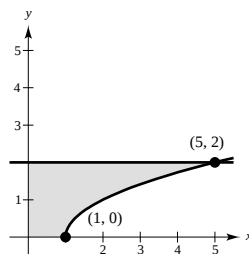


$$18. A = \int_0^1 2 \, dx + \int_1^5 [2 - \sqrt{x-1}] \, dx$$

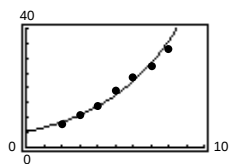
$$x = y^2 + 1$$

$$A = \int_0^2 (y^2 + 1) \, dy$$

$$= \left[\frac{1}{3}y^3 + y \right]_0^2 = \frac{14}{3}$$



$$20. (a) R_1(t) = 5.2834(1.2701)^t = 5.2834 e^{0.2391t}$$

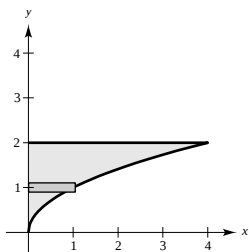


$$(b) R_2(t) = 10 + 5.28 e^{0.2t}$$

$$\text{Difference} = \int_{10}^{15} [R_1(t) - R_2(t)] \, dt \approx 171.25 \text{ billion dollars}$$

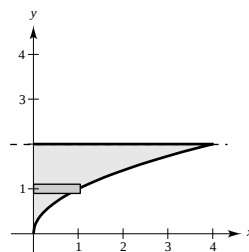
22. (a) Shell

$$V = 2\pi \int_0^2 y^3 \, dy = \left[\frac{\pi}{2} y^4 \right]_0^2 = 8\pi$$



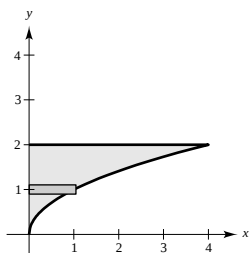
(b) Shell

$$\begin{aligned} V &= 2\pi \int_0^2 (2-y)y^2 \, dy \\ &= 2\pi \int_0^2 (2y^2 - y^3) \, dy \\ &= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 = \frac{8\pi}{3} \end{aligned}$$



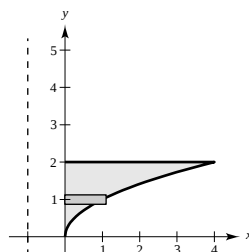
(c) Disk

$$V = \pi \int_0^2 y^4 \, dy = \left[\frac{\pi}{5} y^5 \right]_0^2 = \frac{32\pi}{5}$$



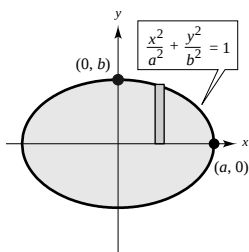
(d) Disk

$$\begin{aligned} V &= \pi \int_0^2 [(y^2 + 1)^2 - 1^2] \, dy \\ &= \pi \int_0^2 (y^4 + 2y^2) \, dy \\ &= \pi \left[\frac{1}{5}y^5 + \frac{2}{3}y^3 \right]_0^2 = \frac{176\pi}{15} \end{aligned}$$



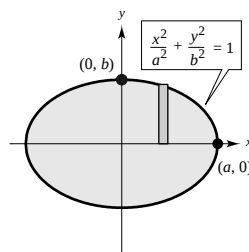
24. (a) Shell

$$\begin{aligned}
 V &= 4\pi \int_0^a (x) \frac{b}{a} \sqrt{a^2 - x^2} dx \\
 &= \frac{-2\pi b}{a} \int_0^a (a^2 - x^2)^{1/2} (-2x) dx \\
 &= \left[\frac{-4\pi b}{3a} (a^2 - x^2)^{3/2} \right]_0^a = \frac{4}{3} \pi a^2 b
 \end{aligned}$$



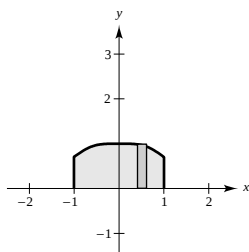
(b) Disk

$$\begin{aligned}
 V &= 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx \\
 &= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a \\
 &= \frac{4}{3} \pi a b^2
 \end{aligned}$$



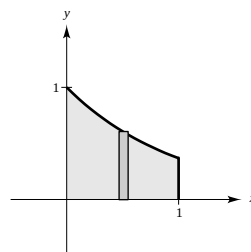
26. Disk

$$\begin{aligned}
 V &= 2\pi \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \right]^2 dx \\
 &= \left[2\pi \arctan x \right]_0^1 \\
 &= 2\pi \left(\frac{\pi}{4} - 0 \right) \\
 &= \frac{\pi^2}{2}
 \end{aligned}$$



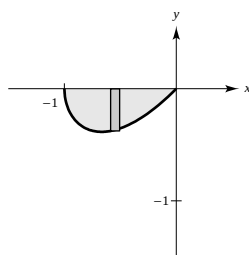
28. Disk

$$\begin{aligned}
 V &= \pi \int_0^1 (e^{-x})^2 dx \\
 &= \pi \int_0^1 e^{-2x} dx = \left[-\frac{\pi}{2} e^{-2x} \right]_0^1 \\
 &= \left(\frac{-\pi}{2e^2} + \frac{\pi}{2} \right) = \frac{\pi}{2} \left(1 - \frac{1}{e^2} \right)
 \end{aligned}$$



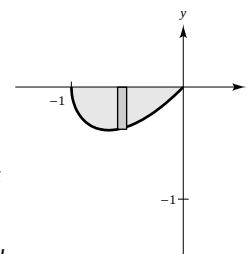
30. (a) Disk

$$\begin{aligned}
 V &= \pi \int_{-1}^0 x^2(x+1) dx \\
 &= \pi \int_{-1}^0 (x^3 + x^2) dx \\
 &= \pi \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_{-1}^0 = \frac{\pi}{12}
 \end{aligned}$$



(b) Shell

$$\begin{aligned}
 u &= \sqrt{x+1} \\
 x &= u^2 - 1 \\
 dx &= 2u du \\
 V &= 2\pi \int_{-1}^0 x^2 \sqrt{x+1} dx \\
 &= 4\pi \int_0^1 (u^2 - 1)^2 u^2 du \\
 &= 4\pi \int_0^1 (u^6 - 2u^4 + u^2) du \\
 &= 4\pi \left[\frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 \right]_0^1 = \frac{32\pi}{105}
 \end{aligned}$$



$$32. A(x) = \frac{1}{2}bh = \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{3}\sqrt{a^2 - x^2})$$

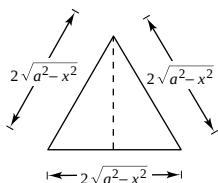
$$= \sqrt{3}(a^2 - x^2)$$

$$V = \sqrt{3} \int_{-a}^a (a^2 - x^2) dx = \sqrt{3} \left[a^2x - \frac{x^3}{3} \right]_{-a}^a$$

$$= \sqrt{3} \left(\frac{4a^3}{3} \right)$$

Since $(4\sqrt{3}a^3)/3 = 10$, we have $a^3 = (5\sqrt{3})/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This length is a little over 1 unit. Answers (b).

40. $F = kx$

$$50 = k(9) \Rightarrow k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x dx = \left[\frac{25}{9}x^2 \right]_0^9$$

$$= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

34. $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x} \right]_1^3 = \frac{14}{3}$$

38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^3 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx = 4\pi \int_0^3 \sqrt{x+1} dx$$

$$= 4\pi \left[\left(\frac{2}{3} \right) (x+1)^{3/2} \right]_0^3 = \frac{56\pi}{3}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3/\text{min}$$

$$V = \pi r^2 h = \pi \left(\frac{1}{9} \right) h$$

$$\frac{dV}{dt} = \frac{\pi}{9} \left(\frac{dh}{dt} \right)$$

$$\frac{dh}{dt} = \frac{9}{\pi} \left(\frac{dV}{dt} \right) = \frac{9}{\pi} \left(-\frac{8}{7.481} \right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49 \text{ minutes}$

$(49)(12) = 588 \text{ gallons pumped}$

Volume of water pumped in Exercise 41: 391.7 gallons

$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi \text{ ft} \cdot \text{ton}$

44. (a) Weight of section of cable:
- $4 \Delta x$

Distance: $200 - x$

$$W = 4 \int_0^{200} (200 - x) dx = \left[-2(200 - x)^2 \right]_0^{200} = 80,000 \text{ ft} \cdot \text{lb} = 40 \text{ ft} \cdot \text{ton}$$

- (b) Work to move 300 pounds 200 feet vertically:
- $200(300) = 60,000 \text{ ft} \cdot \text{lb} = 30 \text{ ft} \cdot \text{ton}$

Total work = work for drawing up the cable + work of lifting the load

$$= 40 \text{ ft} \cdot \text{ton} + 30 \text{ ft} \cdot \text{ton} = 70 \text{ ft} \cdot \text{ton}$$

46. $W = \int_a^b F(x) dx$

$$F(x) = \begin{cases} -(2/9)x + 6, & 0 \leq x \leq 9 \\ -(4/3)x + 16, & 9 \leq x \leq 12 \end{cases}$$

$$\begin{aligned} W &= \int_0^9 \left(-\frac{2}{9}x + 6 \right) dx + \int_9^{12} \left(-\frac{4}{3}x + 16 \right) dx \\ &= \left[-\frac{1}{9}x^2 + 6x \right]_0^9 + \left[-\frac{2}{3}x^2 + 16x \right]_9^{12} \\ &= (-9 + 54) + (-96 + 192 + 54 - 144) = 51 \text{ ft} \cdot \text{lbs} \end{aligned}$$

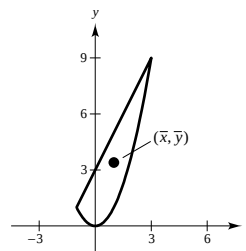
48. $A = \int_{-1}^3 [(2x + 3) - x^2] dx = \left[x^2 + 3x - \frac{1}{3}x^3 \right]_{-1}^3 = \frac{32}{3}$

$$\frac{1}{A} = \frac{3}{32}$$

$$\bar{x} = \frac{3}{32} \int_{-1}^3 x(2x + 3 - x^2) dx = \frac{3}{32} \int_{-1}^3 (3x + 2x^2 - x^3) dx = \frac{3}{32} \left[\frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^3 = 1$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{32} \right) \frac{1}{2} \int_{-1}^3 [(2x + 3)^2 - x^4] dx = \frac{3}{64} \int_{-1}^3 (9 + 12x + 4x^2 - x^4) dx \\ &= \frac{3}{64} \left[9x + 6x^2 + \frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^3 = \frac{17}{5} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(1, \frac{17}{5} \right)$$



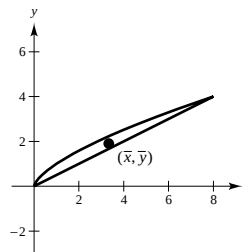
50. $A = \int_0^8 \left(x^{2/3} - \frac{1}{2}x \right) dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{4}x^2 \right]_0^8 = \frac{16}{5}$

$$\frac{1}{A} = \frac{5}{16}$$

$$\begin{aligned} \bar{x} &= \frac{5}{16} \int_0^8 x \left(x^{2/3} - \frac{1}{2}x \right) dx \\ &= \frac{5}{16} \left[\frac{3}{8}x^{8/3} - \frac{1}{6}x^3 \right]_0^8 = \frac{10}{3} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \left(\frac{5}{16} \right) \frac{1}{2} \int_0^8 \left(x^{4/3} - \frac{1}{4}x^2 \right) dx \\ &= \frac{1}{2} \left(\frac{5}{16} \right) \left[\frac{3}{7}x^{7/3} - \frac{1}{12}x^3 \right]_0^8 = \frac{40}{21} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{10}{3}, \frac{40}{21} \right)$$



52. Wall at shallow end:

$$F = 62.4 \int_0^5 y(20) dy = \left[(1248) \frac{y^2}{2} \right]_0^5 = 15,600 \text{ lb}$$

Wall at deep end:

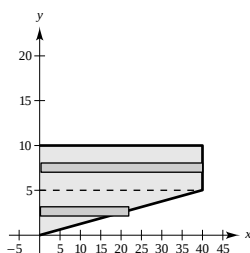
$$F = 62.4 \int_0^{10} y(20) dy = \left[(624) y^2 \right]_0^{10} = 62,400 \text{ lb}$$

Side wall:

$$F_1 = 62.4 \int_0^5 y(40) dy = \left[(1248) y^2 \right]_0^5 = 31,200 \text{ lb}$$

$$F_2 = 62.4 \int_0^5 (10 - y)8y dy = 62.4 \int_0^5 (80y - 8y^2) dy$$

$$F = F_1 + F_2 = 72,800 \text{ lb}$$



54. $F = 62.4(16\pi)5 = 4992\pi \text{ lb}$

Problem Solving for Chapter 6

2. $R = \int_0^1 x(1 - x) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Let (c, mc) be the intersection of the line and the parabola.

Then, $mc = c(1 - c) \Rightarrow m = 1 - c$ or $c = 1 - m$.

$$\frac{1}{2} \left(\frac{1}{6} \right) = \int_0^{1-m} (x - x^2 - mx) dx$$

$$\begin{aligned} \frac{1}{12} &= \left[\frac{x^2}{2} - \frac{x^3}{3} - m \frac{x^2}{2} \right]_0^{1-m} \\ &= \frac{(1-m)^2}{2} - \frac{(1-m)^3}{3} - m \frac{(1-m)^2}{2} \end{aligned}$$

$$1 = 6(1-m)^2 - 4(1-m)^3 - 6m(1-m)^2$$

$$= (1-m)^2(6 - 4(1-m) - 6m)$$

$$= (1-m)^2(2 - 2m)$$

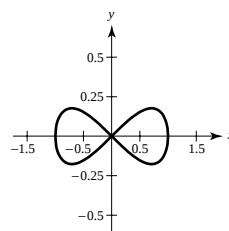
$$\frac{1}{2} = (1-m)^3$$

$$\left(\frac{1}{2} \right)^{1/3} = 1 - m$$

$$m = 1 - \left(\frac{1}{2} \right)^{1/3} \approx 0.2063$$

4. $8y^2 = x^2(1 - x^2)$

$$y = \pm \frac{|x|\sqrt{1-x^2}}{2\sqrt{2}}$$



For $x > 0$, $y' = \frac{1 - 2x^2}{2\sqrt{2}\sqrt{1-x^2}}$

$$\begin{aligned} S &= 2(2\pi) \int_0^1 x \sqrt{1 + \left(\frac{1 - 2x^2}{2\sqrt{2}\sqrt{1-x^2}} \right)^2} dx \\ &= \frac{5\sqrt{2}\pi}{3} \end{aligned}$$

6. By the Theorem of Pappus,

$$\begin{aligned}
 V &= 2\pi r A \\
 &= 2\pi \left[d + \frac{1}{2}\sqrt{w^2 + l^2} \right] lw
 \end{aligned}$$

- 8.
- $f'(x)^2 = e^x$

$$f'(x) = e^{x/2}$$

$$f(x) = 2e^{x/2} + C$$

$$f(0) = 0 \Rightarrow C = -2$$

$$f(x) = 2e^{x/2} - 2$$

10. Let
- ρ_f
- be the density of the fluid and
- ρ_0
- the density of the iceberg. The buoyant force is

$$F = \rho_f g \int_{-h}^0 A(y) dy$$

where $A(y)$ is a typical cross section and g is the acceleration due to gravity. The weight of the object is

$$W = \rho_0 g \int_{-h}^{L-h} A(y) dy.$$

$$F = W$$

$$\rho_f g \int_{-h}^0 A(y) dy = \rho_0 g \int_{-h}^{L-h} A(y) dy$$

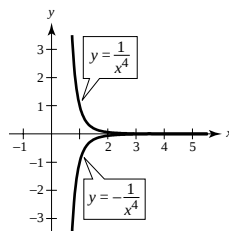
$$\frac{\rho_0}{\rho_f} = \frac{\text{submerged volume}}{\text{total volume}} = \frac{0.92 \times 10^3}{1.03 \times 10^3} = 0.893 \text{ or } 89.3\%$$

12. (a)
- $\bar{y} = 0$
- by symmetry

$$M_y = 2 \int_1^6 x \frac{1}{x^4} dx = 2 \int_1^6 \frac{1}{x^3} dx = \frac{35}{36}$$

$$m = 2 \int_1^6 \frac{1}{x^4} dx = \frac{215}{324}$$

$$\bar{x} = \frac{35/36}{215/324} = \frac{63}{43} \quad (\bar{x}, \bar{y}) = \left(\frac{63}{43}, 0 \right)$$



$$(b) M_y = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$$

$$m = 2 \int_1^b \frac{1}{x^4} dx = \frac{2(b^3 - 1)}{3b^3}$$

$$\bar{x} = \frac{(b^2 - 1)/b^2}{2(b^3 - 1)/3b^3} = \frac{3b(b + 1)}{2(b^2 + b + 1)} \quad (\bar{x}, \bar{y}) = \left(\frac{3b(b + 1)}{2(b^2 + b + 1)}, 0 \right)$$

$$\lim_{b \rightarrow \infty} \bar{x} = \frac{3}{2} \quad (\bar{x}, \bar{y}) = \left(\frac{3}{2}, 0 \right)$$

14. (a) Trapezoidal: Area
- $\approx \frac{160}{2(8)}[0 + 2(50) + 2(54) + 2(82) + 2(82) + 2(73) + 2(75) + 2(80) + 0] = 9920$
- sq ft

$$(b) \text{ Simpson's: Area } \approx \frac{160}{3(8)}[0 + 4(50) + 2(54) + 4(82) + 2(82) + 4(73) + 2(75) + 4(80) + 0] = 10,413\frac{1}{3} \text{ sq ft}$$

16. Point of equilibrium:
- $1000 - 0.4x^2 = 42x$

$$x = 20, p = 840$$

$$(P_0, x_0) = (840, 20)$$

$$\text{Consumer surplus} = \int_0^{20} [(1000 - 0.4x^2) - 840] dx = 2133.33$$

$$\text{Producer surplus} = \int_0^{20} [840 - 42x] dx = 8400$$

PART I

CHAPTER 6

Applications of Integration

Section 6.1	Area of a Region Between Two Curves	2
Section 6.2	Volume: The Disk Method	9
Section 6.3	Volume: The Shell Method	17
Section 6.4	Arc Length and Surfaces of Revolution	22
Section 6.5	Work	27
Section 6.6	Moments, Centers of Mass, and Centroids	30
Section 6.7	Fluid Pressure and Fluid Force	37
Review Exercises		40
Problem Solving		46

CHAPTER 6

Applications of Integration

Section 6.1 Area of a Region Between Two Curves

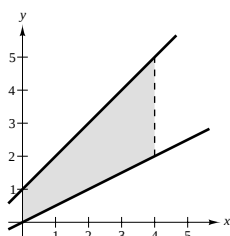
Solutions to Odd-Numbered Exercises

$$1. A = \int_0^6 [0 - (x^2 - 6x)] dx = - \int_0^6 (x^2 - 6x) dx$$

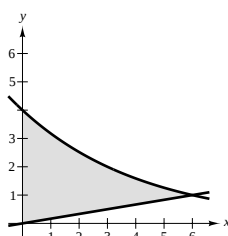
$$3. A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)] dx = \int_0^3 (-2x^2 + 6x) dx$$

$$5. A = 2 \int_{-1}^0 3(x^3 - x) dx = 6 \int_{-1}^0 (x^3 - x) dx \quad \text{or} \quad -6 \int_0^1 (x^3 - x) dx$$

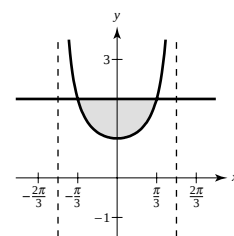
$$7. \int_0^4 \left[(x + 1) - \frac{x}{2} \right] dx$$



$$9. \int_0^6 \left[4(2^{-x/3}) - \frac{x}{6} \right] dx$$



$$11. \int_{-\pi/3}^{\pi/3} [2 - \sec x] dx$$

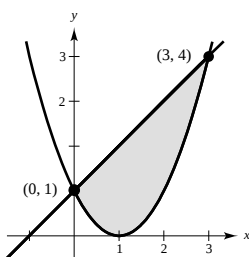


$$13. f(x) = x + 1$$

$$g(x) = (x - 1)^2$$

$$A \approx 4$$

Matches (d)

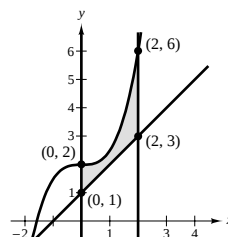


$$15. A = \int_0^2 \left[\left(\frac{1}{2}x^3 + 2 \right) - (x + 1) \right] dx$$

$$= \int_0^2 \left(\frac{1}{2}x^3 - x + 1 \right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^2}{2} + x \right]_0^2$$

$$= \left(\frac{16}{8} - \frac{4}{2} + 2 \right) - 0 = 2$$

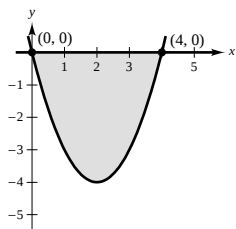


17. The points of intersection are given by:

$$x^2 - 4x = 0$$

$$x(x - 4) = 0 \quad \text{when } x = 0, 4$$

$$\begin{aligned} A &= \int_0^4 [g(x) - f(x)] dx \\ &= -\int_0^4 (x^2 - 4x) dx \\ &= -\left[\frac{x^3}{3} - 2x^2\right]_0^4 \\ &= \frac{32}{3} \end{aligned}$$

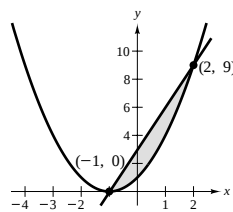


19. The points of intersection are given by:

$$x^2 + 2x + 1 = 3x + 3$$

$$(x - 2)(x + 1) = 0 \quad \text{when } x = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(x) - f(x)] dx \\ &= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)] dx \\ &= \int_{-1}^2 (2 + x - x^2) dx \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$



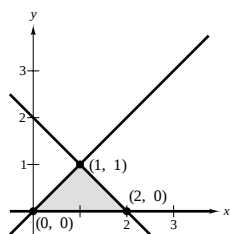
21. The points of intersection are given by:

$$x = 2 - x \quad \text{and} \quad x = 0 \quad \text{and} \quad 2 - x = 0$$

$$x = 1 \quad x = 0 \quad x = 2$$

$$A = \int_0^1 [(2 - y) - (y)] dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if we integrate with respect to x , we need two integrals. Also, note that the region is a triangle.

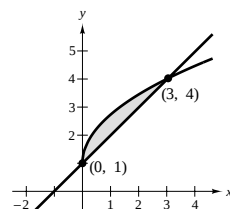


23. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

$$\sqrt{3x} = x \quad \text{when } x = 0, 3$$

$$\begin{aligned} A &= \int_0^3 [f(x) - g(x)] dx \\ &= \int_0^3 [(\sqrt{3x} + 1) - (x + 1)] dx \\ &= \int_0^3 [(3x)^{1/2} - x] dx \\ &= \left[\frac{2}{9} (3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2} \end{aligned}$$

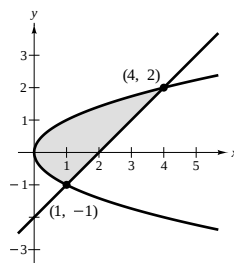


25. The points of intersection are given by:

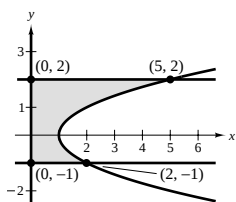
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(y) - f(y)] dy \\ &= \int_{-1}^2 [(y + 2) - y^2] dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$

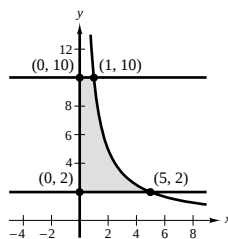


$$\begin{aligned}
 27. A &= \int_{-1}^2 [f(y) - g(y)] dy \\
 &= \int_{-1}^2 [(y^2 + 1) - 0] dy \\
 &= \left[\frac{y^3}{3} + y \right]_{-1}^2 = 6
 \end{aligned}$$



$$29. y = \frac{10}{x} \Rightarrow x = \frac{10}{y}$$

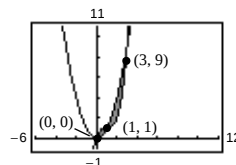
$$\begin{aligned}
 A &= \int_2^{10} \frac{10}{y} dy \\
 &= \left[10 \ln y \right]_2^{10} \\
 &= 10(\ln 10 - \ln 2) \\
 &= 10 \ln 5 \approx 16.0944
 \end{aligned}$$



31. The points of intersection are given by:

$$\begin{aligned}
 x^3 - 3x^2 + 3x &= x^2 \\
 x(x - 1)(x - 3) &= 0 \quad \text{when } x = 0, 1, 3 \\
 A &= \int_0^1 [f(x) - g(x)] dx + \int_1^3 [g(x) - f(x)] dx \\
 &= \int_0^1 [(x^3 - 3x^2 + 3x) - x^2] dx + \int_1^3 [x^2 - (x^3 - 3x^2 + 3x)] dx \\
 &= \int_0^1 (x^3 - 4x^2 + 3x) dx + \int_1^3 (-x^3 + 4x^2 - 3x) dx \\
 &= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 + \left[-\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2 \right]_1^3 = \frac{37}{12}
 \end{aligned}$$

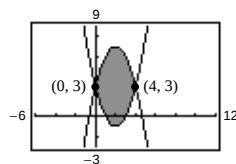
Numerical Approximation: $0.417 + 2.667 \approx 3.083$



33. The points of intersection are given by:

$$\begin{aligned}
 x^2 - 4x + 3 &= 3 + 4x - x^2 \\
 2x(x - 4) &= 0 \quad \text{when } x = 0, 4 \\
 A &= \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)] dx \\
 &= \int_0^4 (-2x^2 + 8x) dx \\
 &= \left[-\frac{2x^3}{3} + 4x^2 \right]_0^4 = \frac{64}{3}
 \end{aligned}$$

Numerical Approximation: 21.333



35. $f(x) = x^4 - 4x^2$, $g(x) = x^2 - 4$

The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$

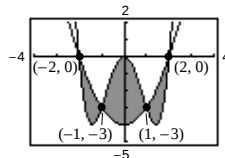
$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$

By symmetry,

$$\begin{aligned} A &= 2 \int_0^1 [(x^4 - 4x^2) - (x^2 - 4)] dx + 2 \int_1^2 [(x^2 - 4) - (x^4 - 4x^2)] dx \\ &= 2 \int_0^1 (x^4 - 5x^2 + 4) dx + 2 \int_1^2 (-x^4 + 5x^2 - 4) dx \\ &= 2 \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_0^1 + 2 \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_1^2 \\ &= 2 \left[\frac{1}{5} - \frac{5}{3} + 4 \right] + 2 \left[\left(-\frac{32}{5} + \frac{40}{3} - 8 \right) - \left(-\frac{1}{5} + \frac{5}{3} - 4 \right) \right] = 8. \end{aligned}$$

Numerical Approximation: $5.067 + 2.933 = 8.0$



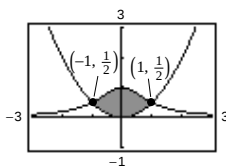
37. The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2 + 2)(x^2 - 1) = 0$$

$$x = \pm 1$$



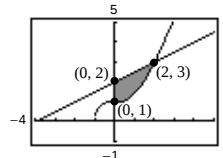
$$\begin{aligned} A &= 2 \int_0^1 [f(x) - g(x)] dx \\ &= 2 \int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2} \right] dx \\ &= 2 \left[\arctan x - \frac{x^3}{6} \right]_0^1 \\ &= 2 \left(\frac{\pi}{4} - \frac{1}{6} \right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237 \end{aligned}$$

Numerical Approximation: 1.237

39. $\sqrt{1+x^3} \leq \frac{1}{2}x + 2$ on $[0, 2]$

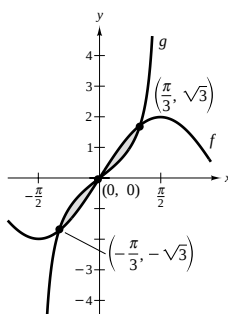
Numerical approximation: 1.759

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1+x^3} \right] dx \approx 1.759$$

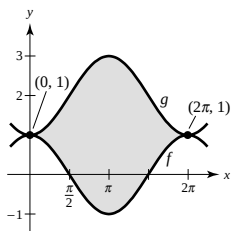


41. $A = 2 \int_0^{\pi/3} [f(x) - g(x)] dx$

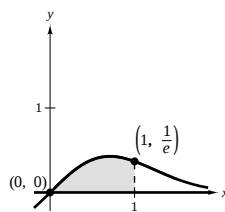
$$\begin{aligned} &= 2 \int_0^{\pi/3} (2 \sin x - \tan x) dx \\ &= 2 \left[-2 \cos x + \ln |\cos x| \right]_0^{\pi/3} \\ &= 2(1 - \ln 2) \approx 0.614 \end{aligned}$$



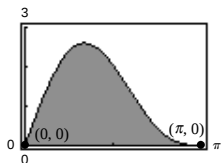
$$\begin{aligned}
 43. A &= \int_0^{2\pi} [(2 - \cos x) - \cos x] dx \\
 &= 2 \int_0^{2\pi} (1 - \cos x) dx \\
 &= 2 \left[x - \sin x \right]_0^{2\pi} = 4\pi \approx 12.566
 \end{aligned}$$



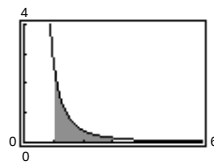
$$\begin{aligned}
 45. A &= \int_0^1 [xe^{-x^2} - 0] dx \\
 &= \left[-\frac{1}{2}e^{-x^2} \right]_0^1 = \frac{1}{2} \left(1 - \frac{1}{e} \right) \approx 0.316
 \end{aligned}$$



$$\begin{aligned}
 47. A &= \int_0^{\pi} [(2 \sin x + \sin 2x) - 0] dx \\
 &= \left[-2 \cos x - \frac{1}{2} \cos 2x \right]_0^{\pi} = 4.0
 \end{aligned}$$



$$\begin{aligned}
 49. A &= \int_1^3 \left[\frac{1}{x^2} e^{1/x} - 0 \right] dx \\
 &= \left[-e^{1/x} \right]_1^3 = e - e^{1/3} \approx 1.323
 \end{aligned}$$

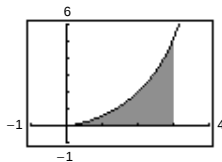


$$51. (a) y = \sqrt{\frac{x^3}{4-x}}, \quad y = 0, \quad x = 3$$

$$(b) A = \int_0^3 \sqrt{\frac{x^3}{4-x}} dx,$$

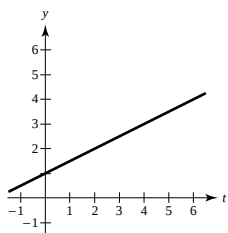
No, it cannot be evaluated by hand.

$$(c) 4.7721$$

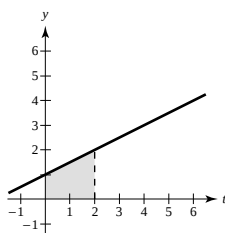


$$53. F(x) = \int_0^x \left(\frac{1}{2}t + 1 \right) dt = \left[\frac{t^2}{4} + t \right]_0^x = \frac{x^2}{4} + x$$

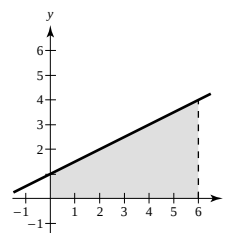
$$(a) F(0) = 0$$



$$(b) F(2) = \frac{2^2}{4} + 2 = 3$$

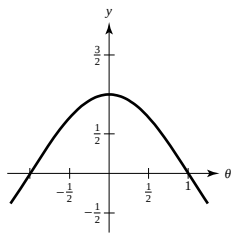


$$(c) F(6) = \frac{6^2}{4} + 6 = 15$$

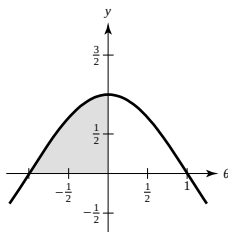


$$55. F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2} d\theta = \left[\frac{2}{\pi} \sin \frac{\pi\theta}{2} \right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$$

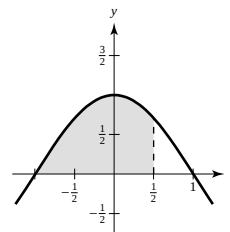
$$(a) F(-1) = 0$$



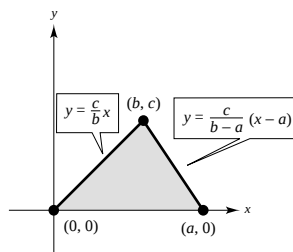
$$(b) F(0) = \frac{2}{\pi} \approx 0.6366$$



$$(c) F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$$



$$\begin{aligned} 57. A &= \int_0^c \left[\left(\frac{b-a}{c}y + a \right) - \frac{b}{c}y \right] dy \\ &= \int_0^c \left(-\frac{a}{c}y + a \right) dy \\ &= \left[-\frac{a}{2c}y^2 + ay \right]_0^c \\ &= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2}(\text{base})(\text{height}) \right) \end{aligned}$$



$$59. f(x) = x^3$$

$$f'(x) = 3x^2$$

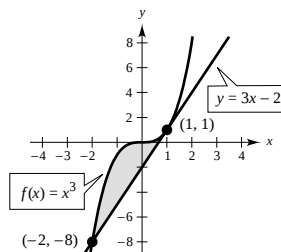
$$\text{At } (1, 1), f'(1) = 3.$$

Tangent line:

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^1 [x^3 - (3x - 2)] dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^1 = \frac{27}{4}$$

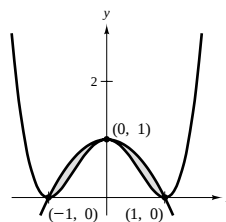


61. The variable is y .

$$63. x^4 - 2x^2 + 1 \leq 1 - x^2 \text{ on } [-1, 1]$$

$$\begin{aligned} A &= \int_{-1}^1 [(1 - x^2) - (x^4 - 2x^2 + 1)] dx \\ &= \int_{-1}^1 (x^2 - x^4) dx \\ &= \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^1 = \frac{4}{15} \end{aligned}$$

You can use a single integral because $x^4 - 2x^2 + 1 \leq 1 - x^2$ on $[-1, 1]$.



65. Offer 2 is better because the accumulated salary (area under the curve) is larger.

$$67. \quad A = \int_{-3}^3 (9 - x^2) dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b] dx = 18$$

$$\int_0^{\sqrt{9-b}} [(9 - b) - x^2] dx = 9$$

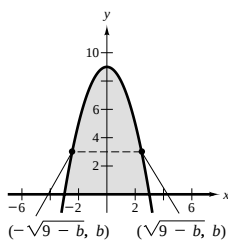
$$\left[(9 - b)x - \frac{x^3}{3} \right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3} (9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

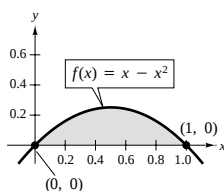
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



$$69. \quad \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (x_i - x_i^2) \Delta x$$

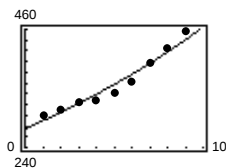
where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

$$\int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

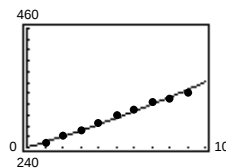


$$71. \quad \int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)] dt = \int_0^5 0.13t dt = \left[\frac{0.13t^2}{2} \right]_0^5 = \$1.625 \text{ billion}$$

$$73. \quad (a) \quad y_1 = (275.0675)(1.0537)^t = (275.0675)e^{0.0523t}$$



$$(b) \quad y_2 = (239.9407)(1.0417)^t = (239.9407)e^{0.0408t}$$



$$(c) \quad \int_{10}^{15} (y_1 - y_2) dt \approx 649.5 \text{ billion dollars}$$

(d) No, model $y_1 > y_2$ forever because $1.0537 > 1.0417$.

No, these models are not accurate. According to news reports, $E > R$ eventually.

75. The total area is 8 times the area of the shaded region to the right. A point (x, y) is on the upper boundary of the region if

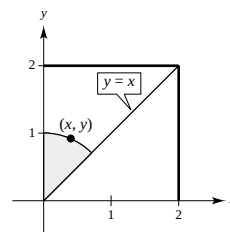
$$\sqrt{x^2 + y^2} = 2 - y$$

$$x^2 + y^2 = 4 - 4y + y^2$$

$$x^2 = 4 - 4y$$

$$4y = 4 - x^2$$

$$y = 1 - \frac{x^2}{4}$$



We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$

$$x^2 + 4x - 4 = 0$$

$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \Rightarrow x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8 \int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx$$

$$= 8 \left[x - \frac{x^3}{12} - \frac{x^2}{2} \right]_0^{-2+2\sqrt{2}} = \frac{16}{3}(4\sqrt{2} - 5) \approx 8(0.4379) = 3.503$$

$$\begin{aligned} 77. \text{ (a) } A &= 2 \left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5-x}\right) dx + \int_5^{5.5} (1-0) dx \right] \\ &= 2 \left(\left[x + \frac{2}{9}(5-x)^{3/2} \right]_0^5 + \left[x \right]_5^{5.5} \right) = 2 \left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5 \right) \approx 6.031 \text{ m}^2 \end{aligned}$$

$$\text{(b) } V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$$

$$\text{(c) } 5000 V \approx 5000(12.062) = 60,310 \text{ pounds}$$

79. True

81. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)] dx = \int_0^2 [x - (2x - x^2)] dx = \frac{2}{3} \neq 0.$$

Section 6.2 Volume: The Disk Method

$$1. V = \pi \int_0^1 (-x + 1)^2 dx = \pi \int_0^1 (x^2 - 2x + 1) dx = \pi \left[\frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{\pi}{3}$$

$$3. V = \pi \int_1^4 (\sqrt{x})^2 dx = \pi \int_1^4 x dx = \pi \left[\frac{x^2}{2} \right]_1^4 = \frac{15\pi}{2}$$

$$5. V = \pi \int_0^1 [(x^2)^2 - (x^3)^2] dx = \pi \int_0^1 (x^4 - x^6) dx = \pi \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{2\pi}{35}$$

$$7. y = x^2 \Rightarrow x = \sqrt{y}$$

$$V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi$$

$$9. y = x^{2/3} \Rightarrow x = y^{3/2}$$

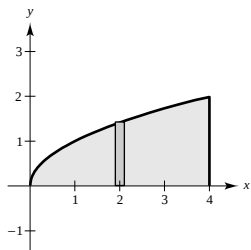
$$V = \pi \int_0^1 (y^{3/2})^2 dy = \pi \int_0^1 y^3 dy = \pi \left[\frac{y^4}{4} \right]_0^1 = \frac{\pi}{4}$$

11. $y = \sqrt{x}$, $y = 0$, $x = 4$

(a) $R(x) = \sqrt{x}$, $r(x) = 0$

$$V = \pi \int_0^4 (\sqrt{x})^2 dx$$

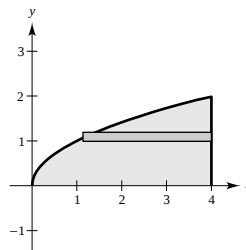
$$= \pi \int_0^4 x dx = \left[\frac{\pi}{2} x^2 \right]_0^4 = 8\pi$$



(b) $R(y) = 4$, $r(y) = y^2$

$$V = \pi \int_0^2 (16 - y^4) dy$$

$$= \pi \left[16y - \frac{1}{5} y^5 \right]_0^2 = \frac{128\pi}{5}$$

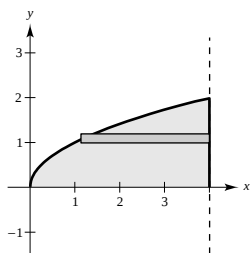


(c) $R(y) = 4 - y^2$, $r(y) = 0$

$$V = \pi \int_0^2 (4 - y^2)^2 dy$$

$$= \pi \int_0^2 (16 - 8y^2 + y^4) dy$$

$$= \pi \left[16y - \frac{8}{3} y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{256\pi}{15}$$

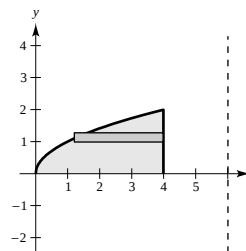


(d) $R(y) = 6 - y^2$, $r(y) = 2$

$$V = \pi \int_0^2 [(6 - y^2)^2 - 4] dy$$

$$= \pi \int_0^2 (32 - 12y^2 + y^4) dy$$

$$= \pi \left[32y - 4y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{192\pi}{5}$$



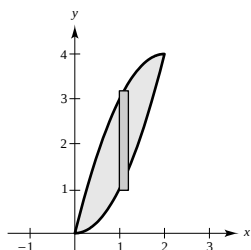
13. $y = x^2$, $y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2$, $r(x) = x^2$

$$V = \pi \int_0^2 [(4x - x^2)^2 - x^4] dx$$

$$= \pi \int_0^2 (16x^2 - 8x^3) dx$$

$$= \pi \left[\frac{16}{3} x^3 - 2x^4 \right]_0^2 = \frac{32\pi}{3}$$

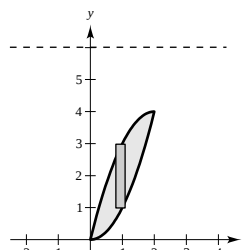


(b) $R(x) = 6 - x^2$, $r(x) = 6 - (4x - x^2)$

$$V = \pi \int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2] dx$$

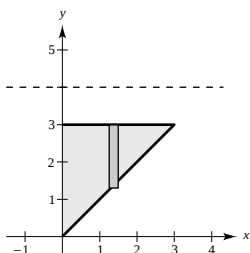
$$= 8\pi \int_0^2 (x^3 - 5x^2 + 6x) dx$$

$$= 8\pi \left[\frac{x^4}{4} - \frac{5}{3} x^3 + 3x^2 \right]_0^2 = \frac{64\pi}{3}$$



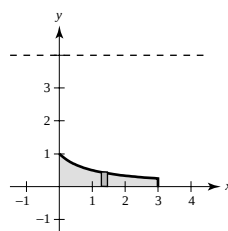
15. $R(x) = 4 - x$, $r(x) = 1$

$$\begin{aligned} V &= \pi \int_0^3 [(4 - x)^2 - (1)^2] dx \\ &= \pi \int_0^3 (x^2 - 8x + 15) dx \\ &= \pi \left[\frac{x^3}{3} - 4x^2 + 15x \right]_0^3 = 18\pi \end{aligned}$$



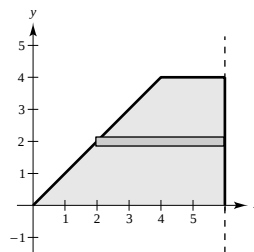
17. $R(x) = 4$, $r(x) = 4 - \frac{1}{1+x}$

$$\begin{aligned} V &= \pi \int_0^3 \left[4^2 - \left(4 - \frac{1}{1+x} \right)^2 \right] dx \\ &= \pi \int_0^3 \left[\frac{8}{1+x} - \frac{1}{(1+x)^2} \right] dx \\ &= \pi \left[8 \ln(1+x) + \frac{1}{1+x} \right]_0^3 \\ &= \pi \left[8 \ln 4 + \frac{1}{4} - 1 \right] \\ &= \left(8 \ln 4 - \frac{3}{4} \right) \pi \approx 32.485 \end{aligned}$$



19. $R(y) = 6 - y$, $r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^4 (6 - y)^2 dy \\ &= \pi \int_0^4 (y^2 - 12y + 36) dy \\ &= \pi \left[\frac{y^3}{3} - 6y^2 + 36y \right]_0^4 \\ &= \frac{208\pi}{3} \end{aligned}$$

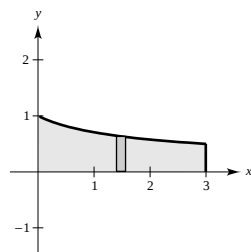


21. $R(y) = 6 - y^2$, $r(y) = 2$

$$\begin{aligned} V &= \pi \int_{-2}^2 [(6 - y^2)^2 - (2)^2] dy \\ &= 2\pi \int_0^2 (y^4 - 12y^2 + 32) dy \\ &= 2\pi \left[\frac{y^5}{5} - 4y^3 + 32y \right]_0^2 \\ &= \frac{384\pi}{5} \end{aligned}$$

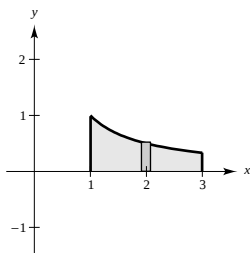
23. $R(x) = \frac{1}{\sqrt{x+1}}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^3 \left(\frac{1}{\sqrt{x+1}} \right)^2 dx \\ &= \pi \int_0^3 \frac{1}{x+1} dx \\ &= \left[\pi \ln|x+1| \right]_0^3 = \pi \ln 4 \end{aligned}$$



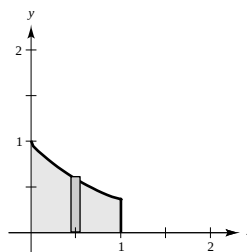
25. $R(x) = \frac{1}{x}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \left[-\frac{1}{x} \right]_1^4 \\ &= \frac{3\pi}{4} \end{aligned}$$



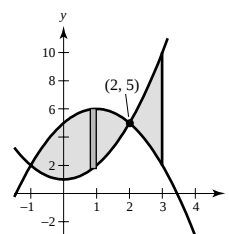
27. $R(x) = e^{-x}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^1 (e^{-x})^2 dx \\ &= \pi \int_0^1 e^{-2x} dx \\ &= \left[-\frac{\pi}{2} e^{-2x} \right]_0^1 \\ &= \frac{\pi}{2} (1 - e^{-2}) \approx 1.358 \end{aligned}$$



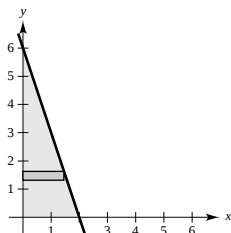
29. $V = \pi \int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2] dx + \pi \int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2] dx$

$$\begin{aligned} &= \pi \int_0^2 (-4x^3 - 8x^2 + 20x + 24) dx + \pi \int_2^3 (4x^3 + 8x^2 - 20x - 24) dx \\ &= \pi \left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x \right]_0^2 + \pi \left[x^3 + \frac{8}{3}x^3 - 10x^2 - 24x \right]_2^3 \\ &= \pi \frac{152}{3} + \pi \frac{125}{3} = \frac{277\pi}{3} \end{aligned}$$

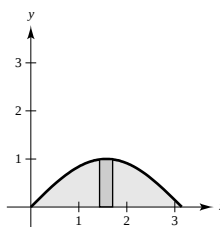


31. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

$$\begin{aligned} V &= \pi \int_0^6 \left[\frac{1}{3}(6 - y) \right]^2 dy \\ &= \frac{\pi}{9} \int_0^6 [36 - 12y + y^2] dy \\ &= \frac{\pi}{9} \left[36y - 6y^2 + \frac{y^3}{3} \right]_0^6 \\ &= \frac{\pi}{9} \left[216 - 216 + \frac{216}{3} \right] \\ &= 8\pi \end{aligned}$$



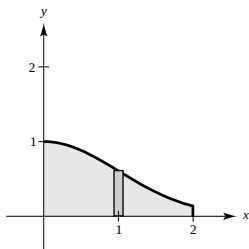
33. $V = \pi \int_0^\pi [\sin x]^2 dx \approx 4.9348$



$$35. V = \pi \int_0^2 [e^{-x^2}]^2 dx \approx 1.9686$$

$$39. A \approx 3$$

Matches (a)



$$37. V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2 dx \approx 49.0218$$

41. Disk Method:

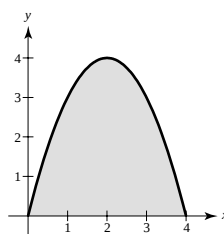
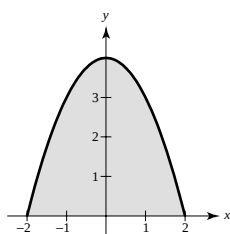
$$V = \pi \int_a^b [R(x)]^2 dx \quad \text{or} \quad V = \pi \int_c^d [R(y)]^2 dy$$

Washer Method:

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx \quad \text{or}$$

$$V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2) dy$$

43.



The volumes are the same because the solid has been translated horizontally.

$$45. R(x) = \frac{1}{2}x, \quad r(x) = 0$$

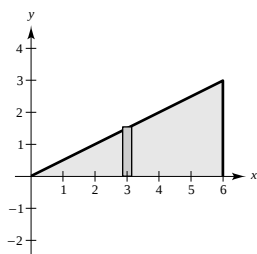
$$V = \pi \int_0^6 \frac{1}{4}x^2 dx$$

$$= \left[\frac{\pi}{12}x^3 \right]_0^6 = 18\pi$$

Note: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$



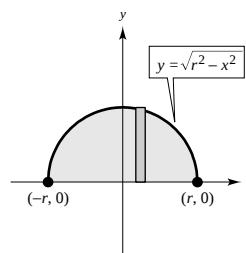
$$47. R(x) = \sqrt{r^2 - x^2}, \quad r(x) = 0$$

$$V = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx$$

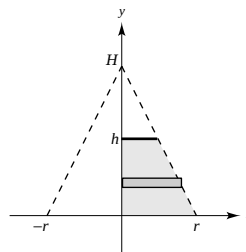
$$= 2\pi \left[r^2x - \frac{1}{3}x^3 \right]_0^r$$

$$= 2\pi \left(r^3 - \frac{1}{3}r^3 \right) = \frac{4}{3}\pi r^3$$



49. $x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right)$, $R(y) = r\left(1 - \frac{y}{H}\right)$, $r(y) = 0$

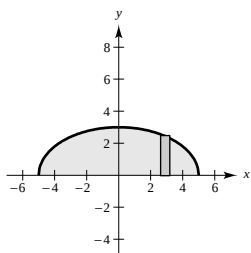
$$\begin{aligned} V &= \pi \int_0^h \left[r\left(1 - \frac{y}{H}\right) \right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2 \right) dy \\ &= \pi r^2 \left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3 \right]_0^h \\ &= \pi r^2 \left(h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right) \\ &= \pi r^2 h \left(1 - \frac{h}{H} + \frac{h^2}{3H^2} \right) \end{aligned}$$



51. $V = \pi \int_0^2 \left(\frac{1}{8}x^2 \sqrt{2-x} \right)^2 dx = \frac{\pi}{64} \int_0^2 x^4(2-x) dx = \frac{\pi}{64} \left[\frac{2x^5}{5} - \frac{x^6}{6} \right]_0^2 = \frac{\pi}{30}$

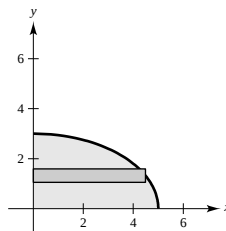
53. (a) $R(x) = \frac{3}{5} \sqrt{25-x^2}$, $r(x) = 0$

$$\begin{aligned} V &= \frac{9\pi}{25} \int_{-5}^5 (25-x^2) dx \\ &= \frac{18\pi}{25} \int_0^5 (25-x^2) dx \\ &= \frac{18\pi}{25} \left[25x - \frac{x^3}{3} \right]_0^5 = 60\pi \end{aligned}$$



(b) $R(y) = \frac{5}{3} \sqrt{9-y^2}$, $r(y) = 0$, $x \geq 0$

$$\begin{aligned} V &= \frac{25\pi}{9} \int_0^3 (9-y^2) dy \\ &= \frac{25\pi}{9} \left[9y - \frac{y^3}{3} \right]_0^3 = 50\pi \end{aligned}$$



55. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500,000}{3} \text{ ft}^3$

Volume of water in the tank:

$$\begin{aligned} \pi \int_{-50}^{y_0} (\sqrt{2500-y^2})^2 dy &= \pi \int_{-50}^{y_0} (2500-y^2) dy \\ &= \pi \left[2500y - \frac{y^3}{3} \right]_{-50}^{y_0} \\ &= \pi \left(2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \end{aligned}$$

When the tank is one-fourth of its capacity:

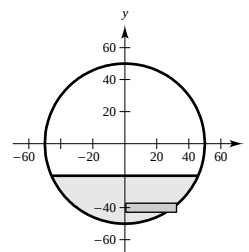
$$\begin{aligned} \frac{1}{4} \left(\frac{500,000\pi}{3} \right) &= \pi \left(2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \\ 125,000 &= 7500y_0 - y_0^3 + 250,000 \end{aligned}$$

$$y_0^3 - 7500y_0 - 125,000 = 0$$

$$y_0 \approx -17.36$$

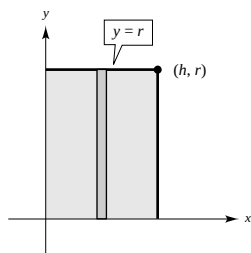
Depth: $-17.36 - (-50) = 32.64$ feet

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.



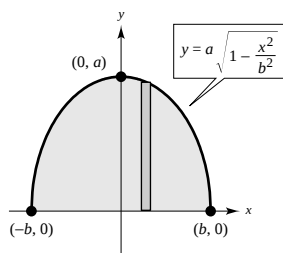
57. (a) $\pi \int_0^h r^2 dx$ (ii)

is the volume of a right circular cylinder with radius r and height h .



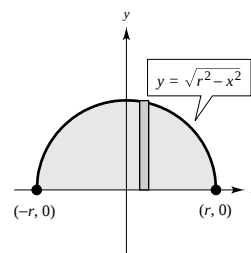
(b) $\pi \int_{-b}^b \left(a \sqrt{1 - \frac{x^2}{b^2}} \right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



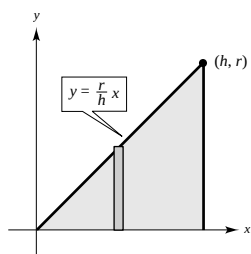
(c) $\pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx$ (iii)

is the volume of a sphere with radius r .



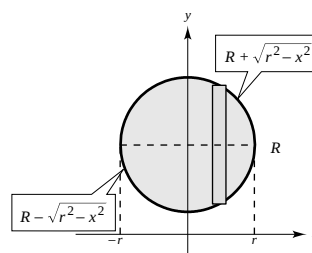
(d) $\pi \int_0^h \left(\frac{rx}{h} \right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h .

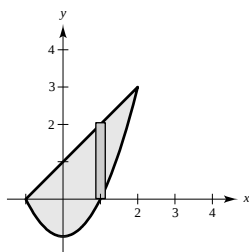


(e) $\pi \int_{-r}^r \left[(R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2 \right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R .



59.



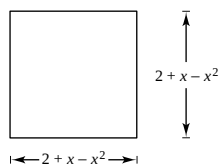
$$\text{Base of Cross Section} = (x + 1) - (x^2 - 1) = 2 + x - x^2$$

(a) $A(x) = b^2 = (2 + x - x^2)^2$

$$= 4 + 4x - 3x^2 - 2x^3 + x^4$$

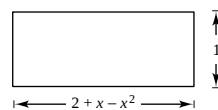
$$V = \int_{-1}^2 (4 + 4x - 3x^2 - 2x^3 + x^4) dx$$

$$= \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_{-1}^2 = \frac{81}{10}$$

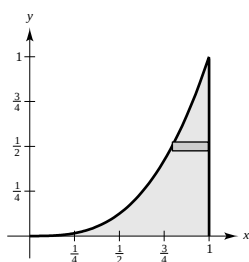


(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



61.

Base of Cross Section = $1 - \sqrt[3]{y}$

$$(b) A(y) = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi \left(\frac{1 - \sqrt[3]{y}}{2} \right)^2 = \frac{1}{8} \pi (1 - \sqrt[3]{y})^2$$

$$V = \frac{1}{8} \pi \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{8} \left(\frac{1}{10} \right) = \frac{\pi}{80}$$

$$(c) A(y) = \frac{1}{2} bh = \frac{1}{2} (1 - \sqrt[3]{y}) \left(\frac{\sqrt{3}}{2} (1 - \sqrt[3]{y}) \right) = \frac{\sqrt{3}}{4} (1 - \sqrt[3]{y})^2$$

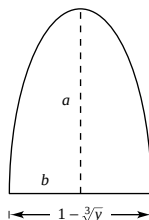
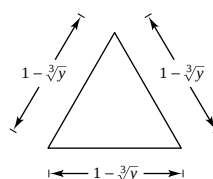
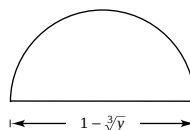
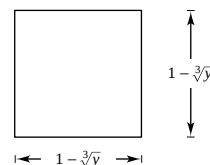
$$V = \frac{\sqrt{3}}{4} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\sqrt{3}}{4} \left(\frac{1}{10} \right) = \frac{\sqrt{3}}{40}$$

$$(d) A(y) = \frac{1}{2} \pi ab = \frac{\pi}{2} (2) (1 - \sqrt[3]{y}) \frac{1 - \sqrt[3]{y}}{2} = \frac{\pi}{2} (1 - \sqrt[3]{y})^2$$

$$V = \frac{\pi}{2} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{2} \left(\frac{1}{10} \right) = \frac{\pi}{20}$$

$$(a) A(y) = b^2 = (1 - \sqrt[3]{y})^2$$

$$V = \int_0^1 (1 - \sqrt[3]{y})^2 dy = \int_0^1 (1 - 2y^{1/3} + y^{2/3}) dy = \left[y - \frac{3}{2} y^{4/3} + \frac{3}{5} y^{5/3} \right]_0^1 = \frac{1}{10}$$



63. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \leq x \leq b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x) dx = \int_a^b A_2(x) dx = V_2$$

Thus, the volumes are the same.

$$65. \frac{4}{3} \pi (25 - r^2)^{3/2} = \frac{1}{2} \left(\frac{4}{3} \right) \pi (125)$$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2} \right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. (a) Since the cross sections are isosceles right triangles:

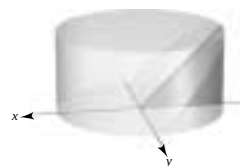
$$A(x) = \frac{1}{2} bh = \frac{1}{2} (\sqrt{r^2 - y^2}) (\sqrt{r^2 - y^2}) = \frac{1}{2} (r^2 - y^2)$$

$$V = \frac{1}{2} \int_{-r}^r (r^2 - y^2) dy = \int_0^r (r^2 - y^2) dy = \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3} r^3$$

$$(b) A(x) = \frac{1}{2} bh = \frac{1}{2} \sqrt{r^2 - y^2} (\sqrt{r^2 - y^2} \tan \theta) = \frac{\tan \theta}{2} (r^2 - y^2)$$

$$V = \frac{\tan \theta}{2} \int_{-r}^r (r^2 - y^2) dy = \tan \theta \int_0^r (r^2 - y^2) dy = \tan \theta \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3} r^3 \tan \theta$$

As $\theta \rightarrow 90^\circ$, $V \rightarrow \infty$.



Section 6.3 Volume: The Shell Method

1. $p(x) = x$

$h(x) = x$

$$V = 2\pi \int_0^2 x(x) dx = \left[\frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

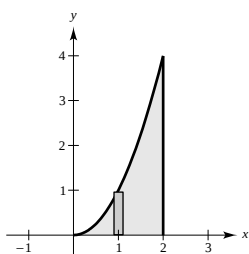
$$= \left[\frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

5. $p(x) = x$

$h(x) = x^2$

$$V = 2\pi \int_0^2 x^3 dx$$

$$= \left[\frac{\pi x^4}{2} \right]_0^2 = 8\pi$$



3. $p(x) = x$

$h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x} dx$$

$$= 2\pi \int_0^4 x^{3/2} dx$$

$$= \left[\frac{4\pi}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}$$

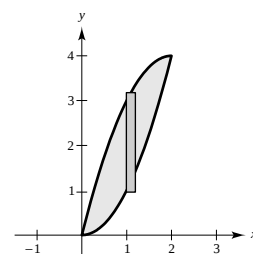
7. $p(x) = x$

$h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x(4x - 2x^2) dx$$

$$= 4\pi \int_0^2 (2x^2 - x^3) dx$$

$$= 4\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \frac{16\pi}{3}$$

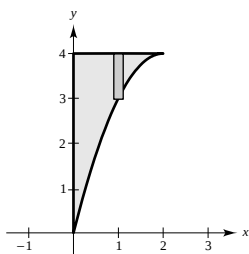


9. $p(x) = x$

$h(x) = 4 - (4x - x^2) = x^2 - 4x + 4$

$$V = 2\pi \int_0^2 (x^3 - 4x^2 + 4x) dx$$

$$= 2\pi \left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2 \right]_0^2 = \frac{8\pi}{3}$$



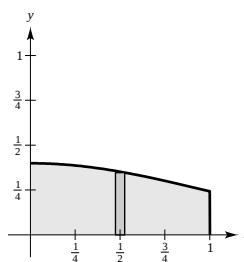
11. $p(x) = x$

$h(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$V = 2\pi \int_0^1 x \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx$$

$$= \sqrt{2\pi} \int_0^1 e^{-x^2/2} x dx$$

$$= \left[-\sqrt{2\pi} e^{-x^2/2} \right]_0^1 = \sqrt{2\pi} \left(1 - \frac{1}{\sqrt{e}} \right) \approx 0.986$$



13. $p(y) = y$

$h(y) = 2 - y$

$$V = 2\pi \int_0^2 y(2 - y) dy$$

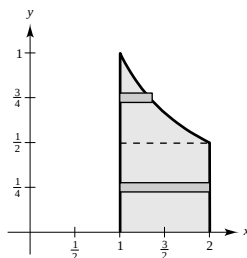
$$= 2\pi \int_0^2 (2y - y^2) dy$$

$$= 2\pi \left[y^2 - \frac{y^3}{3} \right]_0^2 = \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \leq y < \frac{1}{2}$.

$$p(y) = y \text{ and } h(y) = \frac{1}{y} - 1 \text{ if } \frac{1}{2} \leq y \leq 1.$$

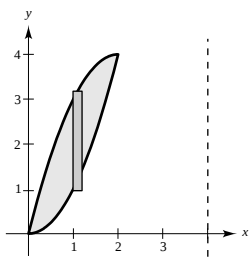
$$\begin{aligned} V &= 2\pi \int_0^{1/2} y \, dy + 2\pi \int_{1/2}^1 (1 - y) \, dy \\ &= 2\pi \left[\frac{y^2}{2} \right]_0^{1/2} + 2\pi \left[y - \frac{y^2}{2} \right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$



17. $p(x) = 4 - x$

$$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$$

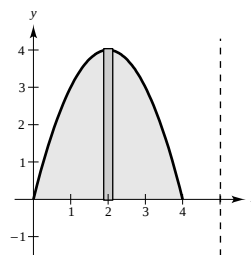
$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)(4x - 2x^2) \, dx \\ &= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x) \, dx \\ &= 4\pi \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 16\pi \end{aligned}$$



19. $p(x) = 5 - x$

$$h(x) = 4x - x^2$$

$$\begin{aligned} V &= 2\pi \int_0^4 (5 - x)(4x - x^2) \, dx \\ &= 2\pi \int_0^4 (x^3 - 9x^2 + 20x) \, dx \\ &= 2\pi \left[\frac{x^4}{4} - 3x^3 + 10x^2 \right]_0^4 = 64\pi \end{aligned}$$

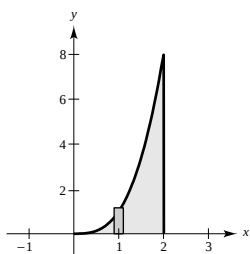


21. (a) Disk

$$R(x) = x^3$$

$$r(x) = 0$$

$$V = \pi \int_0^2 x^6 \, dx = \pi \left[\frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}$$

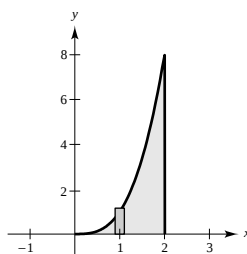


- (b) Shell

$$p(x) = x$$

$$h(x) = x^3$$

$$V = 2\pi \int_0^2 x^4 \, dx = 2\pi \left[\frac{x^5}{5} \right]_0^2 = \frac{64\pi}{5}$$

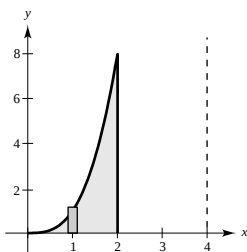


- (c) Shell

$$p(x) = 4 - x$$

$$h(x) = x^3$$

$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)x^3 \, dx \\ &= 2\pi \int_0^2 (4x^3 - x^4) \, dx \\ &= 2\pi \left[x^4 - \frac{1}{5}x^5 \right]_0^2 = \frac{96\pi}{5} \end{aligned}$$

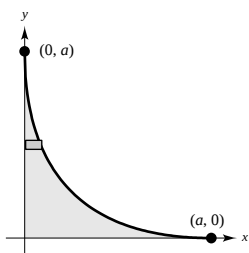


23. (a) Shell

$$p(y) = y$$

$$h(y) = (a^{1/2} - y^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y) dy \\ &= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2) dy \\ &= 2\pi \left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3} \right]_0^a \\ &= 2\pi \left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3} \right] = \frac{\pi a^3}{15} \end{aligned}$$



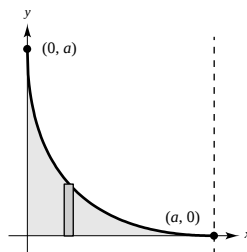
(b) Same as part (a) by symmetry

(c) Shell

$$p(x) = a - x$$

$$h(x) = (a^{1/2} - x^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2 dx \\ &= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2) dx \\ &= 2\pi \left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3 \right]_0^a = \frac{4\pi a^3}{15} \end{aligned}$$



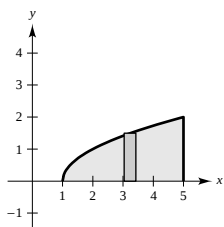
$$25. V = 2\pi \int_x^d p(y)h(y) dy \quad \text{or} \quad V = 2\pi \int_a^b p(x)h(x) dx$$

$$27. \pi \int_1^5 (x - 1) dx = \pi \int_1^5 (\sqrt{x - 1})^2 dx$$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x -axis by using the Disk Method.

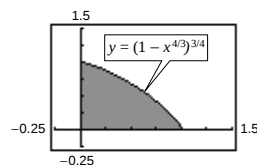
$$2\pi \int_0^2 y[5 - (y^2 + 1)] dy$$

represents this same volume by using the Shell Method.



Disk Method

29. (a)

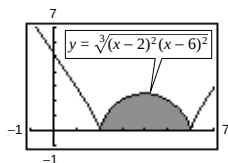


$$(b) x^{4/3} + y^{4/3} = 1, x = 0, y = 0$$

$$y = (1 - x^{4/3})^{3/4}$$

$$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4} dx \approx 1.5056$$

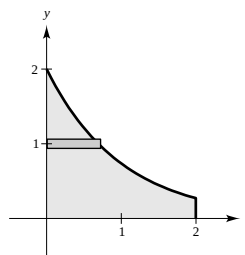
31. (a)



$$(b) V = 2\pi \int_2^6 x \sqrt[3]{(x-2)^2(x-6)^2} dx \approx 187.249$$

33. $y = 2e^{-x}$, $y = 0$, $x = 0$, $x = 2$ Volume ≈ 7.5

Matches (d)

35. $p(x) = x$

$$h(x) = 2 - \frac{1}{2}x^2$$

$$V = 2\pi \int_0^2 x \left(2 - \frac{1}{2}x^2\right) dx = 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi \left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \text{ (total volume)}$$

Now find x_0 such that

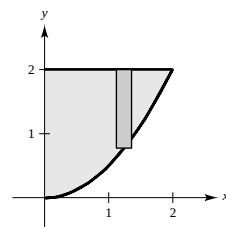
$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2 \left[x^2 - \frac{1}{8}x^4 \right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

$$x_0^2 = 4 \pm 2\sqrt{3} \quad (\text{Quadratic Formula})$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}}$ since the other root is too large.Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$ 

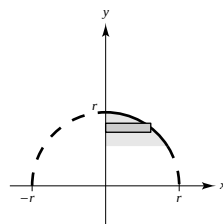
$$\begin{aligned} 37. V &= 4\pi \int_{-1}^1 (2-x)\sqrt{1-x^2} dx \\ &= 8\pi \int_{-1}^1 \sqrt{1-x^2} dx - 4\pi \int_{-1}^1 x\sqrt{1-x^2} dx \\ &= 8\pi \left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^1 x(1-x^2)^{1/2}(-2) dx \\ &= 4\pi^2 + \left[2\pi \left(\frac{2}{3}\right)(1-x^2)^{3/2}\right]_{-1}^1 = 4\pi^2 \end{aligned}$$

39. Disk Method

$$R(y) = \sqrt{r^2 - y^2}$$

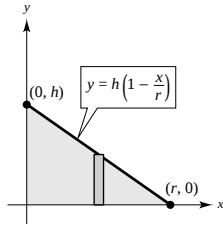
$$r(y) = 0$$

$$\begin{aligned} V &= \pi \int_{r-h}^r (r^2 - y^2) dy \\ &= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r = \frac{1}{3} \pi h^2 (3r - h) \end{aligned}$$



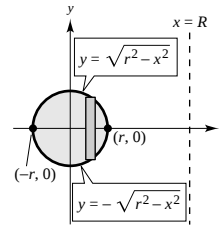
41. (a) $2\pi \int_0^r hx \left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h .

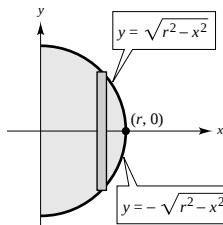


(b) $2\pi \int_{-r}^r (R - x)(2\sqrt{r^2 - x^2}) dx$ (v)

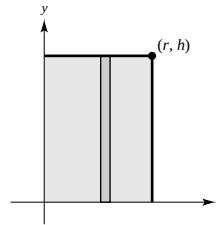
is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R .



(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2} dx$ (iii) is the volume of a sphere with radius r .

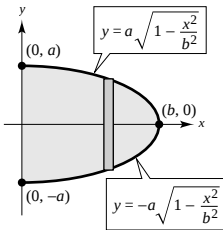


(d) $2\pi \int_0^r hx dx$ (i) is the volume of a right circular cylinder with a radius of r and a height of h .



(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)} dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

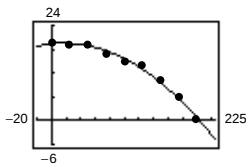


43. (a) $V = 2\pi \int_0^{200} xf(x) dx$

$$\approx \frac{2\pi(200)}{3(8)} [0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$$\approx 1,366,593 \text{ cubic feet}$$

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi \int_0^{200} xd(x) dx \approx 2\pi(213,800) = 1,343,345 \text{ cubic feet}$

(d) Number gallons $\approx V(7.48) = 10,048,221 \text{ gallons}$

Section 6.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b) $y = \frac{12}{5}x$

$$y' = \frac{12}{5}$$

$$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2} dx = \left[\frac{13}{5}x\right]_0^5 = 13$$

3. $y = \frac{2}{3}x^{3/2} + 1$

$$y' = x^{1/2}, [0, 1]$$

$$s = \int_0^1 \sqrt{1+x} dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}(\sqrt{8}-1) \approx 1.219$$

5. $y = \frac{3}{2}x^{2/3}$

$$y' = \frac{1}{x^{1/3}}, [1, 8]$$

$$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx$$

$$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx$$

$$= \frac{3}{2} \int_1^8 \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx$$

$$= \frac{3}{2} \left[\frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^8$$

$$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$$

7. $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, [1, 2]$$

$$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, [1, 2]$$

$$s = \int_a^b \sqrt{1 + (y')^2} dx$$

$$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) dx$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$$

9. $y = \ln(\sin x), \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$$y' = \frac{1}{\sin x} \cos x = \cot x$$

$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$

$$s = \int_{\pi/4}^{3\pi/4} \csc x dx$$

$$= \left[\ln|\csc x - \cot x|\right]_{\pi/4}^{3\pi/4}$$

$$= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \approx 1.763$$

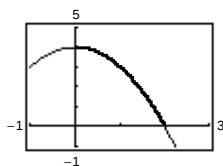
11. (a) $y = 4 - x^2, 0 \leq x \leq 2$

(b) $y' = -2x$

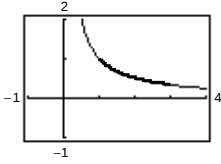
(c) $L \approx 4.647$

$$1 + (y')^2 = 1 + 4x^2$$

$$L = \int_0^2 \sqrt{1 + 4x^2} dx$$



13. (a) $y = \frac{1}{x}, 1 \leq x \leq 3$



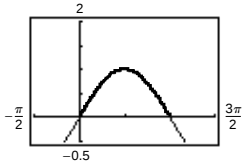
(b) $y' = -\frac{1}{x^2}$

(c) $L \approx 2.147$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}} dx$$

15. (a) $y = \sin x, 0 \leq x \leq \pi$



(b) $y' = \cos x$

(c) $L \approx 3.820$

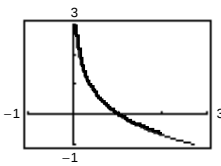
$$1 + (y')^2 = 1 + \cos^2 x$$

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

17. (a) $x = e^{-y}, 0 \leq y \leq 2$

$$y = -\ln x$$

$$1 \geq x \geq e^{-2} \approx 0.135$$



(b) $y' = -\frac{1}{x}$

(c) $L \approx 2.221$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}} dx$$

Alternatively, you can do all the computations with respect to y .

(a) $x = e^{-y}, 0 \leq y \leq 2$

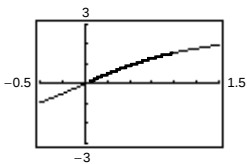
(b) $\frac{dx}{dy} = -e^{-y}$

(c) $L \approx 2.221$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$

$$L = \int_0^2 \sqrt{1 + e^{-2y}} dy$$

19. (a) $y = 2 \arctan x, 0 \leq x \leq 1$



(b) $y' = \frac{2}{1+x^2}$

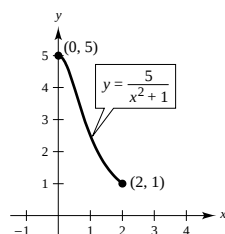
(c) $L \approx 1.871$

$$L = \int_0^1 \sqrt{1 + \frac{4}{(1+x^2)^2}} dx$$

21. $\int_0^2 \sqrt{1 + \left[\frac{d}{dx} \left(\frac{5}{x^2 + 1} \right) \right]^2} dx$

$s \approx 5$

Matches (b)



23. $y = x^3, [0, 4]$

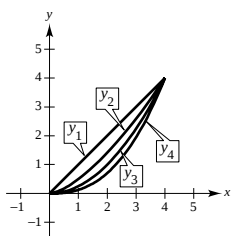
(a) $d = \sqrt{(4 - 0)^2 + (64 - 0)^2} \approx 64.125$

(b) $d = \sqrt{(1 - 0)^2 + (1 - 0)^2} + \sqrt{(2 - 1)^2 + (8 - 1)^2} + \sqrt{(3 - 2)^2 + (27 - 8)^2} + \sqrt{(4 - 3)^2 + (64 - 27)^2}$
 ≈ 64.525

(c) $s = \int_0^4 \sqrt{1 + (3x^2)^2} dx = \int_0^4 \sqrt{1 + 9x^4} dx \approx 64.666$

(d) 64.672

25. (a)



(b) y_1, y_2, y_3, y_4

(c) $y_1' = 1, L_1 = \int_0^4 \sqrt{2} dx \approx 5.657$

$y_2' = \frac{3}{4}x^{1/2}, L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}} dx \approx 5.759$

$y_3' = \frac{1}{2}x, L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}} dx \approx 5.916$

$y_4' = \frac{5}{16}x^{3/2}, L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3} dx \approx 6.063$

27. $y = \frac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$

When $x = 0, y = \frac{2}{3}$. Thus, the fleeing object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3} \left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} \right] = \left(\frac{1}{2} \right) \frac{x - 1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x - 1)^2}{4x} = \frac{(x + 1)^2}{4x}$$

$$s = \int_0^1 \frac{x + 1}{2x^{1/2}} dx = \frac{1}{2} \int_0^1 (x^{1/2} + x^{-1/2}) dx = \frac{1}{2} \left[\frac{2}{3}x^{3/2} + 2x^{1/2} \right]_0^1 = \frac{4}{3} = 2 \left(\frac{2}{3} \right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

29. $y = 20 \cosh \frac{x}{20}, -20 \leq x \leq 20$

$$y' = \sinh \frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2 \frac{x}{20} = \cosh^2 \frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh \frac{x}{20} dx = 2 \int_0^{20} \cosh \frac{x}{20} dx = 2(20) \sinh \frac{x}{20} \Big|_0^{20}$$

$$= 40 \sinh(1) \approx 47.008 \text{ m.}$$

$$31. \quad y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = \frac{9}{9 - x^2}$$

$$s = \int_0^2 \sqrt{\frac{9}{9 - x^2}} dx$$

$$= \int_0^2 \frac{3}{\sqrt{9 - x^2}} dx$$

$$= \left[3 \arcsin \frac{x}{3} \right]_0^2$$

$$= 3 \left(\arcsin \frac{2}{3} - \arcsin 0 \right)$$

$$= 3 \arcsin \frac{2}{3} \approx 2.1892$$

$$35. \quad y = \frac{x^3}{6} + \frac{1}{2x}$$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2} \right)^2, [1, 2]$$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x} \right) \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3} \right) dx$$

$$= 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2} \right]_1^2 = \frac{47\pi}{16}$$

$$33. \quad y = \frac{x^3}{3}$$

$$y' = x^2, [0, 3]$$

$$S = 2\pi \int_0^3 \frac{x^3}{3} \sqrt{1 + x^4} dx$$

$$= \frac{\pi}{6} \int_0^3 (1 + x^4)^{1/2} (4x^3) dx$$

$$= \left[\frac{\pi}{9} (1 + x^4)^{3/2} \right]_0^3$$

$$= \frac{\pi}{9} (82\sqrt{82} - 1) \approx 258.85$$

$$37. \quad y = \sqrt[3]{x} + 2$$

$$y' = \frac{1}{3x^{2/3}}, [1, 8]$$

$$S = 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9x^{4/3}}} dx$$

$$= \frac{2\pi}{3} \int_1^8 x^{1/3} \sqrt{9x^{4/3} + 1} dx$$

$$= \frac{\pi}{18} \int_1^8 (9x^{4/3} + 1)^{1/2} (12x^{1/3}) dx$$

$$= \left[\frac{\pi}{27} (9x^{4/3} + 1)^{3/2} \right]_1^8$$

$$= \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10}) \approx 199.48$$

$$39. \quad y = \sin x$$

$$y' = \cos x, [0, \pi]$$

$$S = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} dx$$

$$\approx 14.4236$$

41. A rectifiable curve is one that has a finite arc length.

43. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(di) \sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(di) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i} \right)^2} \Delta x_i.$$

45. $y = \frac{hx}{r}$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$\begin{aligned} S &= 2\pi \int_0^r x \sqrt{\frac{r^2 + h^2}{r^2}} dx \\ &= \left[\frac{2\pi \sqrt{r^2 + h^2}}{r} \left(\frac{x^2}{2} \right) \right]_0^r = \pi r \sqrt{r^2 + h^2} \end{aligned}$$

47. $y = \sqrt{9 - x^2}$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$\begin{aligned} S &= 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}} dx \\ &= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}} dx \\ &= \left[-6\pi \sqrt{9 - x^2} \right]_0^2 \\ &= 6\pi(3 - \sqrt{5}) \approx 14.40 \end{aligned}$$

See figure in Exercise 48.

49. $y = \frac{1}{3}x^{1/2} - x^{3/2}$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$\begin{aligned} S &= 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2} \right) \sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2} dx = \frac{2\pi}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2} \right) (x^{-1/2} + 9x^{1/2}) dx \\ &= \frac{\pi}{3} \int_0^{1/3} \left(\frac{1}{3} + 2x - 9x^2 \right) dx = \frac{\pi}{3} \left[\frac{1}{3}x + x^2 - 3x^3 \right]_0^{1/3} = \frac{\pi}{27} \text{ ft}^2 \approx 0.1164 \text{ ft}^2 \approx 16.8 \text{ in}^2 \end{aligned}$$

Amount of glass needed: $V = \frac{\pi}{27} \left(\frac{0.015}{12} \right) \approx 0.00015 \text{ ft}^3 \approx 0.25 \text{ in}^3$

51. (a) $y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$

(b) $\text{Area} = \int_0^{400} f(x) dx \approx 131,734.5 \text{ square feet}$

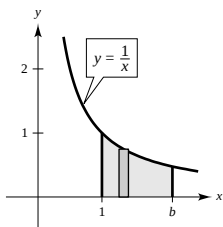
$$\approx 3.0 \text{ acres}$$

(Answers will vary.)

(c) $L = \int_0^{400} \sqrt{1 + f'(x)^2} dx \approx 794.9 \text{ feet}$

(Answers will vary.)

53. (a) $V = \pi \int_1^b \frac{1}{x^2} dx = \left[-\frac{\pi}{x} \right]_1^b = \pi \left(1 - \frac{1}{b} \right)$



(b) $\begin{aligned} S &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2} \right)^2} dx \\ &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx \\ &= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx \end{aligned}$

—CONTINUED—

53. —CONTINUED—

$$(c) \lim_{b \rightarrow \infty} V = \lim_{b \rightarrow \infty} \pi \left(1 - \frac{1}{b} \right) = \pi$$

(d) Since

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx > \int_1^b \frac{1}{x} dx = \left[\ln x \right]_1^b = \ln b$$

and $\lim_{b \rightarrow \infty} \ln b \rightarrow \infty$. Thus,

$$\lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx = \infty.$$

55. (a) Area of circle with radius L : $A = \pi L^2$ Area of sector with central angle θ (in radians)

$$S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi} (\pi L^2) = \frac{1}{2} L^2 \theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left(\frac{s}{L} \right) = \frac{1}{2} L s = \frac{1}{2} L (2\pi r) = \pi r L$$

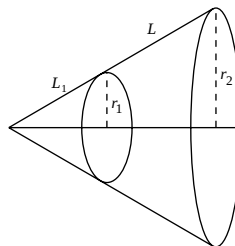
(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$\begin{aligned} S &= \pi r_2(L + L_1) - \pi r_1 L_1 \\ &= \pi r_2 L + \pi L_1(r_2 - r_1) \end{aligned}$$

$$\text{By similar triangles, } \frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow L r_1 = L_1(r_2 - r_1)$$

Hence,

$$\begin{aligned} S &= \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1 \\ &= \pi L(r_1 + r_2). \end{aligned}$$



Section 6.5 Work

1. $W = Fd = (100)(10) = 1000 \text{ ft} \cdot \text{lb}$

3. $W = Fd = (112)(4) = 448 \text{ joules (newton-meters)}$

5. Work equals force times distance, $W = FD$.

7. Since the work equals the area under the force function, you have $(c) < (d) < (a) < (b)$.

9. $F(x) = kx$

$$5 = k(4)$$

$$k = \frac{5}{4}$$

$$W = \int_0^7 \frac{5}{4} x dx = \left[\frac{5}{8} x^2 \right]_0^7$$

$$= \frac{245}{8} \text{ in} \cdot \text{lb}$$

$$= 30.625 \text{ in} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$$

11. $F(x) = kx$

$$250 = k(30) \Rightarrow k = \frac{25}{3}$$

$$W = \int_{20}^{50} F(x) dx = \int_{20}^{50} \frac{25}{3} x dx = \left[\frac{25x^2}{6} \right]_{20}^{50}$$

$$= 8750 \text{ n} \cdot \text{cm} = 87.5 \text{ joules or Nm}$$

13. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$$W = \int_0^{12} \frac{20}{9}x \, dx = \left[\frac{10}{9}x^2 \right]_0^{12} = \frac{40}{3} \text{ ft} \cdot \text{lb}$$

15. $W = 18 = \int_0^{1/3} kx \, dx = \left[\frac{kx^2}{2} \right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$$W = \int_{1/3}^{7/12} 324x \, dx = 162x^2 \Big|_{1/3}^{7/12} = 37.125 \text{ ft} \cdot \text{lbs}$$

[Note: 4 inches = $\frac{1}{3}$ foot]

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$s = \frac{k}{(4000)^2}$

$k = 80,000,000$

$F(x) = \frac{80,000,000}{x^2}$

(a) $W = \int_{4000}^{4100} \frac{80,000,000}{x^2} \, dx = \left[-\frac{80,000,000}{x} \right]_{4000}^{4100} \approx 487.8 \text{ mi} \cdot \text{tons}$

$\approx 5.15 \times 10^9 \text{ ft} \cdot \text{lb}$

(b) $W = \int_{4000}^{4300} \frac{80,000,000}{x^2} \, dx \approx 1395.3 \text{ mi} \cdot \text{ton}$

$\approx 1.47 \times 10^{10} \text{ ft} \cdot \text{lb}$

19. Assume that the earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$10 = \frac{k}{(4000)^2}$

$k = 160,000,000$

$F(x) = \frac{160,000,000}{x^2}$

(a) $W = \int_{4000}^{15,000} \frac{160,000,000}{x^2} \, dx = \left[-\frac{160,000,000}{x} \right]_{4000}^{15,000} \approx -10,666.667 + 40,000$

$= 29,333.333 \text{ mi} \cdot \text{ton}$

$\approx 2.93 \times 10^4 \text{ mi} \cdot \text{ton}$

$\approx 3.10 \times 10^{11} \text{ ft} \cdot \text{lb}$

(b) $W = \int_{4000}^{26,000} \frac{160,000,000}{x^2} \, dx = \left[-\frac{160,000,000}{x} \right]_{4000}^{26,000} \approx -6,153.846 + 40,000$

$= 33,846.154 \text{ mi} \cdot \text{ton}$

$\approx 3.38 \times 10^4 \text{ mi} \cdot \text{ton}$

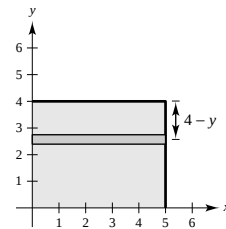
$\approx 3.57 \times 10^{11} \text{ ft} \cdot \text{lb}$

21. Weight of each layer: $62.4(20) \Delta y$

Distance: $4 - y$

(a) $W = \int_2^4 62.4(20)(4 - y) \, dy = \left[4992y - 624y^2 \right]_2^4 = 2496 \text{ ft} \cdot \text{lb}$

(b) $W = \int_0^4 62.4(20)(4 - y) \, dy = \left[4992y - 624y^2 \right]_0^4 = 9984 \text{ ft} \cdot \text{lb}$

23. Volume of disk: $\pi(2)^2 \Delta y = 4\pi \Delta y$

Weight of disk of water: $9800(4\pi) \Delta y$

Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi \, dy = 39,200\pi \int_0^4 (5 - y) \, dy$$

$$= 39,200\pi \left[5y - \frac{y^2}{2} \right]_0^4$$

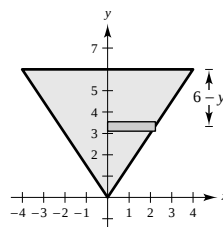
$$= 39,200\pi(12) = 470,400\pi \text{ newton-meters}$$

25. Volume of disk: $\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk: $62.4\pi\left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9} \int_0^6 (6 - y)y^2 dy = \frac{4}{9}(62.4)\pi \left[2y^3 - \frac{1}{4}y^4 \right]_0^6 = 2995.2\pi \text{ ft} \cdot \text{lb}$$

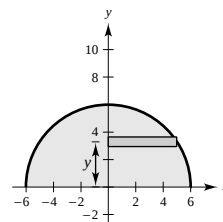


27. Volume of disk: $\pi(\sqrt{36 - y^2})^2 \Delta y$

Weight of disk: $62.4\pi(36 - y^2) \Delta y$

Distance: y

$$\begin{aligned} W &= 62.4\pi \int_0^6 y(36 - y^2) dy \\ &= 62.4\pi \int_0^6 (36y - y^3) dy = 62.4\pi \left[18y^2 - \frac{1}{4}y^4 \right]_0^6 \\ &= 20,217.6\pi \text{ ft} \cdot \text{lb} \end{aligned}$$



29. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2} \Delta y$

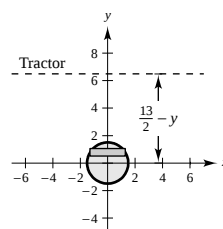
Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2} \Delta y$

Distance: $\frac{13}{2} - y$

$$\begin{aligned} W &= \int_{-1.5}^{1.5} 42(8)\sqrt{(9/4) - y^2} \left(\frac{13}{2} - y \right) dy \\ &= 336 \left[\frac{13}{2} \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} dy - \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} y dy \right] \end{aligned}$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336 \left(\frac{13}{2} \right) \pi \left(\frac{3}{2} \right)^2 \left(\frac{1}{2} \right) = 2457\pi \text{ ft} \cdot \text{lb}$$



31. Weight of section of chain: $3 \Delta y$

Distance: $15 - y$

$$\begin{aligned} W &= 3 \int_0^{15} (15 - y) dy \\ &= \left[-\frac{3}{2}(15 - y)^2 \right]_0^{15} \\ &= 337.5 \text{ ft} \cdot \text{lb} \end{aligned}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section: $3 \Delta y$

Distance: $10 - y$

$$\begin{aligned} W_2 &= 3 \int_0^{10} (10 - y) dy = \left[-\frac{3}{2}(10 - y)^2 \right]_0^{10} \\ &= 150 \text{ ft} \cdot \text{lb} \end{aligned}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

35. Weight of section of chain: $3 \Delta y$ Distance: $15 - 2y$

$$W = 3 \int_0^{7.5} (15 - 2y) dy = \left[-\frac{3}{4}(15 - 2y)^2 \right]_0^{7.5} \\ = \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

37. Work to pull up the ball: $W_1 = 500(15) = 7500 \text{ ft} \cdot \text{lb}$

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1 \Delta y$ Distance: $15 - x$

$$W_2 = \int_0^{15} (15 - x) dx = \left[-\frac{1}{2}(15 - x)^2 \right]_0^{15} \\ = 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375 \\ = 7987.5 \text{ ft} \cdot \text{lb}$$

$$39. \quad p = \frac{k}{V}$$

$$1000 = \frac{k}{2}$$

$$k = 2000$$

$$W = \int_2^3 \frac{2000}{V} dV = \left[2000 \ln |V| \right]_2^3 \\ = 2000 \ln \left(\frac{3}{2} \right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

$$41. F(x) = \frac{k}{(2 - x)^2}$$

$$W = \int_{-2}^1 \frac{k}{(2 - x)^2} dx = \left[\frac{k}{2 - x} \right]_{-2}^1 = k \left(1 - \frac{1}{4} \right) \\ = \frac{3k}{4} (\text{units of work})$$

$$43. W = \int_0^5 1000[1.8 - \ln(x + 1)] dx \approx 3249.44 \text{ ft} \cdot \text{lb}$$

$$45. W = \int_0^5 100x\sqrt{125 - x^3} dx \approx 10,330.3 \text{ ft} \cdot \text{lb}$$

Section 6.6 Moments, Centers of Mass, and Centroids

$$1. \quad \bar{x} = \frac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\frac{6}{7}$$

$$3. \quad \bar{x} = \frac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$$

$$5. (a) \quad \bar{x} = \frac{(7 + 5) + (8 + 5) + (12 + 5) + (15 + 5) + (18 + 5)}{5} = 17 = 12 + 5$$

$$(b) \quad \bar{x} = \frac{12(-6 - 3) + 1(-4 - 3) + 6(-2 - 3) + 3(0 - 3) + 11(8 - 3)}{12 + 1 + 6 + 3 + 11} = \frac{-99}{33} = -3$$

$$7. \quad 50x = 75(L - x) = 75(10 - x)$$

$$50x = 750 - 75x$$

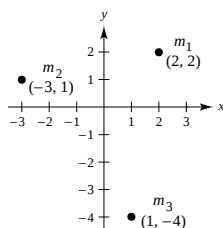
$$125x = 750$$

$$x = 6 \text{ feet}$$

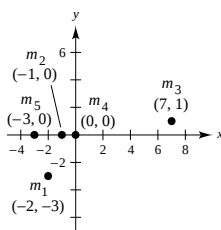
$$9. \quad \bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$$

$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$

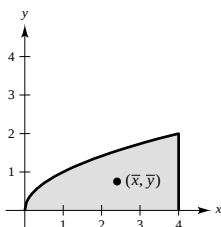
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9} \right)$$



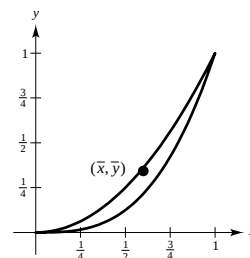
$$\begin{aligned}
 11. \quad \bar{x} &= \frac{3(-2) + 4(-1) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{8} \\
 \bar{y} &= \frac{3(-3) + 4(0) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{16} \\
 (\bar{x}, \bar{y}) &= \left(-\frac{7}{8}, -\frac{7}{16}\right)
 \end{aligned}$$



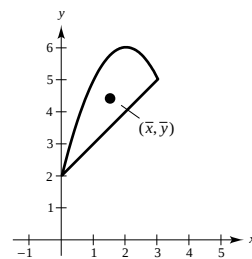
$$\begin{aligned}
 13. \quad m &= \rho \int_0^4 \sqrt{x} \, dx = \left[\frac{2\rho}{3} x^{3/2} \right]_0^4 = \frac{16\rho}{3} \\
 M_x &= \rho \int_0^4 \frac{\sqrt{x}}{2} (\sqrt{x}) \, dx = \left[\rho \frac{x^2}{4} \right]_0^4 = 4\rho \\
 \bar{y} &= \frac{M_x}{m} = 4\rho \left(\frac{3}{16\rho} \right) = \frac{3}{4} \\
 M_y &= \rho \int_0^4 x \sqrt{x} \, dx = \left[\rho \frac{2}{5} x^{5/2} \right]_0^4 = \frac{64\rho}{5} \\
 \bar{x} &= \frac{M_y}{m} = \frac{64\rho}{5} \left(\frac{3}{16\rho} \right) = \frac{12}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{12}{5}, \frac{3}{4} \right)
 \end{aligned}$$



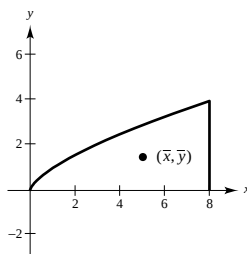
$$\begin{aligned}
 15. \quad m &= \rho \int_0^1 (x^2 - x^3) \, dx = \rho \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\rho}{12} \\
 M_x &= \rho \int_0^1 \frac{(x^2 + x^3)}{2} (x^2 - x^3) \, dx = \frac{\rho}{2} \int_0^1 (x^4 - x^6) \, dx = \frac{\rho}{2} \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{\rho}{35} \\
 \bar{y} &= \frac{M_x}{m} = \frac{\rho}{35} \left(\frac{12}{\rho} \right) = \frac{12}{35} \\
 M_y &= \rho \int_0^1 x(x^2 - x^3) \, dx = \rho \int_0^1 (x^3 - x^4) \, dx = \rho \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{\rho}{20} \\
 \bar{x} &= \frac{M_y}{m} = \frac{\rho}{20} \left(\frac{12}{\rho} \right) = \frac{3}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{3}{5}, \frac{12}{35} \right)
 \end{aligned}$$



$$\begin{aligned}
 17. \quad m &= \rho \int_0^3 [(-x^2 + 4x + 2) - (x + 2)] \, dx = -\rho \left[\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \frac{9\rho}{2} \\
 M_x &= \rho \int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2} \right] [(-x^2 + 4x + 2) - (x + 2)] \, dx \\
 &= \frac{\rho}{2} \int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x) \, dx = \frac{\rho}{2} \int_0^3 (x^4 - 8x^3 + 11x^2 + 12x) \, dx \\
 &= \frac{\rho}{2} \left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2 \right]_0^3 = \frac{99\rho}{5} \\
 \bar{y} &= \frac{M_x}{m} = \frac{99\rho}{5} \left(\frac{2}{9\rho} \right) = \frac{22}{5} \\
 M_y &= \rho \int_0^3 x[(-x^2 + 4x + 2) - (x + 2)] \, dx = \rho \int_0^3 (-x^3 + 3x^2) \, dx = \rho \left[-\frac{x^4}{4} + x^3 \right]_0^3 = \frac{27\rho}{4} \\
 \bar{x} &= \frac{M_y}{m} = \frac{27\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{3}{2} \\
 (\bar{x}, \bar{y}) &= \left(\frac{3}{2}, \frac{22}{5} \right)
 \end{aligned}$$

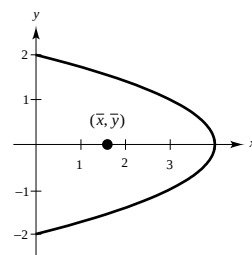


$$\begin{aligned}
 19. \quad m &= \rho \int_0^8 x^{2/3} dx = \rho \left[\frac{3}{5} x^{5/3} \right]_0^8 = \frac{96\rho}{5} \\
 M_x &= \rho \int_0^8 \frac{x^{2/3}}{2} (x^{2/3}) dx = \frac{\rho}{2} \left[\frac{3}{7} x^{7/3} \right]_0^8 = \frac{192\rho}{7} \\
 \bar{y} &= \frac{M_x}{m} = \frac{192\rho}{7} \left(\frac{5}{96\rho} \right) = \frac{10}{7} \\
 M_y &= \rho \int_0^8 x(x^{2/3}) dx = \rho \left[\frac{3}{8} x^{8/3} \right]_0^8 = 96\rho \\
 \bar{x} &= \frac{M_y}{m} = 96\rho \left(\frac{5}{96\rho} \right) = 5 \\
 (\bar{x}, \bar{y}) &= \left(5, \frac{10}{7} \right)
 \end{aligned}$$

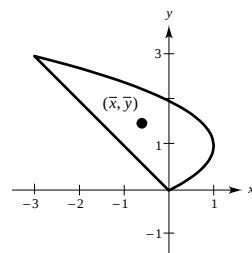


$$\begin{aligned}
 21. \quad m &= 2\rho \int_0^2 (4 - y^2) dy = 2\rho \left[4y - \frac{y^3}{3} \right]_0^2 = \frac{32\rho}{3} \\
 M_y &= 2\rho \int_0^2 \left(\frac{4 - y^2}{2} \right) (4 - y^2) dy = \rho \left[16y - \frac{8}{3}y^3 + \frac{y^5}{5} \right]_0^2 = \frac{256\rho}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{256\rho}{15} \left(\frac{3}{32\rho} \right) = \frac{8}{5} \\
 (\bar{x}, \bar{y}) &= \left(\frac{8}{5}, 0 \right)
 \end{aligned}$$

By symmetry, M_x and $\bar{y} = 0$.



$$\begin{aligned}
 23. \quad m &= \rho \int_0^3 [(2y - y^2) - (-y)] dy = \rho \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = \frac{9\rho}{2} \\
 M_y &= \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2} [(2y - y^2) - (-y)] dy = \frac{\rho}{2} \int_0^3 (y - y^2)(3y - y^2) dy \\
 &= \frac{\rho}{2} \int_0^3 (y^4 - 4y^3 + 3y^2) dy = \frac{\rho}{2} \left[\frac{y^5}{5} - y^4 + y^3 \right]_0^3 = -\frac{27\rho}{10} \\
 \bar{x} &= \frac{M_y}{m} = -\frac{27\rho}{10} \left(\frac{2}{9\rho} \right) = -\frac{3}{5} \\
 M_x &= \rho \int_0^3 y[(2y - y^2) - (-y)] dy = \rho \int_0^3 (3y^2 - y^3) dy = \rho \left[y^3 - \frac{y^4}{4} \right]_0^3 = \frac{27\rho}{4} \\
 \bar{y} &= \frac{M_x}{m} = \frac{27\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{3}{2} \\
 (\bar{x}, \bar{y}) &= \left(-\frac{3}{5}, \frac{3}{2} \right)
 \end{aligned}$$



$$\begin{aligned}
 25. \quad A &= \int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \\
 M_x &= \frac{1}{2} \int_0^1 (x^2 - x^4) dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15} \\
 M_y &= \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{12}
 \end{aligned}$$

$$27. A = \int_0^3 (2x + 4) dx = \left[x^2 + 4x \right]_0^3 = 9 + 12 = 21$$

$$M_x = \frac{1}{2} \int_0^3 (2x + 4)^2 dx = \int_0^3 (2x^2 + 8x + 8) dx = \left[\frac{2x^3}{3} + 4x^2 + 8x \right]_0^3 = 18 + 36 + 24 = 78$$

$$M_y = \int_0^3 (2x^2 + 4x) dx = \left[\frac{2x^3}{3} + 2x^2 \right]_0^3 = 18 + 18 = 36$$

$$29. m = \rho \int_0^5 10x \sqrt{125 - x^3} dx \approx 1033.0\rho$$

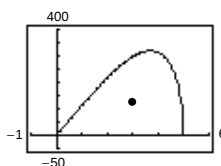
$$M_x = \rho \int_0^5 \left(\frac{10x \sqrt{125 - x^3}}{2} \right) (10x \sqrt{125 - x^3}) dx = 50\rho \int_0^5 x^2 (125 - x^3) dx = \frac{3,124,375\rho}{24} \approx 130,208\rho$$

$$M_y = \rho \int_0^5 10x^2 \sqrt{125 - x^3} dx = -\frac{10\rho}{3} \int_0^5 \sqrt{125 - x^3} (-3x^2) dx = \frac{12,500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is (3.0, 126.0).



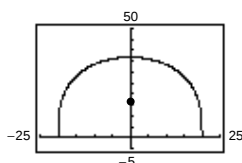
$$31. m = \rho \int_{-20}^{20} 5 \sqrt[3]{400 - x^2} dx \approx 1239.76\rho$$

$$M_x = \rho \int_{-20}^{20} \frac{5 \sqrt[3]{400 - x^2}}{2} (5 \sqrt[3]{400 - x^2}) dx$$

$$= \frac{25\rho}{2} \int_{-20}^{20} (400 - x^2)^{2/3} dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 16.2).



$$33. A = \frac{1}{2}(2a)c = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac} \right) \frac{1}{2} \int_0^c \left[\left(\frac{b-a}{c}y + a \right)^2 - \left(\frac{b+a}{c}y - a \right)^2 \right] dy$$

$$= \frac{1}{2ac} \int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2 \right] dy$$

$$= \frac{1}{2ac} \left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3 \right]_0^c = \frac{1}{2ac} \left(\frac{2}{3}abc \right) = \frac{b}{3}$$

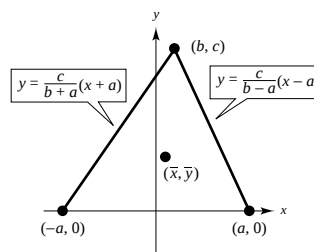
$$\bar{y} = \frac{1}{ac} \int_0^c y \left[\left(\frac{b-a}{c}y + a \right) - \left(\frac{b+a}{c}y - a \right) \right] dy$$

$$= \frac{1}{ac} \int_0^c y \left(-\frac{2a}{c}y + 2a \right) dy = \frac{2}{c} \int_0^c \left(y - \frac{y^2}{c} \right) dy$$

$$= \frac{2}{c} \left[\frac{y^2}{2} - \frac{y^3}{3c} \right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3} \right)$$

In Exercise 566 of Section P.2, you found that $(b/3, c/3)$ is the point of intersection of the medians.



35. $A = \frac{c}{2}(a + b)$

$$\frac{1}{A} = \frac{2}{c(a + b)}$$

$$\bar{x} = \frac{2}{c(a + b)} \int_0^c x \left(\frac{b-a}{c}x + a \right) dx = \frac{2}{c(a + b)} \int_0^c \left(\frac{b-a}{c}x^2 + ax \right) dx = \frac{2}{c(a + b)} \left[\frac{b-a}{c} \frac{x^3}{3} + \frac{ax^2}{2} \right]_0^c$$

$$= \frac{2}{c(a + b)} \left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2} \right] = \frac{2}{c(a + b)} \left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6} \right] = \frac{c(2b + a)}{3(a + b)} = \frac{(a + 2b)c}{3(a + b)}$$

$$\bar{y} = \frac{2}{c(a + b)} \frac{1}{2} \int_0^c \left(\frac{b-a}{c}x + a \right)^2 dx = \frac{1}{c(a + b)} \int_0^c \left[\left(\frac{b-a}{c} \right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2 \right] dx$$

$$= \frac{1}{c(a + b)} \left[\left(\frac{b-a}{c} \right)^2 \frac{x^3}{3} + \frac{2a(b-a)}{c} \frac{x^2}{2} + a^2x \right]_0^c = \frac{1}{c(a + b)} \left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c \right]$$

$$= \frac{1}{3c(a + b)} [(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$$

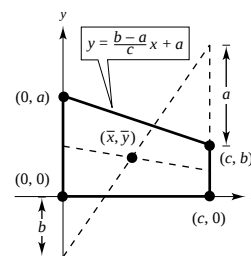
$$= \frac{1}{3(a + b)} [b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a + b)}$$

Thus, $(\bar{x}, \bar{y}) = \left(\frac{(a + 2b)c}{3(a + b)}, \frac{a^2 + ab + b^2}{3(a + b)} \right)$.

The one line passes through $(0, a/2)$ and $(c, b/2)$. Its equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$.

The other line passes through $(0, -b)$ and $(c, a + b)$. Its equation is $y = \frac{a + 2b}{c}x - b$.

$(\bar{x}, \bar{y})$ is the point of intersection of these two lines.



37. $\bar{x} = 0$ by symmetry

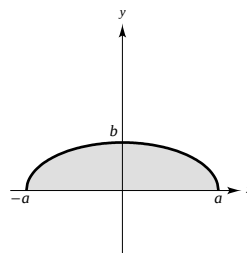
$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

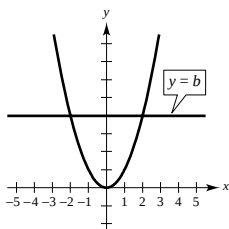
$$\bar{y} = \frac{2}{\pi ab} \frac{1}{2} \int_{-a}^a \left(\frac{b}{a} \sqrt{a^2 - x^2} \right)^2 dx$$

$$= \frac{1}{\pi ab} \left(\frac{b^2}{a^2} \right) \left[a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{b}{\pi a^3} \left[\frac{4a^3}{3} \right] = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi} \right)$$



39. (a)



(b) $\bar{x} = 0$ by symmetry

(c) $M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2) dx = 0$ because $bx - x^3$ is odd

(d) $\bar{y} > \frac{b}{2}$ since there is more area above $y = \frac{b}{2}$ than below

(e) $M_x = \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b + x^2)(b - x^2)}{2} dx$

$$= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2} \left[b^2x - \frac{x^5}{5} \right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5}$$

$$A = \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2) dx = \left[bx - \frac{x^3}{3} \right]_{-\sqrt{b}}^{\sqrt{b}}$$

$$= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3} \right) 2 = 4 \frac{b\sqrt{b}}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b.$$

41. (a)
- $\bar{x} = 0$
- by symmetry

$$A = 2 \int_0^{40} f(x) dx = \frac{2(40)}{3(4)} [30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3} (278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2} dx = \frac{40}{3(4)} [30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3} (7216) = \frac{72160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72160/3}{5560/3} = \frac{72160}{5560} \approx 12.98$$

$$(\bar{x}, \bar{y}) = (0, 12.98)$$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$

(c) $\bar{y} = \frac{M_x}{A} \approx \frac{23697.68}{1843.54} \approx 12.85$

$$(\bar{x}, \bar{y}) = (0, 12.85)$$

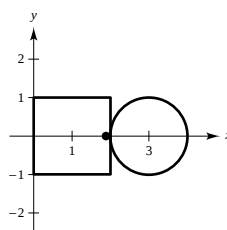
43. Centroids of the given regions: (1, 0) and (3, 0)

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0 \right) \approx (1.88, 0)$$



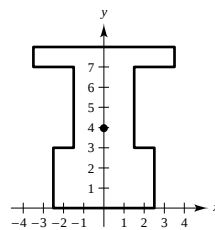
45. Centroids of the given regions:
- $\left(0, \frac{3}{2}\right)$
- , (0, 5), and
- $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34}\right)$$



47. Centroids of the given regions: (1, 0) and (3, 0)

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0 \right) \approx (2.22, 0)$$

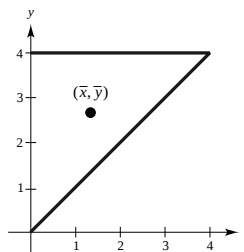
49. $V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14$

$$51. A = \frac{1}{2}(4)(4) = 8$$

$$\bar{y} = \left(\frac{1}{8}\right) \frac{1}{2} \int_0^4 (4+x)(4-x) dx = \frac{1}{16} \left[16x - \frac{x^3}{3} \right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi r A = 2\pi \left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



$$53. m = m_1 + \cdots + m_n$$

$$M_y = m_1 x_1 + \cdots + m_n x_n$$

$$M_x = m_1 y_1 + \cdots + m_n y_n$$

$$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$$

$$55. (a) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$$

$$(b) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$$

$$(c) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$$

$$(d) \text{ No.}$$

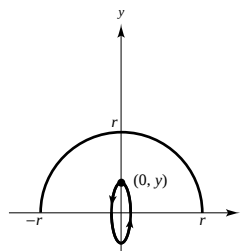
57. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius y .

$$d = 2\pi y$$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.



$$59. A = \int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$$

$$m = \rho A = \frac{\rho}{n+1}$$

$$M_x = \frac{\rho}{2} \int_0^1 (x^n)^2 dx = \left[\frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n+1} \right]_0^1 = \frac{\rho}{2(2n+1)}$$

$$M_y = \rho \int_0^1 x(x^n) dx = \left[\rho \cdot \frac{x^{n+2}}{n+2} \right]_0^1 = \frac{\rho}{n+2}$$

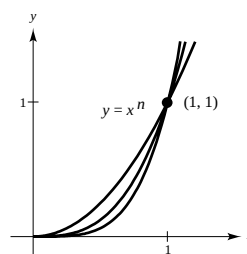
$$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$$

$$\text{Centroid: } \left(\frac{n+1}{n+2}, \frac{n+1}{4n+2}\right)$$

$$\text{As } n \rightarrow \infty, (\bar{x}, \bar{y}) \rightarrow \left(1, \frac{1}{4}\right).$$

The graph approaches the x -axis and the line $x = 1$ as $n \rightarrow \infty$.



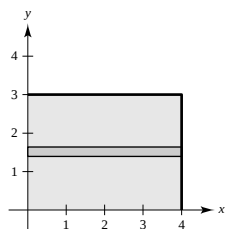
Section 6.7 Fluid Pressure and Fluid Force

$$1. F = PA = [62.4(5)](3) = 936 \text{ lb}$$

$$5. h(y) = 3 - y$$

$$L(y) = 4$$

$$\begin{aligned} F &= 62.4 \int_0^3 (3 - y)(4) dy \\ &= 249.6 \int_0^3 (3 - y) dy \\ &= 249.6 \left[3y - \frac{y^2}{2} \right]_0^3 = 1123.2 \text{ lb} \end{aligned}$$

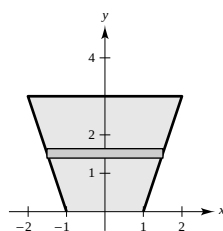


$$\begin{aligned} 3. F &= 62.4(h + 2)(6) - (62.4)(h)(6) \\ &= 62.4(2)(6) = 748.8 \text{ lb} \end{aligned}$$

$$7. h(y) = 3 - y$$

$$L(y) = 2\left(\frac{y}{3} + 1\right)$$

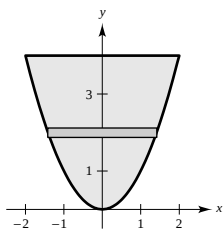
$$\begin{aligned} F &= 2(62.4) \int_0^3 (3 - y)\left(\frac{y}{3} + 1\right) dy \\ &= 124.8 \int_0^3 \left(3 - \frac{y^2}{3}\right) dy \\ &= 124.8 \left[3y - \frac{y^3}{9} \right]_0^3 = 748.8 \text{ lb} \end{aligned}$$



$$9. h(y) = 4 - y$$

$$L(y) = 2\sqrt{y}$$

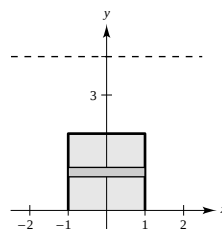
$$\begin{aligned} F &= 2(62.4) \int_0^4 (4 - y)\sqrt{y} dy \\ &= 124.8 \int_0^4 (4y^{1/2} - y^{3/2}) dy \\ &= 124.8 \left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5} \right]_0^4 = 1064.96 \text{ lb} \end{aligned}$$



$$11. h(y) = 4 - y$$

$$L(y) = 2$$

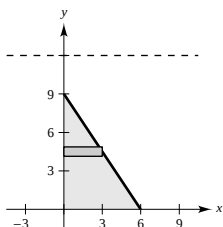
$$\begin{aligned} F &= 9800 \int_0^2 2(4 - y) dy \\ &= 9800 \left[8y - y^2 \right]_0^2 = 117,600 \text{ Newtons} \end{aligned}$$



13. $h(y) = 12 - y$

$$L(y) = 6 - \frac{2y}{3}$$

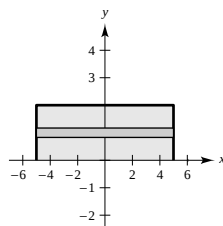
$$\begin{aligned}
 F &= 9800 \int_0^9 (12 - y) \left(6 - \frac{2y}{3} \right) dy \\
 &= 9800 \left[72y - 7y^2 + \frac{2y^3}{9} \right]_0^9 = 2,381,400 \text{ Newtons}
 \end{aligned}$$



15. $h(y) = 2 - y$

$$L(y) = 10$$

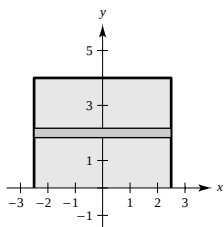
$$\begin{aligned}
 F &= 140.7 \int_0^2 (2 - y)(10) dy \\
 &= 1407 \int_0^2 (2 - y) dy \\
 &= 1407 \left[2y - \frac{y^2}{2} \right]_0^2 = 2814 \text{ lb}
 \end{aligned}$$



17. $h(y) = 4 - y$

$$L(y) = 6$$

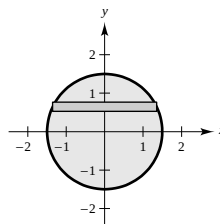
$$\begin{aligned}
 F &= 140.7 \int_0^4 (4 - y)(6) dy \\
 &= 844.2 \int_0^4 (4 - y) dy \\
 &= 844.2 \left[4y - \frac{y^2}{2} \right]_0^4 = 6753.6 \text{ lb}
 \end{aligned}$$



19. $h(y) = -y$

$$L(y) = 2 \left(\frac{1}{2} \right) \sqrt{9 - 4y^2}$$

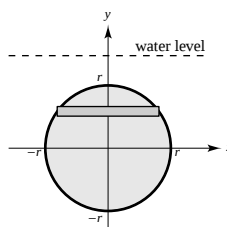
$$\begin{aligned}
 F &= 42 \int_{-3/2}^0 (-y) \sqrt{9 - 4y^2} dy \\
 &= \frac{42}{8} \int_{-3/2}^0 (9 - 4y^2)^{1/2} (-8y) dy \\
 &= \left[\left(\frac{21}{4} \right) \left(\frac{2}{3} \right) (9 - 4y^2)^{3/2} \right]_{-3/2}^0 = 94.5 \text{ lb}
 \end{aligned}$$



21. $h(y) = k - y$

$$L(y) = 2\sqrt{r^2 - y^2}$$

$$\begin{aligned}
 F &= w \int_{-r}^r (k - y) \sqrt{r^2 - y^2} (2) dy \\
 &= w \left[2k \int_{-r}^r \sqrt{r^2 - y^2} dy + \int_{-r}^r \sqrt{r^2 - y^2} (-2y) dy \right]
 \end{aligned}$$



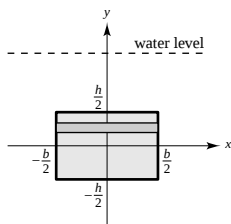
The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r .

$$F = w \left[(2k) \frac{\pi r^2}{2} + 0 \right] = wk\pi r^2$$

23. $h(y) = k - y$

$$L(y) = b$$

$$\begin{aligned} F &= w \int_{-h/2}^{h/2} (k - y)b \, dy \\ &= wb \left[ky - \frac{y^2}{2} \right]_{-h/2}^{h/2} = wb(hk) = wkhb \end{aligned}$$



27. $h(y) = 4 - y$

$$F = 62.4 \int_0^4 (4 - y)L(y) \, dy$$

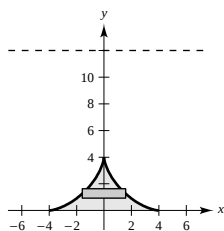
Using Simpson's Rule with $n = 8$ we have:

$$\begin{aligned} F &\approx 62.4 \left(\frac{4 - 0}{3(8)} \right) [0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0] \\ &= 3010.8 \text{ lb} \end{aligned}$$

29. $h(y) = 12 - y$

$$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$$

$$\begin{aligned} F &= 62.4 \int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2} \, dy \\ &\approx 6448.73 \text{ lb} \end{aligned}$$



25. From Exercise 22:

$$F = 64(15)(1)(1) = 960 \text{ lb}$$

 31. (a) If the fluid force is one half of 1123.2 lb, and the height of the water is b , then

$$h(y) = b - y$$

$$L(y) = 4$$

$$F = 62.4 \int_0^b (b - y)(4) \, dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y) \, dy = 2.25$$

$$\left[by - \frac{y^2}{2} \right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

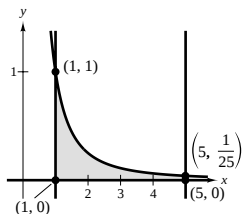
$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

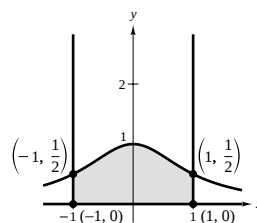
33. $F = F_w = w \int_c^d h(y)L(y) \, dy$, see page 471.

Review Exercises for Chapter 6

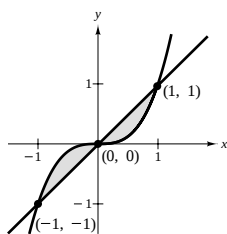
$$1. A = \int_1^5 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^5 = \frac{4}{5}$$



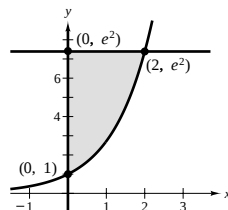
$$\begin{aligned} 3. A &= \int_{-1}^1 \frac{1}{x^2 + 1} dx \\ &= \left[\arctan x \right]_{-1}^1 \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) = \frac{\pi}{2} \end{aligned}$$



$$\begin{aligned} 5. A &= 2 \int_0^1 (x - x^3) dx \\ &= 2 \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

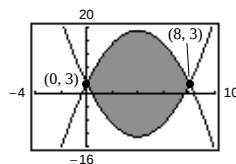


$$\begin{aligned} 7. A &= \int_0^2 (e^2 - e^x) dx \\ &= \left[xe^2 - e^x \right]_0^2 \\ &= e^2 + 1 \end{aligned}$$



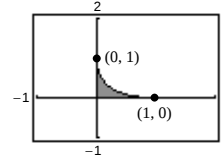
$$\begin{aligned} 9. A &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx \\ &= \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\ &= \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} 11. A &= \int_0^8 [(3 + 8x - x^2) - (x^2 - 8x + 3)] dx \\ &= \int_0^8 (16x - 2x^2) dx \\ &= \left[8x^2 - \frac{2}{3}x^3 \right]_0^8 = \frac{512}{3} \approx 170.667 \end{aligned}$$



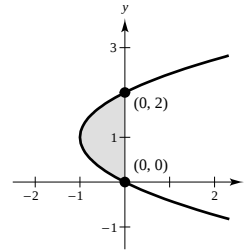
13. $y = (1 - \sqrt{x})^2$

$$\begin{aligned} A &= \int_0^1 (1 - \sqrt{x})^2 dx \\ &= \int_0^1 (1 - 2x^{1/2} + x) dx \\ &= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{1}{6} \approx 0.1667 \end{aligned}$$



15. $x = y^2 - 2y \Rightarrow x + 1 = (y - 1)^2 \Rightarrow y = 1 \pm \sqrt{x + 1}$

$$\begin{aligned} A &= \int_{-1}^0 [(1 + \sqrt{x+1}) - (1 - \sqrt{x+1})] dx = \int_{-1}^0 2\sqrt{x+1} dx \\ A &= \int_0^2 [0 - (y^2 - 2y)] dy = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{1}{3}y^3 \right]_0^2 = \frac{4}{3} \end{aligned}$$



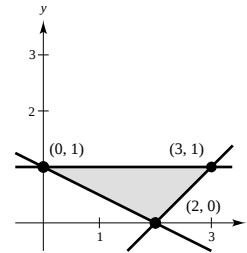
17. $A = \int_0^2 \left[1 - \left(1 - \frac{x}{2} \right) \right] dx + \int_2^3 [1 - (x - 2)] dx$

$$= \int_0^2 \frac{x}{2} dx + \int_2^3 (3 - x) dx$$

$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$

$$y = x - 2 \Rightarrow x = y + 2, y = 1$$

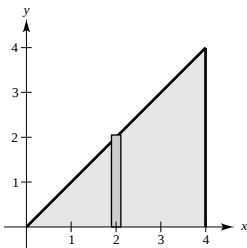
$$\begin{aligned} A &= \int_0^1 [(y + 2) - (2 - 2y)] dy \\ &= \int_0^1 3y dy = \left[\frac{3}{2}y^2 \right]_0^1 = \frac{3}{2} \end{aligned}$$



19. Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

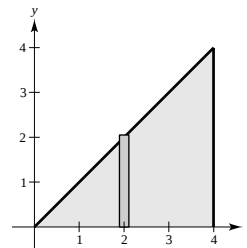
21. (a) **Disk**

$$V = \pi \int_0^4 x^2 dx = \left[\frac{\pi x^3}{3} \right]_0^4 = \frac{64\pi}{3}$$



(b) **Shell**

$$V = 2\pi \int_0^4 x^2 dx = \left[\frac{2\pi}{3} x^3 \right]_0^4 = \frac{128\pi}{3}$$

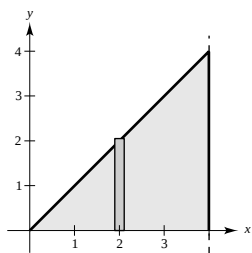


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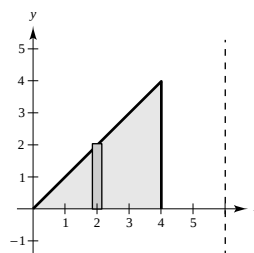
21. —CONTINUED—

(c) **Shell**

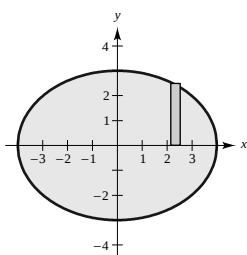
$$\begin{aligned}
 V &= 2\pi \int_0^4 (4-x)x \, dx \\
 &= 2\pi \int_0^4 (4x - x^2) \, dx \\
 &= 2\pi \left[2x^2 - \frac{x^3}{3} \right]_0^4 = \frac{64\pi}{3}
 \end{aligned}$$

(d) **Shell**

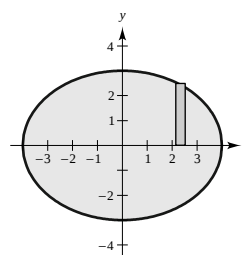
$$\begin{aligned}
 V &= 2\pi \int_0^4 (6-x)x \, dx \\
 &= 2\pi \int_0^4 (6x - x^2) \, dx \\
 &= 2\pi \left[3x^2 - \frac{1}{3}x^3 \right]_0^4 = \frac{160\pi}{3}
 \end{aligned}$$

23. (a) **Shell**

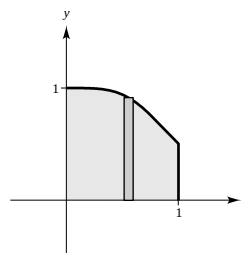
$$\begin{aligned}
 V &= 4\pi \int_0^4 x \left(\frac{3}{4} \right) \sqrt{16 - x^2} \, dx \\
 &= \left[3\pi \left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (16 - x^2)^{3/2} \right]_0^4 = 64\pi
 \end{aligned}$$

(b) **Disk**

$$\begin{aligned}
 V &= 2\pi \int_0^4 \left[\frac{3}{4} \sqrt{16 - x^2} \right]^2 dx \\
 &= \frac{9\pi}{8} \left[16x - \frac{x^3}{3} \right]_0^4 = 48\pi
 \end{aligned}$$

25. **Shell**

$$\begin{aligned}
 V &= 2\pi \int_0^1 \frac{x}{x^4 + 1} \, dx \\
 &= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1} \, dx \\
 &= \left[\pi \arctan(x^2) \right]_0^1 \\
 &= \pi \left[\frac{\pi}{4} - 0 \right] = \frac{\pi^2}{4}
 \end{aligned}$$



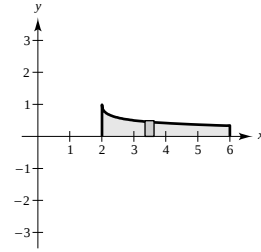
27. Shell

$$u = \sqrt{x-2}$$

$$x = u^2 + 2$$

$$dx = 2u \, du$$

$$\begin{aligned} V &= 2\pi \int_2^6 \frac{x}{1 + \sqrt{x-2}} dx = 4\pi \int_0^2 \frac{(u^2 + 2)u}{1 + u} du \\ &= 4\pi \int_0^2 \frac{u^3 + 2u}{1 + u} du = 4\pi \int_0^2 \left(u^2 - u + 3 - \frac{3}{1 + u} \right) du \\ &= 4\pi \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3 \ln(1 + u) \right]_0^2 = \frac{4\pi}{3}(20 - 9 \ln 3) \approx 42.359 \end{aligned}$$



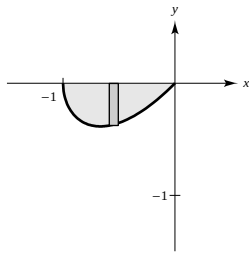
29. Since $y \leq 0$, $A = -\int_{-1}^0 x\sqrt{x+1} \, dx$.

$$u = x + 1$$

$$x = u - 1$$

$$dx = du$$

$$\begin{aligned} A &= -\int_0^1 (u - 1)\sqrt{u} \, du = -\int_0^1 (u^{3/2} - u^{1/2}) \, du \\ &= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_0^1 = \frac{4}{15} \end{aligned}$$



31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disk: $\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2) \, dy = 16\pi$

$$\frac{1}{9} \int_{-3}^{y_0} (9 - y^2) \, dy = 1$$

$$\left[9y - \frac{1}{3}y^3 \right]_{-3}^{y_0} = 9$$

$$\left(9y_0 - \frac{1}{3}y_0^3 \right) - (-27 + 9) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958 \text{ ft}$.

33. $f(x) = \frac{4}{5}x^{5/4}$

$$f'(x) = x^{1/4}$$

$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u - 1)^2$$

$$dx = 2(u - 1) \, du$$

$$\begin{aligned} s &= \int_0^4 \sqrt{1 + \sqrt{x}} \, dx = 2 \int_1^3 \sqrt{u}(u - 1) \, du \\ &= 2 \int_1^3 (u^{3/2} - u^{1/2}) \, du \\ &= 2 \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^3 = \frac{4}{15} \left[u^{3/2}(3u - 5) \right]_1^3 \\ &= \frac{8}{15}(1 + 6\sqrt{3}) \approx 6.076 \end{aligned}$$

35. $y = 300 \cosh\left(\frac{x}{2000}\right) - 280, -2000 \leq x \leq 2000$

$$y' = \frac{3}{20} \sinh\left(\frac{x}{2000}\right)$$

$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20} \sinh\left(\frac{x}{2000}\right) \right]^2} \, dx$$

$$= \frac{1}{20} \int_{-2000}^{2000} \sqrt{400 + 9 \sinh^2\left(\frac{x}{2000}\right)} \, dx$$

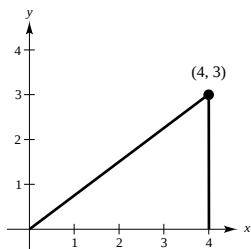
$$\approx 4018.2 \text{ ft (by Simpson's Rule or graphing utility)}$$

$$37. y = \frac{3}{4}x$$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right) \sqrt{\frac{25}{16}} dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



$$39. F = kx$$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x dx = \left[2x^2\right]_0^5 = 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

$$41. \text{Volume of disk: } \pi\left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Weight of disk: } 62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Distance: } 175 - y$$

$$W = \frac{62.4\pi}{9} \int_0^{150} (175 - y) dy = \frac{62.4\pi}{9} \left[175y - \frac{y^2}{2}\right]_0^{150} = 104,000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

$$43. \text{Weight of section of chain: } 5 \Delta x$$

$$\text{Distance moved: } 10 - x$$

$$W = 5 \int_0^{10} (10 - x) dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10} = 250 \text{ ft} \cdot \text{lb}$$

$$45. W = \int_a^b F(x) dx$$

$$80 = \int_0^4 ax^2 dx = \left[\frac{ax^3}{3}\right]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

$$47. A = \int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \int_0^a (a - 2\sqrt{a}x^{1/2} + x) dx = \left[ax - \frac{4}{3}\sqrt{a}x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$$

$$\frac{1}{A} = \frac{6}{a^2}$$

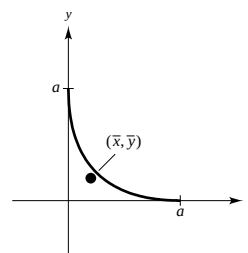
$$\bar{x} = \frac{6}{a^2} \int_0^a x(\sqrt{a} - \sqrt{x})^2 dx = \frac{6}{a^2} \int_0^a (ax - 2\sqrt{a}x^{3/2} + x^2) dx$$

$$\bar{y} = \left(\frac{6}{a^2}\right) \frac{1}{2} \int_0^a (\sqrt{a} - \sqrt{x})^4 dx$$

$$= \frac{3}{a^2} \int_0^a (a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2) dx$$

$$= \frac{3}{a^2} \left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



49. By symmetry, $\bar{x} = 0$.

$$A = 2 \int_0^1 (a^2 - x^2) dx = 2 \left[a^2x - \frac{x^3}{3} \right]_0^1 = \frac{4a^3}{3}$$

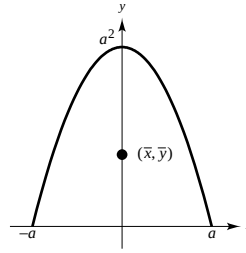
$$\frac{1}{A} = \frac{3}{4a^3}$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{4a^3} \right) \frac{1}{2} \int_{-a}^a (a^2 - x^2)^2 dx \\ &= \frac{6}{8a^3} \int_0^a (a^4 - 2a^2x^2 + x^4) dx \end{aligned}$$

$$= \frac{6}{8a^3} \left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5 \right]_0^a$$

$$= \frac{6}{8a^3} \left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5} \right)$$

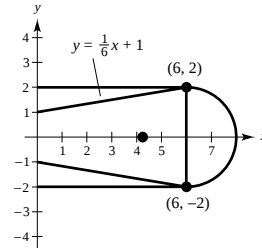


51. $\bar{y} = 0$ by symmetry

For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$\begin{aligned} M_y &= \rho \int_0^6 x \left[\left(\frac{1}{6}x + 1 \right) - \left(-\frac{1}{6}x - 1 \right) \right] dx \\ &= \rho \int_0^6 \left(\frac{1}{3}x^2 + 2x \right) dx = \rho \left[\frac{x^3}{9} + x^2 \right]_0^6 = 60\rho \end{aligned}$$



For the semicircle:

$$m = \left(\frac{1}{2} \right) (\pi)(2)^2 \rho = 2\pi\rho$$

$$M_y = \rho \int_6^8 x \left[\sqrt{4 - (x-6)^2} - (-\sqrt{4 - (x-6)^2}) \right] dx = 2\rho \int_6^8 x \sqrt{4 - (x-6)^2} dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$\begin{aligned} M_y &= 2\rho \int_0^2 (u+6) \sqrt{4-u^2} du = 2\rho \int_0^2 u \sqrt{4-u^2} du + 12\rho \int_0^2 \sqrt{4-u^2} du \\ &= 2\rho \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (4-u^2)^{3/2} \right]_0^2 + 12\rho \left[\frac{\pi(2)^2}{4} \right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4+9\pi)}{3} \end{aligned}$$

Thus, we have:

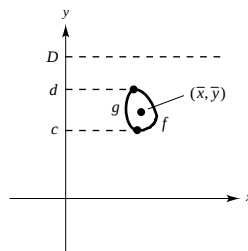
$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4+9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4+9\pi)}{3} \cdot \frac{1}{2\rho(9+\pi)} = \frac{2(9\pi+49)}{3(\pi+9)}$$

The centroid of the blade is $\left(\frac{2(9\pi+49)}{3(\pi+9)}, 0 \right)$.

53. Let D = surface of liquid; ρ = weight per cubic volume.

$$\begin{aligned}
 F &= \rho \int_c^d (D - y)[f(y) - g(y)] dy \\
 &= \rho \left[\int_c^d D[f(y) - g(y)] dy - \int_c^d y[f(y) - g(y)] dy \right] \\
 &= \rho \left[\int_c^d [f(y) - g(y)] dy \right] \left[D - \frac{\int_c^d y[f(y) - g(y)] dy}{\int_c^d [f(y) - g(y)] dy} \right] \\
 &= \rho(\text{Area})(D - \bar{y}) \\
 &= \rho(\text{Area})(\text{depth of centroid})
 \end{aligned}$$



Problem Solving for Chapter 6

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2) dx = \left[\frac{cx^2}{2} - \frac{x^3}{3} \right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c \rightarrow 0^+} \frac{T}{R} = \lim_{c \rightarrow 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

3. (a) $\frac{1}{2}V = \int_0^1 [\pi(2 + \sqrt{1 - y^2})^2 - \pi(2 - \sqrt{1 - y^2})^2] dy$

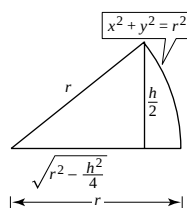
$$\begin{aligned}
 &= \pi \int_0^1 [(4 + 4\sqrt{1 - y^2} + (1 - y^2)) - (4 - 4\sqrt{1 - y^2} + (1 - y^2))] dy \\
 &= 8\pi \int_0^1 \sqrt{1 - y^2} dy \quad (\text{Integral represents } 1/4 \text{ (area of circle)}) \\
 &= 8\pi \left(\frac{\pi}{4} \right) = 2\pi^2 \Rightarrow V = 4\pi^2
 \end{aligned}$$

(b) $(x - R)^2 + y^2 = r^2 \Rightarrow x = R \pm \sqrt{r^2 - y^2}$

$$\begin{aligned}
 \frac{1}{2}V &= \int_0^r [\pi(R + \sqrt{r^2 - y^2})^2 - \pi(R - \sqrt{r^2 - y^2})^2] dy \\
 &= \pi \int_0^r 4R\sqrt{r^2 - y^2} dy \\
 &= \pi(4R) \frac{1}{4} \pi r^2 - \pi^2 r^2 R \\
 V &= 2\pi^2 r^2 R
 \end{aligned}$$

5. $V = 2(2\pi) \int_{\sqrt{r^2 - (h^2/4)}}^r x\sqrt{r^2 - x^2} dx$

$$\begin{aligned}
 &= -2\pi \left[\frac{2}{3}(r^2 - x^2)^{3/2} \right]_{\sqrt{r^2 - (h^2/4)}}^r \\
 &= \frac{-4\pi}{3} \left[-\frac{h^3}{8} \right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!
 \end{aligned}$$



7. (a) Tangent at A:
- $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2$$

$$\text{To find point B: } x^3 = 3x - 2$$

$$x^3 - 3x + 2 = 0$$

$$(x - 1)^2(x + 2) = 0 \Rightarrow B = (-2, -8)$$

$$\text{Tangent at B: } y = x^3, y' = 3x^2$$

$$y + 8 = 12(x + 2)$$

$$y = 12x + 16$$

$$\text{To find point C: } x^3 = 12x + 16$$

$$x^3 - 12x - 16 = 0$$

$$(x + 2)^2(x - 4) = 0 \Rightarrow C = (4, 64)$$

$$\text{Area of } R = \int_{-2}^1 (x^3 - 3x + 2) dx = \frac{27}{4}$$

$$\text{Area of } S = \int_{-2}^4 (12x + 16 - x^3) dx = 108$$

$$\text{Area of } S = 16(\text{area of } R) \quad \left[\frac{\text{area } S}{\text{area } R} = 16 \right]$$

- (b) Tangent at A(a, a
- ³
-):
- $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

$$\text{To find point B: } x^3 - 3a^2x + 2a^3 = 0$$

$$(x - a)^2(x + 2a) = 0 \Rightarrow$$

$$B = (-2a, -8a^3)$$

$$\text{Tangent at B: } y + 8a^3 = 12a^2(x + 2a)$$

$$y = 12a^2x + 16a^3$$

$$\text{To find point C: } x^3 - 12a^2x - 16a^3 = 0$$

$$(x + 2a)^2(x - 4a) = 0 \Rightarrow$$

$$C = (4a, 64a^3)$$

$$\text{Area of } R = \int_{-2a}^a [x^3 - 3a^2x + 2a^3] dx = \frac{27}{4}a^4$$

$$\text{Area of } S = \int_{-2a}^{4a} [12a^2x + 16a^3 - x^3] dx = 108a^4$$

$$\text{Area of } S = 16(\text{area of } R)$$

$$9. s(x) = \int_{\alpha}^x \sqrt{1 + f'(t)^2} dt$$

$$(a) s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2}$$

$$(b) ds = \sqrt{1 + f'(x)^2} dx$$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right] (dx)^2 = (dx)^2 + (dy)^2$$

$$(c) s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2} \right)^2} dt = \int_1^x \sqrt{1 + \frac{9}{4}t} dt$$

$$(d) s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t} dt = \left[\frac{8}{27} \left(1 + \frac{9}{4}t \right)^{3/2} \right]_1^2 = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$$

This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

11. (a)
- $\bar{y} = 0$
- by symmetry

$$M_y = \int_1^6 x \left(\frac{1}{x^3} - \left(-\frac{1}{x^3} \right) \right) dx = \int_1^6 \frac{2}{x^2} dx = \left[-2\frac{1}{x} \right]_1^6 = \frac{5}{3}$$

$$m = 2 \int_1^6 \frac{1}{x^3} dx = \left[-\frac{1}{x^2} \right]_1^6 = \frac{35}{36}$$

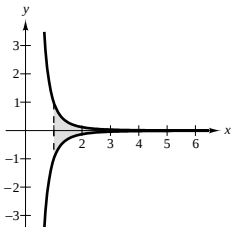
$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \quad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0 \right)$$

$$(b) m = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$$

$$M_y = 2 \int_1^b \frac{1}{x^2} dx = \frac{2(b - 1)}{b}$$

$$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \quad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0 \right)$$

$$(c) \lim_{b \rightarrow \infty} \bar{x} = \lim_{b \rightarrow \infty} \frac{2b}{b + 1} = 2 \quad (\bar{x}, \bar{y}) = (2, 0)$$



13. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

17. (a) Wall at shallow end

From Exercise 22: $F = 62.4(2)(4)(20) = 9984 \text{ lb}$

(b) Wall at deep end

From Exercise 22: $F = 62.4(4)(8)(20) = 39,936 \text{ lb}$

(c) Side wall

From Exercise 22: $F_1 = 62.4(2)(4)(40) = 19,968 \text{ lb}$

$$\begin{aligned} F_2 &= 62.4 \int_0^4 (8 - y)(10y) dy \\ &= 624 \int_0^4 (8y - y^2) dy = 624 \left[4y^2 - \frac{y^3}{3} \right]_0^4 \\ &= 26,624 \text{ lb} \end{aligned}$$

Total force: $F_1 + F_2 = 46,592 \text{ lb}$

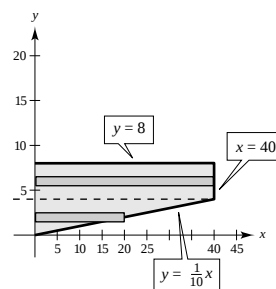
15. Point of equilibrium: $50 - 0.5x = 0.125x$

$$x = 80, p = 10$$

$$(P_0, x_0) = (10, 80)$$

$$\text{Consumer surplus} = \int_0^{80} [(50 - 0.5x) - 10] dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} [10 - 0.125x] dx = 400$$



C H A P T E R 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1	Basic Integration Rules	308
Section 7.2	Integration by Parts	312
Section 7.3	Trigonometric Integrals	321
Section 7.4	Trigonometric Substitution	328
Section 7.5	Partial Fractions	336
Section 7.6	Integration by Tables and Other Integration Techniques . .	343
Section 7.7	Indeterminate Forms and L'Hôpital's Rule	348
Section 7.8	Improper Integrals	353
	Review Exercises	358
	Problem Solving	363

CHAPTER 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Even-Numbered Exercises

$$2. (a) \frac{d}{dx}[\ln\sqrt{x^2+1} + C] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$$

$$(b) \frac{d}{dx}\left[\frac{2x}{(x^2+1)^2} + C\right] = \frac{(x^2+1)^2(2) - (2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$$(c) \frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{x}{x^2+1} dx \text{ matches (a).}$$

$$4. (a) \frac{d}{dx}[2x \sin(x^2+1) + C] = 2x[\cos(x^2+1)(2x)] + 2 \sin(x^2+1) = 2[2x^2 \cos(x^2+1) + \sin(x^2+1)]$$

$$(b) \frac{d}{dx}\left[-\frac{1}{2} \sin(x^2+1) + C\right] = -\frac{1}{2} \cos(x^2+1)(2x) = -x \cos(x^2+1)$$

$$(c) \frac{d}{dx}\left[\frac{1}{2} \sin(x^2+1) + C\right] = \frac{1}{2} \cos(x^2+1)(2x) = x \cos(x^2+1)$$

$$(d) \frac{d}{dx}[-2x \sin(x^2+1) + C] = -2x[\cos(x^2+1)(2x)] - 2 \sin(x^2+1) = -2[2x^2 \cos(x^2+1) + \sin(x^2+1)]$$

$$\int x \cos(x^2+1) dx \text{ matches (c).}$$

$$6. \int \frac{2t-1}{t^2-t+2} dt$$

$$u = t^2 - t + 2, du = (2t-1) dt$$

$$\text{Use } \int \frac{du}{u}.$$

$$8. \int \frac{2}{(2t-1)^2+4} dt$$

$$u = 2t-1, du = 2dt, a = 2$$

$$\text{Use } \int \frac{du}{u^2+a^2}.$$

$$10. \int \frac{-2x}{\sqrt{x^2-4}} dx$$

$$u = x^2 - 4, du = 2x dx, n = -\frac{1}{2}$$

$$\text{Use } \int u^n du.$$

$$12. \int \sec 3x \tan 3x dx$$

$$u = 3x, du = 3 dx$$

$$\text{Use } \int \sec u \tan u du.$$

$$14. \int \frac{1}{x\sqrt{x^2-4}} dx$$

$$u = x, du = dx, a = 2$$

$$\text{Use } \int \frac{du}{u\sqrt{u^2-a^2}}.$$

16. Let $u = x - 4$, $du = dx$.

$$\begin{aligned}\int 6(x-4)^5 dx &= 6 \int (x-4)^5 dx = 6 \frac{(x-4)^6}{6} + C \\ &= (x-4)^6 + C\end{aligned}$$

20. Let $u = 4 - 2x^2$, $du = -4x dx$.

$$\begin{aligned}\int x \sqrt{4 - 2x^2} dx &= -\frac{1}{4} \int (4 - 2x^2)^{1/2} (-4x) dx \\ &= -\frac{1}{6} (4 - 2x^2)^{3/2} + C\end{aligned}$$

24. Let $u = x^2 + 2x - 4$, $du = 2(x+1) dx$.

$$\begin{aligned}\int \frac{x+1}{\sqrt{x^2+2x-4}} dx &= \frac{1}{2} \int (x^2+2x-4)^{-1/2} (2)(x+1) dx \\ &= \sqrt{x^2+2x-4} + C\end{aligned}$$

26. $\int \frac{2x}{x-4} dx = \int 2 dx + \int \frac{8}{x-4} dx = 2x + 8 \ln|x-4| + C$

28. $\int \left(\frac{1}{3x-1} - \frac{1}{3x+1} \right) dx = \frac{1}{3} \int \frac{1}{3x-1} (3) dx - \frac{1}{3} \int \frac{1}{3x+1} (3) dx$
 $= \frac{1}{3} \ln|3x-1| - \frac{1}{3} \ln|3x+1| + C = \frac{1}{3} \ln \left| \frac{3x-1}{3x+1} \right| + C$

30. $\int x \left(1 + \frac{1}{x} \right)^3 dx = \int x \left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3} \right) dx = \int \left(x + 3 + \frac{3}{x} + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 + 3x + 3 \ln|x| - \frac{1}{x} + C$

32. $\int \sec 4x dx = \frac{1}{4} \int \sec(4x)(4) dx$
 $= \frac{1}{4} \ln|\sec 4x + \tan 4x| + C$

36. Let $u = \cot x$, $du = -\csc^2 x dx$.

$$\int \csc^2 x e^{\cot x} dx = - \int e^{\cot x} (-\csc^2 x) dx = -e^{\cot x} + C$$

18. Let $u = t - 9$, $du = dt$.

$$\int \frac{2}{(t-9)^2} dt = 2 \int (t-9)^{-2} dt = \frac{-2}{t-9} + C$$

22. $\int \left[x - \frac{3}{(2x+3)^2} \right] dx = \int x dx - \frac{3}{2} \int (2x+3)^{-2} (2) dx$
 $= \frac{x^2}{2} - \frac{3}{2} \frac{(2x+3)^{-1}}{-1} + C$
 $= \frac{x^2}{2} + \frac{3}{2(2x+3)} + C$

34. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned}\int \frac{\sin x}{\sqrt{\cos x}} dx &= - \int (\cos x)^{-1/2} (-\sin x) dx \\ &= -2\sqrt{\cos x} + C\end{aligned}$$

38. $\int \frac{5}{3e^x - 2} dx = 5 \int \left(\frac{1}{3e^x - 2} \right) \left(\frac{e^{-x}}{e^{-x}} \right) dx$
 $= 5 \int \frac{e^{-x}}{3 - 2e^{-x}} dx$
 $= \frac{5}{2} \int \frac{1}{3 - 2e^{-x}} (2e^{-x}) dx$
 $= \frac{5}{2} \ln|3 - 2e^{-x}| + C$

40. Let $u = \ln(\cos x)$, $du = \frac{-\sin x}{\cos x} dx$
 $= -\tan x dx$

$$\int (\tan x)(\ln \cos x) dx = -\int (\ln \cos x)(-\tan x) dx$$

$$= -\frac{[\ln(\cos x)]^2}{2} + C$$

44. $\int \frac{2}{3(\sec x - 1)} dx = \frac{2}{3} \int \frac{1}{\sec x - 1} \cdot \left(\frac{\sec x + 1}{\sec x + 1} \right) dx$

$$= \frac{2}{3} \int \frac{\sec x + 1}{\tan^2 x} dx$$

$$= \frac{2}{3} \int \frac{\sec x}{\tan^2 x} dx + \frac{2}{3} \int \cot^2 x dx$$

$$= \frac{2}{3} \int \frac{\cos x}{\sin^2 x} dx + \frac{2}{3} \int (\csc^2 x - 1) dx$$

$$= \frac{2}{3} \left(-\frac{1}{\sin x} \right) - \frac{2}{3} \cot x - \frac{2}{3}x + C$$

$$= -\frac{2}{3} [\csc x + \cot x + x] + C$$

42. $\int \frac{1 + \cos \alpha}{\sin \alpha} d\alpha = \int \csc \alpha d\alpha + \int \cot \alpha d\alpha$
 $= -\ln|\csc \alpha + \cot \alpha| + \ln|\sin \alpha| + C$

46. $\int \frac{3}{t^2 + 1} dt = 3 \arctan t + C$

48. Let $u = \sqrt{3}x$, $du = \sqrt{3} dx$.

$$\int \frac{1}{4 + 3x^2} dx = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{4 + (\sqrt{3}x)^2} dx$$

$$= \frac{1}{2\sqrt{3}} \arctan\left(\frac{\sqrt{3}x}{2}\right) + C$$

50. Let $u = \frac{1}{t}$, $du = \frac{-1}{t^2} dt$.

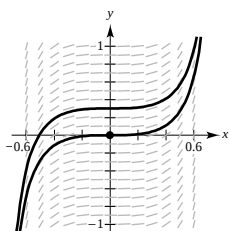
$$\int \frac{e^{1/t}}{t^2} dt = -\int e^{1/t} \left(\frac{-1}{t^2} \right) dt = -e^{1/t} + C$$

52. $\int \frac{1}{(x-1)\sqrt{4x^2 - 8x + 3}} dx = \int \frac{2}{[2(x-1)]\sqrt{[2(x-1)]^2 - 1}} dx = \operatorname{arcsec}|2(x-1)| + C$

54. $\int \frac{1}{\sqrt{1-4x-x^2}} dx = \int \frac{1}{\sqrt{5-(x^2+4x+4)}} dx = \int \frac{1}{\sqrt{5-(x+2)^2}} dx = \arcsin\left(\frac{x+2}{\sqrt{5}}\right) + C \quad (a = \sqrt{5})$

56. $\frac{dy}{dx} = \tan^2(2x)$, $(0, 0)$

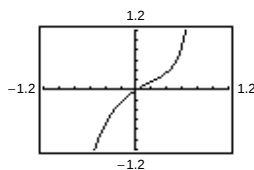
(a)



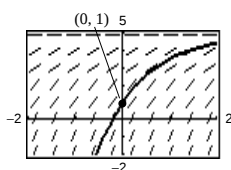
(b) $\int \tan^2(2x) dx = \int (\sec^2(2x) - 1) dx = \frac{1}{2} \tan(2x) - x + C$

$(0, 0): 0 = C$

$$y = \frac{1}{2} \tan(2x) - x$$



58.



60. $r = \int \frac{(1 + e^t)^2}{e^t} dt = \int \frac{1 + 2e^t + e^{2t}}{e^t} dt$
 $= \int (e^{-t} + 2 + e^t) dt = -e^{-t} + 2t + e^t + C$

62. Let $u = 2x$, $du = 2 dx$.

$$y = \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx$$

$$= \operatorname{arcsec}|2x| + C$$

66. Let $u = 1 - \ln x$, $du = \frac{-1}{x} dx$.

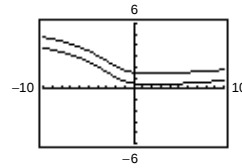
$$\int_1^e \frac{1 - \ln x}{x} dx = - \int_1^e (1 - \ln x) \left(\frac{-1}{x} \right) dx$$

$$= \left[-\frac{1}{2}(1 - \ln x)^2 \right]_1^e = \frac{1}{2}$$

70. $\int_0^4 \frac{1}{\sqrt{25 - x^2}} dx = \left[\arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5} \approx 0.927$

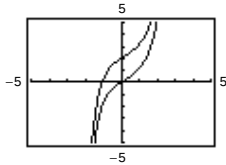
72. $\int \frac{x - 2}{x^2 + 4x + 13} dx = \frac{1}{2} \ln(x^2 + 4x + 13) - \frac{4}{3} \arctan\left(\frac{x + 2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



74. $\int \left(\frac{e^x + e^{-x}}{2} \right)^3 dx = \frac{1}{24} [e^{3x} + 9e^x - 9e^{-x} - e^{-3x}] + C$

The antiderivatives are vertical translations of each other.



76. $\int \sec u \tan u du = \sec u + C$

78. Arctan Rule: $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$

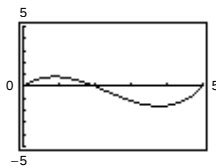
80. They differ by a constant:

$$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C.$$

82. $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$

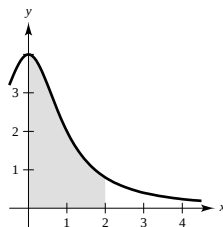
$$\int_0^5 f(x) dx < 0 \text{ because}$$

more area is below the x -axis than above.

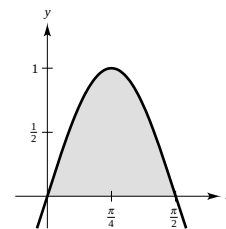


84. $\int_0^2 \frac{4}{x^2 + 1} dx \approx 4$

Matches (d).



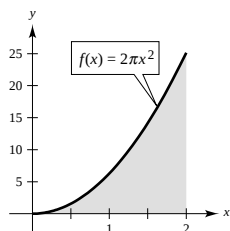
86. $A = \int_0^{\pi/2} \sin 2x dx$
- $$= \left[-\frac{1}{2} \cos 2x \right]_0^{\pi/2} = 1$$



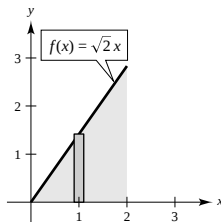
88. $\int_0^2 2\pi x^2 dx$

 (a) Let $f(x) = 2\pi x^2$ over the interval $[0, 2]$.

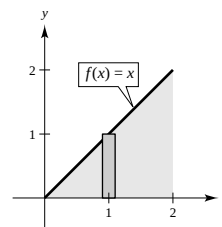
$$A = \int_0^2 (2\pi x^2) dx$$


 (b) Let $f(x) = \sqrt{2}x$ over the interval $[0, 2]$. Revolve this region about the x -axis.

$$\begin{aligned} V &= \pi \int_0^2 (\sqrt{2}x)^2 dx \\ &= \int_0^2 2\pi x^2 dx \end{aligned}$$


 (c) Let $f(x) = x$ over the interval $[0, 2]$. Revolve this region about the y -axis.

$$\begin{aligned} V &= 2\pi \int_0^2 x(x) dx \\ &= \int_0^2 2\pi x^2 dx \end{aligned}$$



90. (a) $\frac{1}{(\pi/n) - 0} \int_0^{\pi/n} \sin(nx) dx = \frac{1}{\pi} \int_0^{\pi/n} \sin(nx)(n) dx = \left[-\frac{1}{\pi} \cos(nx) \right]_0^{\pi/n} = \frac{2}{\pi}$

(b) $\frac{1}{3 - (-3)} \int_{-3}^3 \frac{1}{1 + x^2} dx = \left[\frac{1}{6} \arctan x \right]_{-3}^3 = \frac{1}{3} \arctan 3$

92. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

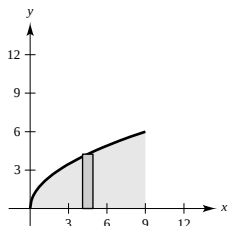
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^9 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx$$

$$= 2\pi \int_0^9 2\sqrt{x+1} dx$$

$$= \left[4\pi \left(\frac{2}{3} \right) (x+1)^{3/2} \right]_0^9$$

$$= \frac{8\pi}{3} (10\sqrt{10} - 1) \approx 256.545$$



94. $y = x^{2/3}$

$$y' = \frac{2}{3x^{1/3}}$$

$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$

$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}} dx \approx 7.6337$$

Section 7.2 Integration by Parts

2. $\frac{d}{dx} [x^2 \sin x + 2x \cos x - 2 \sin x] = x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$. Matches (d)

4. $\frac{d}{dx}[-x + x \ln x] = -1 + x\left(\frac{1}{x}\right) + \ln x = \ln x$. Matches (a)

6. $\int x^2 e^{2x} dx$

$u = x^2, dv = e^{2x} dx$

8. $\int \ln 3x dx$

$u = \ln 3x, dv = dx$

10. $\int x^2 \cos x dx$

$u = x^2, dv = \cos x dx$

12. $dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$

$u = x \Rightarrow du = dx$

14. $\int \frac{e^{1/t}}{t^2} dt = -\int e^{1/t} \left(\frac{-1}{t^2}\right) dt = -e^{1/t} + C$

$$\begin{aligned} 2 \int \frac{x}{e^x} dx &= 2 \int x e^{-x} dx \\ &= 2 \left[-x e^{-x} - \int -e^{-x} dx \right] = 2[-x e^{-x} - e^{-x}] + C \\ &= -2x e^{-x} - 2e^{-x} + C \end{aligned}$$

16. $dv = x^4 dx \Rightarrow v = \frac{x^5}{5}$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\begin{aligned} \int x^4 \ln x dx &= \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \left(\frac{1}{x}\right) dx = \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 dx \\ &= \frac{x^5}{5} \ln x - \frac{1}{25} x^5 + C = \frac{x^5}{25} (5 \ln x - 1) + C \end{aligned}$$

18. Let $u = \ln x, du = \frac{1}{x} dx$.

$$\int \frac{1}{x(\ln x)^3} dx = \int (\ln x)^{-3} \left(\frac{1}{x}\right) dx = \frac{-1}{2(\ln x)^2} + C$$

20. $dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

22. $dv = \frac{x}{(x^2 + 1)^2} dx \Rightarrow v = \int (x^2 + 1)^{-2} x dx = -\frac{1}{2(x^2 + 1)}$

$u = x^2 e^{x^2} \Rightarrow du = (2x^3 e^{x^2} + 2x e^{x^2}) dx = 2x e^{x^2} (x^2 + 1) dx$

$$\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2} dx = -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \int x e^{x^2} dx = -\frac{x^2 e^{x^2}}{2(x^2 + 1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2 + 1)} + C$$

24. $dv = \frac{1}{x^2} dx \Rightarrow v = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$u = \ln 2x \Rightarrow du = \frac{1}{x} dx$

$$\int \frac{\ln(2x)}{x^2} dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2} dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C = -\frac{\ln(2x) + 1}{x} + C$$

$$26. dv = \frac{1}{\sqrt{2+3x}} dx \Rightarrow v = \int (2+3x)^{-1/2} dx = \frac{2}{3} \sqrt{2+3x}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int \frac{x}{\sqrt{2+3x}} dx &= \frac{2x\sqrt{2+3x}}{3} - \frac{2}{3} \int \sqrt{2+3x} dx \\ &= \frac{2x\sqrt{2+3x}}{3} - \frac{4}{27}(2+3x)^{3/2} + C = \frac{2\sqrt{2+3x}}{27}[9x - 2(2+3x)] + C = \frac{2\sqrt{2+3x}}{27}(3x-4) + C \end{aligned}$$

$$28. dv = \sin x dx \Rightarrow v = -\cos x$$

$$u = x \Rightarrow du = dx$$

$$\int x \sin x dx = -x \cos x - \int -\cos x dx = -x \cos x + \sin x + C$$

30. Use integration by parts twice.

$$(1) u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$

$$(2) u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\begin{aligned} \int x^2 \cos x dx &= x^2 \sin x - 2 \left[-x \cos x + \int \cos x dx \right] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

$$32. dv = \sec \theta \tan \theta d\theta \Rightarrow v = \int \sec \theta \tan \theta d\theta = \sec \theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned} \int \theta \sec \theta \tan \theta d\theta &= \theta \sec \theta - \int \sec \theta d\theta \\ &= \theta \sec \theta - \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

$$34. dv = dx \Rightarrow v = \int dx = x$$

$$u = \arccos x \Rightarrow du = -\frac{1}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} 4 \int \arccos x dx &= 4 \left[x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx \right] \\ &= 4 \left[x \arccos x - \sqrt{1-x^2} \right] + C \end{aligned}$$

36. Use integration by parts twice.

$$(1) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \cos 2x \Rightarrow du = -2 \sin 2x dx$$

$$\int e^x \cos 2x dx = e^x \cos 2x + 2 \int e^x \sin 2x dx = e^x \cos 2x + 2 \left(e^x \sin 2x - 2 \int e^x \cos 2x dx \right)$$

$$5 \int e^x \cos 2x dx = e^x \cos 2x + 2e^x \sin 2x$$

$$\int e^x \cos 2x dx = \frac{e^x}{5} (\cos 2x + 2 \sin 2x) + C$$

$$(2) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \sin 2x \Rightarrow du = 2 \cos 2x dx$$

$$38. dv = dx \Rightarrow v = x$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$y' = \ln x$$

$$y = \int \ln x dx = x \ln x - \int x \left(\frac{1}{x} \right) dx = x \ln x - x + C = x(-1 + \ln x) + C$$

40. Use integration by parts twice.

$$(1) \quad dv = \sqrt{x-1} \, dx \Rightarrow v = \int (x-1)^{1/2} \, dx = \frac{2}{3}(x-1)^{3/2}$$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$(2) \quad dv = (x-1)^{3/2} \, dx \Rightarrow v = \int (x-1)^{3/2} \, dx = \frac{2}{5}(x-1)^{5/2}$$

$$u = x \Rightarrow du = dx$$

$$y = \int x^2 \sqrt{x-1} \, dx$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3} \int x(x-1)^{3/2} \, dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3} \left[\frac{2}{5}x(x-1)^{5/2} - \frac{2}{5} \int (x-1)^{5/2} \, dx \right]$$

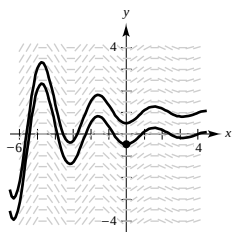
$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C = \frac{2(x-1)^{3/2}}{105}(15x^2 + 12x + 8) + C$$

$$42. \quad dv = dx \Rightarrow v = \int dx = x$$

$$u = \arctan \frac{x}{2} \Rightarrow du = \frac{1}{1 + (x/2)^2} \left(\frac{1}{2} \right) dx = \frac{2}{4 + x^2} dx$$

$$y = \int \arctan \frac{x}{2} \, dx = x \arctan \frac{x}{2} - \int \frac{2x}{4 + x^2} \, dx = x \arctan \frac{x}{2} - \ln(4 + x^2) + C$$

44. (a)



$$(b) \quad \frac{dy}{dx} = e^{-x/3} \sin 2x, \left(0, -\frac{18}{37} \right)$$

$$y = \int e^{-x/3} \sin 2x \, dx$$

Use integration by parts twice.

$$(1) \quad u = \sin 2x, \quad du = 2 \cos 2x$$

$$dv = e^{-x/3} \, dx, \quad v = -3e^{-x/3}$$

$$\int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x + \int 6e^{-x/3} \cos 2x \, dx$$

$$(2) \quad u = \cos 2x, \quad du = -2 \sin 2x$$

$$dv = e^{-x/3} \, dx, \quad v = -3e^{-x/3}$$

$$\int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x + 6 \left[-3e^{-x/3} \cos 2x - \int 6e^{-x/3} \sin 2x \, dx \right] + C$$

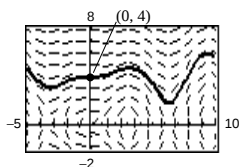
$$37 \int e^{-x/3} \sin 2x \, dx = -3e^{-x/3} \sin 2x - 18e^{-x/3} \cos 2x + C$$

$$y = \int e^{-x/3} \sin 2x \, dx = \frac{1}{37} \left[-3e^{-x/3} \sin 2x - 18e^{-x/3} \cos 2x \right] + C$$

$$\left(0, -\frac{18}{37} \right): \frac{-18}{37} = \frac{1}{37} [0 - 18] + C \Rightarrow C = 0$$

$$y = \frac{-1}{37} [3e^{-x/3} \sin 2x + 18e^{-x/3} \cos 2x]$$

46. $\frac{dy}{dx} = \frac{x}{y} \sin x, y(0) = 4$



48. See Exercise 3.

$$\int_0^1 x^2 e^x dx = \left[x^2 e^x - 2x e^x + 2e^x \right]_0^1 = e - 2 \approx 0.718$$

50. $dv = \sin 2x dx \Rightarrow v = \int \sin 2x dx = -\frac{1}{2} \cos 2x$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sin 2x dx &= \frac{-1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx \\ &= \frac{-1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C \\ &= \frac{1}{4} (\sin 2x - 2x \cos 2x) + C \end{aligned}$$

$$\text{Thus, } \int_0^\pi x \sin 2x dx = \left[\frac{1}{4} (\sin 2x - 2x \cos 2x) \right]_0^\pi = -\frac{\pi}{2}.$$

52. $dv = x dx \Rightarrow v = \int x dx = \frac{x^2}{2}$

$$u = \arcsin x^2 \Rightarrow du = \frac{2x}{\sqrt{1-x^4}} dx$$

$$\begin{aligned} \int x \arcsin x^2 dx &= \frac{x^2}{2} \arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}} dx \\ &= \frac{x^2}{2} \arcsin x^2 + \frac{1}{4} (2)(1-x^4)^{1/2} + C \\ &= \frac{1}{2} [x^2 \arcsin x^2 + \sqrt{1-x^4}] + C \end{aligned}$$

$$\begin{aligned} \text{Thus, } \int_0^1 x \arcsin x^2 dx &= \frac{1}{2} [x^2 \arcsin x^2 + \sqrt{1-x^4}]_0^1 \\ &= \frac{1}{4} (\pi - 2). \end{aligned}$$

54. Use integration by parts twice.

(1) $dv = e^{-x}, v = -e^{-x}, u = \cos x, du = -\sin x dx$

$$\int e^{-x} \cos x dx = -e^{-x} \cos x - \int e^{-x} \sin x dx$$

(2) $dv = e^{-x} dx, v = -e^{-x}, u = \sin x, du = \cos x dx$

$$\int e^{-x} \cos x dx = -e^{-x} \cos x - \left[-e^{-x} \sin x + \int e^{-x} \cos x dx \right] \Rightarrow 2 \int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x$$

Thus,

$$\begin{aligned} \int_0^2 e^{-x} \cos x dx &= \left[\frac{e^{-x} \sin x - e^{-x} \cos x}{2} \right]_0^2 \\ &= \frac{-e^{-2}}{2} [\sin 2 - \cos 2] + \frac{1}{2} \end{aligned}$$

56. $dv = dx \Rightarrow v = \int dx = x$

$$u = \ln(1+x^2) \Rightarrow du = \frac{2x}{1+x^2} dx$$

$$\begin{aligned} \int \ln(1+x^2) dx &= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx \\ &= x \ln(1+x^2) - 2 \int \left[1 - \frac{1}{1+x^2} \right] dx = x \ln(1+x^2) - 2x + 2 \arctan x + C \end{aligned}$$

$$\text{Thus, } \int_0^1 \ln(1+x^2) dx = \left[x \ln(1+x^2) - 2x + 2 \arctan x \right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

58. $u = x, du = dx, dv = \sec^2 x dx, v = \tan x$

Hence,

$$\begin{aligned} \int x \sec^2 x dx &= x \tan x - \int \tan x dx \\ \int_0^{\pi/4} x \sec^2 x dx &= \left[x \tan x + \ln |\cos x| \right]_0^{\pi/4} \\ &= \left(\frac{\pi}{4} + \ln \frac{\sqrt{2}}{2} \right) - 0 \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \end{aligned}$$

60.
$$\begin{aligned} \int x^3 e^{-2x} dx &= x^3 \left(-\frac{1}{2} e^{-2x} \right) - 3x^2 \left(\frac{1}{4} e^{-2x} \right) + 6x \left(-\frac{1}{8} e^{-2x} \right) - 6 \left(\frac{1}{16} e^{-2x} \right) + C \\ &= -\frac{1}{8} e^{-2x} (4x^3 + 6x^2 + 6x + 3) + C \end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	e^{-2x}
−	$3x^2$	$-\frac{1}{2}e^{-2x}$
+	$6x$	$\frac{1}{4}e^{-2x}$
−	6	$-\frac{1}{8}e^{-2x}$
+	0	$\frac{1}{16}e^{-2x}$

62.
$$\begin{aligned} \int x^3 \cos 2x dx &= x^3 \left(\frac{1}{2} \sin 2x \right) - 3x^2 \left(-\frac{1}{4} \cos 2x \right) + 6x \left(-\frac{1}{8} \sin 2x \right) - 6 \left(\frac{1}{16} \cos 2x \right) + C \\ &= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C \\ &= \frac{1}{8} [4x^3 \sin 2x + 6x^2 \cos 2x - 6x \sin 2x - 3 \cos 2x] + C \end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\cos 2x$
−	$3x^2$	$\frac{1}{2} \sin 2x$
+	$6x$	$-\frac{1}{4} \cos 2x$
−	6	$-\frac{1}{8} \sin 2x$
+	0	$\frac{1}{16} \cos 2x$

64.
$$\begin{aligned} \int x^2 (x-2)^{3/2} dx &= \frac{2}{5} x^2 (x-2)^{5/2} - \frac{8}{35} x (x-2)^{7/2} + \frac{16}{315} (x-2)^{9/2} + C \\ &= \frac{2}{315} (x-2)^{5/2} (35x^2 + 40x + 32) + C \end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	$(x-2)^{3/2}$
−	$2x$	$\frac{2}{5} (x-2)^{5/2}$
+	2	$\frac{4}{35} (x-2)^{7/2}$
−	0	$\frac{8}{315} (x-2)^{9/2}$

 66. Answers will vary.
See pages 488, 493.

 68. Yes.
 $u = \ln x, dv = x dx$

70. No. Substitution.

72. No. Substitution.

74.
$$\int \alpha^4 \sin \pi \alpha d\alpha = \frac{1}{\pi^5} [-(\alpha\pi)^4 \cos \pi\alpha + 4(\alpha\pi)^3 \sin \pi\alpha + 12(\alpha\pi)^2 \cos \pi\alpha - 24(\alpha\pi) \sin \pi\alpha - 24 \cos \pi\alpha] + C$$

76.
$$\begin{aligned} \int_0^5 x^4 (25 - x^2)^{3/2} dx &= \left[\frac{1,171,875 \arcsin(x/5)}{128} - \frac{x(2x^2 + 25)(25 - x^2)^{5/2}}{16} + \frac{625x(25 - x^2)^{3/2}}{64} + \frac{46,875x\sqrt{25 - x^2}}{128} \right]_0^5 \\ &\approx 14,381.0699 \end{aligned}$$

$$78. (a) dv = \sqrt{4+x} dx \Rightarrow v = \int (4+x)^{1/2} dx = \frac{2}{3}(4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{4+x} dx &= \frac{2}{3}x(4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} dx \\ &= \frac{2}{3}x(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C \end{aligned}$$

$$(b) u = 4+x \Rightarrow x = u-4 \text{ and } dx = du$$

$$\begin{aligned} \int x\sqrt{4+x} dx &= \int (u-4)u^{1/2} du = \int (u^{3/2} - 4u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{15}u^{3/2}(3u-20) + C \\ &= \frac{2}{15}(4+x)^{3/2}[3(4+x)-20] + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C \end{aligned}$$

$$80. (a) dv = \sqrt{4-x} dx \Rightarrow v = \int (4-x)^{1/2} dx$$

$$= -\frac{2}{3}(4-x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{4-x} dx &= -\frac{2}{3}x(4-x)^{3/2} + \frac{2}{3} \int (4-x)^{3/2} dx \\ &= -\frac{2}{3}x(4-x)^{3/2} - \frac{4}{15}(4-x)^{5/2} + C \\ &= -\frac{2}{15}(4-x)^{3/2}[5x+2(4-x)] + C \\ &= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C \end{aligned}$$

$$(b) u = 4-x \Rightarrow x = 4-u \text{ and } dx = -du$$

$$\begin{aligned} \int x\sqrt{4-x} dx &= -\int (4-u)\sqrt{u} du \\ &= -\int (4u^{1/2} - u^{3/2}) du \\ &= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C \\ &= -\frac{2}{15}u^{3/2}(20-3u) + C \\ &= -\frac{2}{15}(4-x)^{3/2}[20-3(4-x)] + C \\ &= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C \end{aligned}$$

$$84. dv = \cos x dx \Rightarrow v = \sin x$$

$$u = x^n \Rightarrow du = nx^{n-1} dx$$

$$\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$$

$$82. n = 0: \int e^x dx = e^x + C$$

$$n = 1: \int xe^x dx = xe^x - e^x + C = xe^x - \int e^x dx$$

$$\begin{aligned} n = 2: \int x^2 e^x dx &= x^2 e^x - 2xe^x + 2e^x + C \\ &= x^2 e^x - 2 \int xe^x dx \end{aligned}$$

$$\begin{aligned} n = 3: \int x^3 e^x dx &= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C \\ &= x^3 e^x - 3 \int x^2 e^x dx \end{aligned}$$

$$\begin{aligned} n = 4: \int x^4 e^x dx &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C \\ &= x^4 e^x - 4 \int x^3 e^x dx \end{aligned}$$

$$\text{In general, } \int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

(See Exercise 86)

$$86. dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$$

$$u = x^n \Rightarrow du = nx^{n-1} dx$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

88. Use integration by parts twice.

$$(1) \quad dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$$

$$u = \cos bx \Rightarrow du = -b \sin bx$$

$$\begin{aligned} \int e^{ax} \cos bx \, dx &= \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx \, dx = \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \left[\frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx \right] \\ &= \frac{e^{ax} \cos bx}{a} + \frac{be^{ax} \sin bx}{a^2} - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \cos bx \, dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2} \\ \int e^{ax} \cos bx \, dx &= \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C. \end{aligned}$$

$$(2) \quad dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$$

$$u = \sin bx \Rightarrow du = b \cos bx$$

90. $n = 2$ (Use formula in Exercise 84.)

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - 2 \int x \sin x \, dx \quad (\text{Use formula in Exercise 83.}) \quad (n = 1) \\ &= x^2 \sin x - 2 \left[-x \cos x + \int \cos x \, dx \right] = x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

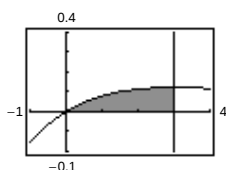
92. $n = 3, a = 2$ (Use formula in Exercise 86 three times.)

$$\begin{aligned} \int x^3 e^{2x} \, dx &= \frac{x^3 e^{2x}}{2} - \frac{3}{2} \int x^2 e^{2x} \, dx \quad (n = 3, a = 2) \\ &= \frac{x^3 e^{2x}}{2} - \frac{3}{2} \left[\frac{x^2 e^{2x}}{2} - \int x e^{2x} \, dx \right] \quad (n = 2, a = 2) \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3}{2} \left[\frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} \, dx \right] = \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3x e^{2x}}{4} - \frac{3e^{2x}}{8} + C \quad (n = 1, a = 2) \\ &= \frac{e^{2x}}{8} (4x^3 - 6x^2 + 6x - 3) + C \end{aligned}$$

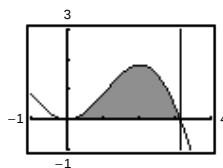
$$94. \quad dv = e^{-x/3} dx \Rightarrow v = -3e^{-x/3}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} A &= \frac{1}{9} \int_0^3 x e^{-x/3} \, dx \\ &= \frac{1}{9} \left(\left[-3x e^{-x/3} \right]_0^3 + 3 \int_0^3 e^{-x/3} \, dx \right) \\ &= \frac{1}{9} \left(\frac{-9}{e} - \left[9e^{-x/3} \right]_0^3 \right) \\ &= -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e} \approx 0.264 \end{aligned}$$



$$\begin{aligned} 96. \quad A &= \int_0^\pi x \sin x \, dx = \left[-x \cos x + \sin x \right]_0^\pi \\ &= \pi \quad (\text{See Exercise 83.}) \end{aligned}$$



98. In Example 6, we showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$.

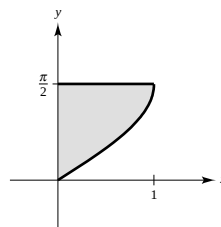
You can also solve this problem directly.

$$A = \int_0^1 \left(\frac{\pi}{2} - \arcsin x \right) dx = \left[\frac{\pi}{2}x - x \arcsin x - \sqrt{1-x^2} \right]_0^1 \quad (\text{Example 3})$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0 \right) - (-1) = 1$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x \left[\frac{\pi}{2} - \arcsin x \right] dx = \frac{\pi}{8}$$

$$\bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2} \left[\frac{\pi}{2} - \arcsin x \right] dx = 1$$



100. (a) Average $= \int_1^2 (1.6t \ln t + 1) dt = \left[0.8t^2 \ln t - 0.4t^2 + t \right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$

(b) Average $= \int_3^4 (1.6t \ln t + 1) dt = \left[0.8t^2 \ln t - 0.4t^2 + t \right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$

102. $c(t) = 30,000 + 500t$, $r = 7\%$, $t_1 = 5$

$$P \int_0^5 (30,000 + 500t)e^{-0.07t} dt = 500 \int_0^5 (60 + t)e^{-0.07t} dt$$

Let $u = 60 + t$, $dv = e^{-0.07t} dt$, $du = dt$, $v = -\frac{100}{7}e^{-0.07t}$.

$$P = 500 \left\{ \left[(60 + t) \left(-\frac{100}{7} e^{-0.07t} \right) \right]_0^5 + \frac{100}{7} \int_0^5 e^{-0.07t} dt \right\}$$

$$= 500 \left\{ \left[(60 + t) \left(-\frac{100}{7} e^{-0.07t} \right) \right]_0^5 - \left[\frac{10,000}{49} e^{-0.07t} \right]_0^5 \right\} \approx \$131,528.68$$

104. $\int_{-\pi}^{\pi} x^2 \cos nx dx = \left[\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx \right]_{-\pi}^{\pi}$

$$= \frac{2\pi}{n^2} \cos n\pi + \frac{2\pi}{n^2} \cos(-n\pi)$$

$$= \frac{4\pi}{n^2} \cos n\pi$$

$$= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$$

$$= \frac{(-1)^n 4\pi}{n^2}$$

106. For any integrable function, $\int f(x) dx = C + \int f(x) dx$, but this cannot be used to imply that $C = 0$.

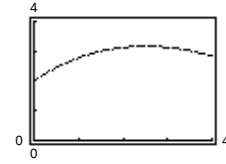
108. On $\left[0, \frac{\pi}{2}\right]$, $\sin x \leq 1 \Rightarrow x \sin x \leq x \Rightarrow \int_0^{\pi/2} x \sin x dx \leq \int_0^{\pi/2} x dx$.

110. $f'(x) = \cos \sqrt{x}, f(0) = 2$

(a) It cannot be solved by integration.

(b) You obtain the points

n	x_n	y_n
0	0	2
1	0.05	2.05
2	0.10	2.098755
3	0.15	2.146276
$\vdots$	$\vdots$	$\vdots$
80	4.0	2.8403565



Section 7.3 Trigonometric Integrals

2. (a) $y = \sec x \Rightarrow y' = \sec x \tan x = \sin x \sec^2 x$.

Matches (iii)

(b) $y = \cos x + \sec x \Rightarrow y' = -\sin x + \sec x \tan x$

$$= -\sin x + \sec^2 x \sin x$$

$$= \sin x(-1 + \sec^2 x)$$

$$= \sin x \tan^2 x \quad \text{Matches (i)}$$

(c) $y = x - \tan x + \frac{1}{3} \tan^3 x \Rightarrow y' = 1 - \sec^2 x + \tan^2 x \sec^2 x$
 $= -\tan^2 x + \tan^2 x(1 + \tan^2 x)$
 $= \tan^4 x \quad \text{Matches (iv)}$

(d) $y = 3x + 2 \sin x \cos^3 x + 3 \sin x \cos x \Rightarrow$
 $y' = 3 + 2 \cos x(\cos^3 x) + 6 \sin x \cos^2 x(-\sin x) + 3 \cos^2 x - 3 \sin^2 x$
 $= 3 + 2 \cos^4 x - 6 \cos^2 x(1 - \cos^2 x) + 3 \cos^2 x - 3(1 - \cos^2 x)$
 $= 8 \cos^4 x \quad \text{Matches (ii)}$

4. $\int \cos^3 x \sin^4 x \, dx = \int \cos x(1 - \sin^2 x) \sin^4 x \, dx$
 $= \int (\sin^4 x - \sin^6 x) \cos x \, dx$
 $= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

6. Let $u = \cos x, du = -\sin x \, dx$.

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin x(1 - \cos^2 x) \, dx \\ &= \int \cos^2 x(-\sin x) \, dx + \int \sin x \, dx \\ &= \frac{1}{3} \cos^3 x - \cos x + C \end{aligned}$$

8. Let $u = \sin \frac{x}{3}, du = \frac{1}{3} \cos \frac{x}{3} \, dx$.

$$\begin{aligned} \int \cos^3 \frac{x}{3} \, dx &= \int \left(\cos \frac{x}{3} \right) \left(1 - \sin^2 \frac{x}{3} \right) \, dx \\ &= 3 \int \left(1 - \sin^2 \frac{x}{3} \right) \left(\frac{1}{3} \cos \frac{x}{3} \right) \, dx \\ &= 3 \left(\sin \frac{x}{3} - \frac{1}{3} \sin^3 \frac{x}{3} \right) + C \\ &= 3 \sin \frac{x}{3} - \sin^3 \frac{x}{3} + C \end{aligned}$$

$$\begin{aligned}
 10. \int \frac{\sin^5 t}{\sqrt{\cos t}} dt &= \int \sin t (1 - \cos^2 t)^2 (\cos t)^{-1/2} dt \\
 &= \int \sin t (1 - 2 \cos^2 t + \cos^4 t) (\cos t)^{-1/2} dt \\
 &= \int [(\cos t)^{-1/2} - 2(\cos t)^{3/2} + (\cos t)^{7/2}] \sin t dt = -2(\cos t)^{1/2} + \frac{4}{5}(\cos t)^{5/2} - \frac{2}{9}(\cos t)^{9/2} + C
 \end{aligned}$$

$$\begin{aligned}
 12. \int \sin^2 2x dx &= \int \frac{1 - \cos 4x}{2} dx = \frac{1}{2} \left(x - \frac{1}{4} \sin 4x \right) + C \\
 &= \frac{1}{8} (4x - \sin 4x) + C
 \end{aligned}$$

$$\begin{aligned}
 14. \int \sin^4 2\theta d\theta &= \int \frac{1 - \cos 4\theta}{2} \cdot \frac{1 - \cos 4\theta}{2} d\theta \\
 &= \frac{1}{4} \int (1 - 2 \cos 4\theta + \cos^2 4\theta) d\theta \\
 &= \frac{1}{4} \int \left(1 - 2 \cos 4\theta + \frac{1 + \cos 8\theta}{2} \right) d\theta \\
 &= \frac{1}{4} \int \left(\frac{3}{2} - 2 \cos 4\theta + \frac{1}{2} \cos 8\theta \right) d\theta \\
 &= \frac{1}{4} \left[\frac{3}{2} \theta - \frac{1}{2} \sin 4\theta + \frac{1}{16} \sin 8\theta \right] + C \\
 &= \frac{3}{8} \theta - \frac{1}{8} \sin 4\theta + \frac{1}{64} \sin 8\theta + C
 \end{aligned}$$

16. Use integration by parts twice.

$$dv = \sin^2 x dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}
 \int x^2 \sin^2 x dx &= \frac{1}{4} x^2 (2x - \sin 2x) - \frac{1}{2} \int (2x^2 - x \sin 2x) dx \\
 &= \frac{1}{2} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{3} x^3 + \frac{1}{2} \int x \sin 2x dx \\
 &= \frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x + \frac{1}{2} \left[-\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx \right] \\
 &= \frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C \\
 &= \frac{1}{24} (4x^3 - 6x^2 \sin 2x - 6x \cos 2x + 3 \sin 2x) + C
 \end{aligned}$$

18. Let $u = \sin x$, $du = \cos x dx$.

$$\begin{aligned}
 \int_0^{\pi/2} \cos^5 x dx &= \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x dx \\
 &= \int_0^{\pi/2} (1 - 2 \sin^2 x + \sin^4 x) \cos x dx \\
 &= \left[\sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x \right]_0^{\pi/2} \\
 &= \frac{8}{15}
 \end{aligned}$$

$$\begin{aligned}
 20. \int_0^{\pi/2} \sin^2 x dx &= \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) dx \\
 &= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{4}
 \end{aligned}$$

$$22. \int \sec^2(2x - 1) dx = \frac{1}{2} \tan(2x - 1) + C$$

$$\begin{aligned} 24. \int \sec^6 3x dx &= \int (1 + \tan^2 3x)^2 \sec^2 3x dx \\ &= \int (1 + 2 \tan^2 3x + \tan^4 3x) \sec^2 3x dx \\ &= \frac{1}{3} \tan 3x + \frac{2}{9} \tan^3 3x + \frac{1}{15} \tan^5 3x + C \end{aligned}$$

$$26. \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$28. \int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} dx = \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C$$

$$30. \text{ Let } u = \sec 2t, du = 2 \sec 2t \tan 2t$$

$$\begin{aligned} \int \tan^3 2t \cdot \sec^3 2t dt &= \int (\sec^2 2t - 1) \sec^3 2t \cdot \tan 2t dt \\ &= \int (\sec^4 2t - \sec^2 2t)(\sec 2t \tan 2t) dt \\ &= \frac{\sec^5 2t}{5} - \frac{\sec^3 2t}{3} + C \end{aligned}$$

$$32. \int \tan^5 2x \sec^2 2x dx = \frac{1}{12} \tan^6 2x + C$$

$$\begin{aligned} 34. \int \sec^2 \frac{x}{2} \tan \frac{x}{2} dx &= 2 \int \sec \frac{x}{2} \left(\frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2} \right) dx \\ &= \sec^2 \frac{x}{2} + C \\ \text{or } \int \sec^2 \frac{x}{2} \tan \frac{x}{2} dx &= 2 \int \tan \frac{x}{2} \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) dx \\ &= \tan^2 \frac{x}{2} + C \end{aligned}$$

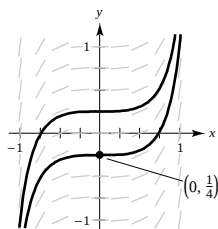
$$\begin{aligned} 36. \int \tan^3 3x dx &= \int (\sec^2 3x - 1) \tan 3x dx \\ &= \frac{1}{3} \int \tan 3x (3 \sec^2 3x) dx + \frac{1}{3} \int \frac{-3 \sin 3x}{\cos 3x} dx \\ &= \frac{1}{6} \tan^2 3x + \frac{1}{3} \ln |\cos 3x| + C \end{aligned}$$

$$\begin{aligned} 38. \int \frac{\tan^2 x}{\sec^5 x} dx &= \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x dx \\ &= \int \sin^2 x \cdot \cos^3 x dx \\ &= \int \sin^2 x (1 - \sin^2 x) \cos x dx \\ &= \int (\sin^2 x - \sin^4 x) \cos x dx \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \end{aligned}$$

$$\begin{aligned} 40. s &= \int \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} d\alpha \\ &= \int \left(\frac{1 - \cos \alpha}{2} \right) \left(\frac{1 + \cos \alpha}{2} \right) d\alpha = \int \frac{1 - \cos^2 \alpha}{4} d\alpha \\ &= \frac{1}{4} \int \sin^2 \alpha d\alpha = \frac{1}{8} \int (1 - \cos 2\alpha) d\alpha \\ &= \frac{1}{8} \left[\theta - \frac{\sin 2\alpha}{2} \right] + C \\ &= \frac{1}{16} (2\alpha - \sin 2\alpha) + C \end{aligned}$$

$$\begin{aligned} 42. y &= \int \sqrt{\tan x} \sec^4 x dx \\ &= \int \tan^{1/2} x (\tan^2 x + 1) \sec^2 x dx \\ &= \int (\tan^{5/2} x + \tan^{1/2} x) \sec^2 x dx \\ &= \frac{2}{7} \tan^{7/2} x + \frac{2}{3} \tan^{3/2} x + C \end{aligned}$$

44. (a)



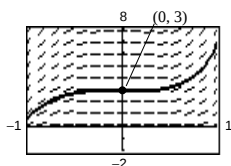
$$(b) \frac{dy}{dx} = \sec^2 x \tan^2 x, \left(0, -\frac{1}{4}\right)$$

$$y = \int \sec^2 x \tan^2 x \, dx \quad u = \tan x, \, du = \sec^2 x \, dx$$

$$y = \frac{\tan^3 x}{3} + C$$

$$\left(0, -\frac{1}{4}\right): -\frac{1}{4} = C \Rightarrow y = \frac{1}{3} \tan^3 x - \frac{1}{4}$$

$$46. \frac{dy}{dx} = 3\sqrt{y} \tan^2 x, y(0) = 3$$



$$\begin{aligned} 48. \int \cos 4\theta \cos(-3\theta) \, d\theta &= \int \cos 4\theta \cos 3\theta \, d\theta \\ &= \frac{1}{2} \int (\cos 7\theta + \cos \theta) \, d\theta \\ &= \frac{\sin 7\theta}{14} + \frac{\sin \theta}{2} + C \end{aligned}$$

$$\begin{aligned} 50. \int \sin(-4x) \cos 3x \, dx &= - \int \sin 4x \cos 3x \, dx \\ &= -\frac{1}{2} \int (\sin x + \sin 7x) \, dx \\ &= -\frac{1}{2} \left[-\cos x - \frac{1}{7} \cos 7x \right] + C \\ &= \frac{1}{14} [7 \cos x + \cos 7x] + C \end{aligned}$$

$$52. \text{ Let } u = \tan \frac{x}{2}, \, du = \frac{1}{2} \sec^2 \frac{x}{2} \, dx.$$

$$\begin{aligned} \int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2} \, dx &= \int \tan^4 \frac{x}{2} \left(\tan^2 \frac{x}{2} + 1 \right) \sec^2 \frac{x}{2} \, dx \\ &= 2 \int \left(\tan^6 \frac{x}{2} + \tan^4 \frac{x}{2} \right) \left(\frac{1}{2} \sec^2 \frac{x}{2} \right) dx \\ &= \frac{2}{7} \tan^7 \frac{x}{2} + \frac{2}{5} \tan^5 \frac{x}{2} + C \end{aligned}$$

$$\begin{aligned} 54. \, u = \cot 3x, \, du = -3 \csc^2 3x \, dx \\ \int \csc^2 3x \cot 3x \, dx &= -\frac{1}{3} \int \cot 3x (-3 \csc^2 3x) \, dx \\ &= -\frac{1}{6} \cot^2 3x + C \end{aligned}$$

$$\begin{aligned} 56. \int \frac{\cot^3 t}{\csc t} \, dt &= \int \frac{\cos^3 t}{\sin^2 t} \, dt = \int \frac{(1 - \sin^2 t) \cos t}{\sin^2 t} \, dt \\ &= \int \frac{\cos t}{\sin^2 t} \, dt - \int \cos t \, dt \\ &= \frac{-1}{\sin t} - \sin t + C \\ &= -\csc t - \sin t + C \end{aligned}$$

$$58. \int \frac{\sin^2 x - \cos^2 x}{\cos x} \, dx = \int \frac{1 - 2 \cos^2 x}{\cos x} \, dx = \int (\sec x - 2 \cos x) \, dx = \ln|\sec x + \tan x| - 2 \sin x + C$$

$$\begin{aligned} 60. \int \frac{1 - \sec t}{\cos t - 1} \, dt &= \int \frac{\cos t - 1}{(\cos t - 1) \cos t} \, dt \\ &= \int \sec t \, dt = \ln|\sec t + \tan t| + C \end{aligned}$$

$$\begin{aligned} 62. \int_0^{\pi/3} \tan^2 x \, dx &= \int_0^{\pi/3} (\sec^2 x - 1) \, dx \\ &= \left[\tan x - x \right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \end{aligned}$$

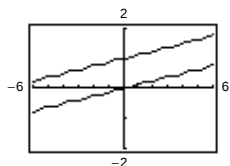
$$64. \text{ Let } u = \tan t, \, du = \sec^2 t \, dt.$$

$$\int_0^{\pi/4} \sec^2 t \sqrt{\tan t} \, dt = \left[\frac{2}{3} \tan^{3/2} t \right]_0^{\pi/4} = \frac{2}{3}$$

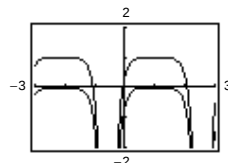
$$\begin{aligned} 66. \int_{-\pi}^{\pi} \sin 3\theta \cos \theta \, d\theta &= \frac{1}{2} \int_{-\pi}^{\pi} (\sin 4\theta + \sin 2\theta) \, d\theta \\ &= -\frac{1}{2} \left[\frac{1}{4} \cos 4\theta + \frac{1}{2} \cos 2\theta \right]_{-\pi}^{\pi} = 0 \end{aligned}$$

$$\begin{aligned}
 68. \int_{-\pi/2}^{\pi/2} (\sin^2 x + 1) dx &= \int_{-\pi/2}^{\pi/2} \left(\frac{1 - \cos 2x}{2} + 1 \right) dx \\
 &= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2} \cos 2x \right) dx = \left[\frac{3}{2}x - \frac{1}{4} \sin 2x \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}
 \end{aligned}$$

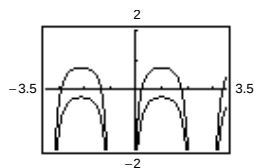
$$70. \int \sin^2 x \cos^2 x dx = \frac{1}{32} [4x - \sin 4x] + C$$



$$72. \int \tan^3(1-x) dx = -\frac{\tan^2(1-x)}{2} - \ln|\cos(1-x)| + C$$



$$74. \int \sec^4(1-x) \tan(1-x) dx = -\frac{\sec^4(1-x)}{4} + C$$



$$\begin{aligned}
 76. \int_0^{\pi/2} (1 - \cos \theta)^2 d\theta &= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{\pi/2} \\
 &= \frac{3\pi}{4} - 2
 \end{aligned}$$

$$78. \int_0^{\pi/2} \sin^6 x dx = \frac{1}{8} \left[\frac{5x}{2} - 2\sin 2x + \frac{3}{8}\sin 4x + \frac{1}{6}\sin^3 2x \right]_0^{\pi/2} = \frac{5\pi}{32}$$

80. See guidelines on page 500.

82. (a) Let $u = \tan x$, $du = \sec^2 x dx$.

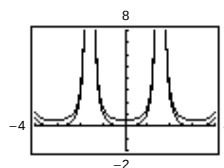
$$\int \sec^2 x \tan x dx = \frac{1}{2} \tan^2 x + C_1$$

Or let $u = \sec x$, $du = \sec x \tan x dx$.

$$\int \sec x (\sec x \tan x) dx = \frac{1}{2} \sec^2 x + C$$

$$(c) \frac{1}{2} \sec^2 x + C = \frac{1}{2} (\tan^2 x + 1) + C = \frac{1}{2} \tan^2 x + \left(\frac{1}{2} + C \right) = \frac{1}{2} \tan^2 x + C_2$$

(b)



84. Disks

$$R(x) = \tan x$$

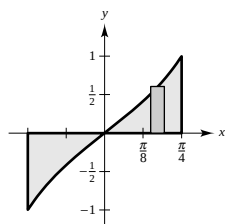
$$r(x) = 0$$

$$V = 2\pi \int_0^{\pi/4} \tan^2 x dx$$

$$= 2\pi \int_0^{\pi/4} (\sec^2 x - 1) dx$$

$$= 2\pi \left[\tan x - x \right]_0^{\pi/4}$$

$$= 2\pi \left(1 - \frac{\pi}{4} \right) \approx 1.348$$



$$86. (a) V = \pi \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx = \frac{\pi}{2} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi^2}{4}$$

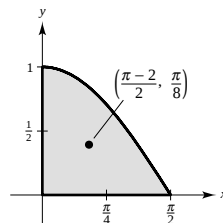
$$(b) A = \int_0^{\pi/2} \cos x \, dx = \left[\sin x \right]_0^{\pi/2} = 1$$

Let $u = x$, $dv = \cos x \, dx$, $du = dx$, $v = \sin x$.

$$\bar{x} = \int_0^{\pi/2} x \cos x \, dx = \left[x \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx = \left[x \sin x + \cos x \right]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\begin{aligned} \bar{y} &= \frac{1}{2} \int_0^{\pi/2} \cos^2 x \, dx \\ &= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx \\ &= \frac{1}{4} \left[x + \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{8} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8} \right)$$



$$88. dv = \cos x \, dx \Rightarrow v = \sin x$$

$$u = \cos^{n-1} x \Rightarrow du = -(n-1) \cos^{n-2} x \sin x \, dx$$

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \end{aligned}$$

$$\text{Therefore, } n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

$$90. \text{ Let } u = \sec^{n-2} x, du = (n-2) \sec^{n-2} x \tan x \, dx, dv = \sec^2 x \, dx, v = \tan x.$$

$$\begin{aligned} \int \sec^n x \, dx &= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \left[\int \sec^n x \, dx - \int \sec^{n-2} x \, dx \right] \end{aligned}$$

$$(n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

$$92. \int \cos^4 x \, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \int \cos^2 x \, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left[\frac{\cos x \sin x}{2} + \frac{1}{2} \int dx \right]$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C = \frac{1}{8} [2 \cos^3 x \sin x + 3 \cos x \sin x + 3x] + C$$

$$\begin{aligned}
94. \int \sin^4 x \cos^2 x \, dx &= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2} \int \cos^2 x \sin^2 x \, dx \\
&= -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2} \left[-\frac{\cos^3 x \sin x}{4} + \frac{1}{4} \int \cos^2 x \, dx \right] \\
&= -\frac{1}{6} \cos^3 x \sin^3 x - \frac{1}{8} \cos^3 x \sin x + \frac{1}{8} \left[\frac{\cos x \sin x}{2} + \frac{x}{2} \right] + C \\
&= -\frac{1}{48} [8 \cos^3 x \sin^3 x + 6 \cos^3 x \sin x - 3 \cos x \sin x - 3x] + C
\end{aligned}$$

96. (a) n is odd and $n \geq 3$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x \, dx &= \left[\frac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx \\
&= \frac{n-1}{n} \left[\left[\frac{\cos^{n-3} x \sin x}{n-2} \right]_0^{\pi/2} + \frac{n-3}{n-2} \int_0^{\pi/2} \cos^{n-4} x \, dx \right] \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \left[\left[\frac{\cos^{n-5} x \sin x}{n-4} \right]_0^{\pi/2} + \frac{n-5}{n-4} \int_0^{\pi/2} \cos^{n-6} x \, dx \right] \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \int_0^{\pi/2} \cos^{n-6} x \, dx \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos x \, dx \\
&= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots (\sin x) \right]_0^{\pi/2} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots 1 \quad (\text{Reverse the order}) \\
&= (1) \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \cdots \left(\frac{n-1}{n} \right) \\
&= \left(\frac{2}{3} \right) \left(\frac{4}{5} \right) \left(\frac{6}{7} \right) \cdots \left(\frac{n-1}{n} \right)
\end{aligned}$$

(b) n is even and $n \geq 2$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^n x \, dx &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \int_0^{\pi/2} \cos^2 x \, dx \quad (\text{From part (a).}) \\
&= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \left(\frac{x}{2} + \frac{1}{4} \sin 2x \right) \right]_0^{\pi/2} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{\pi}{4} \quad (\text{Reverse the order}) \\
&= \left(\frac{\pi}{2} \cdot \frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \cdots \left(\frac{n-1}{n} \right) \\
&= \left(\frac{1}{2} \right) \left(\frac{3}{4} \right) \left(\frac{5}{6} \right) \cdots \left(\frac{n-1}{n} \right) \left(\frac{\pi}{2} \right)
\end{aligned}$$

Section 7.4 Trigonometric Substitution

$$\begin{aligned}
2. \frac{d}{dx} \left[8 \ln |\sqrt{x^2 - 16} + x| + \frac{1}{2} x \sqrt{x^2 - 16} + C \right] &= 8 \left[\frac{(x/\sqrt{x^2 - 16}) + 1}{\sqrt{x^2 - 16} + x} \right] + \frac{1}{2} x \left(\frac{x}{\sqrt{x^2 - 16}} \right) + \frac{1}{2} \sqrt{x^2 - 16} \\
&= \frac{8(x + \sqrt{x^2 - 16})}{\sqrt{x^2 - 16}(\sqrt{x^2 - 16} + x)} + \frac{x^2}{2\sqrt{x^2 - 16}} + \frac{\sqrt{x^2 - 16}}{2} \\
&= \frac{16 + x^2 + x^2 - 16}{2\sqrt{x^2 - 16}} \\
&= \frac{x^2}{\sqrt{x^2 - 16}}
\end{aligned}$$

Indefinite integral: $\int \frac{x^2}{\sqrt{x^2 - 16}} dx$ Matches (d)

$$\begin{aligned}
4. \frac{d}{dx} \left[8 \arcsin \frac{x-3}{4} + \frac{(x-3)\sqrt{7+6x-x^2}}{2} + C \right] &= 8 \left[\frac{1}{\sqrt{1 - [(x-3)/4]^2}} \cdot \frac{1}{4} \right] + \frac{1}{2} (x-3) \frac{3-x}{\sqrt{7+6x-x^2}} + \frac{1}{2} \sqrt{7+6x-x^2} \\
&= \frac{8}{\sqrt{16 - (x-3)^2}} - \frac{(x-3)^2}{2\sqrt{16 - (x-3)^2}} + \frac{\sqrt{16 - (x-3)^2}}{2} \\
&= \frac{16 - (x^2 - 6x + 9) + 16 - (x^2 - 6x + 9)}{2\sqrt{16 - (x-3)^2}} \\
&= \frac{2[16 - (x-3)^2]}{2\sqrt{16 - (x-3)^2}} \\
&= \sqrt{16 - (x-3)^2} \\
&= \sqrt{7+6x-x^2}
\end{aligned}$$

Indefinite integral: $\int \sqrt{7+6x-x^2} dx$ Matches (c)

6. Same substitution as in Exercise 5.

$$\int \frac{10}{x^2 \sqrt{25 - x^2}} dx = 10 \int \frac{5 \cos \theta d\theta}{(25 \sin^2 \theta)(5 \cos \theta)} = \frac{2}{5} \int \csc^2 \theta d\theta = -\frac{2}{5} \cot \theta + C = \frac{-2\sqrt{25 - x^2}}{5x} + C$$

8. Same substitution as in Exercise 5

$$\begin{aligned}
\int \frac{x^2}{\sqrt{25 - x^2}} dx &= \int \frac{25 \sin^2 \theta}{5 \cos \theta} (5 \cos \theta) d\theta = \frac{25}{2} \int (1 - \cos 2\theta) d\theta \\
&= \frac{25}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C = \frac{25}{2} (\theta - \sin \theta \cos \theta) + C \\
&= \frac{25}{2} \left[\arcsin \left(\frac{x}{5} \right) - \left(\frac{x}{5} \right) \left(\frac{\sqrt{25 - x^2}}{5} \right) \right] + C = \frac{1}{2} \left[25 \arcsin \left(\frac{x}{5} \right) - x \sqrt{25 - x^2} \right] + C
\end{aligned}$$

10. Same substitution as in Exercise 9

$$\begin{aligned}
\int \frac{\sqrt{x^2 - 4}}{x} dx &= \int \frac{2 \tan \theta}{2 \sec \theta} (2 \sec \theta \tan \theta) d\theta = 2 \int \tan^2 \theta d\theta = 2 \int (\sec^2 \theta - 1) d\theta \\
&= 2(\tan \theta - \theta) + C = 2 \left[\frac{\sqrt{x^2 - 4}}{2} - \operatorname{arcsec} \left(\frac{x}{2} \right) \right] + C = \sqrt{x^2 - 4} - 2 \operatorname{arcsec} \left(\frac{x}{2} \right) + C
\end{aligned}$$

12. Same substitution as in Exercise 9

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{x^2 - 4}} dx &= \int \frac{8 \sec^3 \theta}{2 \tan \theta} (2 \sec \theta \tan \theta) d\theta = 8 \int \sec^4 \theta d\theta \\
 &= 8 \int (1 + \tan^2 \theta) \sec^2 \theta d\theta = 8 \left(\tan \theta + \frac{\tan^3 \theta}{3} \right) + C = \frac{8}{3} \tan \theta (3 + \tan^2 \theta) + C \\
 &= \frac{8}{3} \left(\frac{\sqrt{x^2 - 4}}{2} \right) \left(3 + \frac{x^2 - 4}{4} \right) + C = \frac{1}{3} \sqrt{x^2 - 4} (12 + x^2 - 4) + C = \frac{1}{3} \sqrt{x^2 - 4} (x^2 + 8) + C
 \end{aligned}$$

14. Same substitution as in Exercise 13.

$$\begin{aligned}
 \int \frac{9x^3}{\sqrt{1 + x^2}} dx &= 9 \int \frac{\tan^3 \theta}{\sec \theta} \sec^2 \theta d\theta = 9 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta = 9 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 3 \sec \theta (\sec^2 \theta - 3) + C = 3 \sqrt{1 + x^2} [(1 + x^2) - 3] + C = 3 \sqrt{1 + x^2} (x^2 - 2) + C
 \end{aligned}$$

16. Same substitution as in Exercise 13

$$\begin{aligned}
 \int \frac{x^2}{(1 + x^2)^2} dx &= \int \frac{x^2}{(\sqrt{1 + x^2})^4} dx = \int \frac{\tan^2 \theta \sec^2 \theta d\theta}{\sec^4 \theta} = \int \sin^2 \theta d\theta \\
 &= \frac{1}{2} \int (1 - \cos 2\theta) d\theta = \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] = \frac{1}{2} [\theta - \sin \theta \cos \theta] + C \\
 &= \frac{1}{2} \left[\arctan x - \left(\frac{x}{\sqrt{1 + x^2}} \right) \left(\frac{1}{\sqrt{1 + x^2}} \right) \right] + C = \frac{1}{2} \left[\arctan x - \frac{x}{1 + x^2} \right] + C
 \end{aligned}$$

18. Let $u = x$, $a = 1$, and $du = dx$.

$$\int \sqrt{1 + x^2} dx = \frac{1}{2} (x \sqrt{1 + x^2} + \ln |x + \sqrt{1 + x^2}|) + C$$

$$\begin{aligned}
 20. \int \frac{x}{\sqrt{9 - x^2}} dx &= -\frac{1}{2} \int (9 - x^2)^{-1/2} (-2x) dx \\
 &= -(9 - x^2)^{1/2} + C \quad (\text{Power Rule})
 \end{aligned}$$

$$22. \int \frac{1}{\sqrt{25 - x^2}} dx = \arcsin \frac{x}{5} + C$$

24. Let $u = 16 - 4x^2$, $du = -8x dx$.

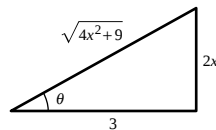
$$\int x \sqrt{16 - 4x^2} dx = -\frac{1}{8} \int (16 - 4x^2)^{1/2} (-8x) dx = \left[-\frac{1}{12} (16 - 4x^2)^{3/2} \right] + C = -\frac{2}{3} (4 - x^2)^{3/2} + C$$

26. Let $u = 1 - t^2$, $du = -2t dt$.

$$\int \frac{t}{(1 - t^2)^{3/2}} dt = -\frac{1}{2} \int (1 - t^2)^{-3/2} (-2t) dt = \frac{1}{\sqrt{1 - t^2}} + C$$

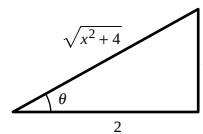
28. Let $2x = 3 \tan \theta$, $dx = \frac{3}{2} \sec^2 \theta d\theta$, $\sqrt{4x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned}
 \int \frac{\sqrt{4x^2 + 9}}{x^4} dx &= \int \frac{3 \sec \theta [(3/2) \sec^2 \theta d\theta]}{(3/2)^4 \tan^4 \theta} \\
 &= \frac{8}{9} \int \frac{\cos \theta}{\sin^4 \theta} d\theta \\
 &= \frac{-8}{27 \sin^3 \theta} + C \\
 &= -\frac{8}{27} \csc^3 \theta + C \\
 &= \frac{-(4x^2 + 9)^{3/2}}{27x^3} + C
 \end{aligned}$$



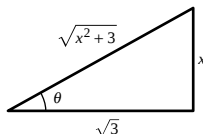
30. Let $2x = 4 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{4x^2 + 16} = 4 \sec \theta$.

$$\begin{aligned} \int \frac{1}{x\sqrt{4x^2 + 16}} dx &= \int \frac{2 \sec^2 \theta d\theta}{2 \tan \theta (4 \sec \theta)} \\ &= \frac{1}{4} \int \frac{\sec \theta}{\tan \theta} d\theta = \frac{1}{4} \int \csc \theta d\theta \\ &= -\frac{1}{4} \ln |\csc \theta + \cot \theta| + C = -\frac{1}{4} \ln \left| \frac{\sqrt{x^2 + 4} + 2}{x} \right| + C \end{aligned}$$



32. Let $x = \sqrt{3} \tan \theta$, $dx = \sqrt{3} \sec^2 \theta d\theta$, $x^2 + 3 = 3 \sec^2 \theta$.

$$\begin{aligned} \int \frac{1}{(x^2 + 3)^{3/2}} dx &= \int \frac{\sqrt{3} \sec^2 \theta d\theta}{3 \sqrt{3} \sec^3 \theta} \\ &= \frac{1}{3} \int \cos \theta d\theta \\ &= \frac{1}{3} \sin \theta + C \\ &= \frac{x}{3\sqrt{x^2 + 3}} + C \end{aligned}$$

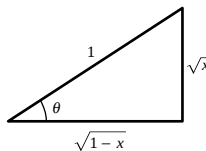


34. Let $u = x^2 + 2x + 2$, $du = (2x + 2) dx$.

$$\int (x + 1) \sqrt{x^2 + 2x + 2} dx = \frac{1}{2} \int (x^2 + 2x + 2)^{1/2} (2x + 2) dx = \frac{1}{3} (x^2 + 2x + 2)^{3/2} + C$$

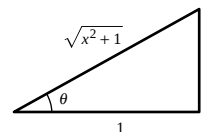
36. Let $\sqrt{x} = \sin \theta$, $x = \sin^2 \theta$, $dx = 2 \sin \theta \cos \theta d\theta$, $\sqrt{1 - x} = \cos \theta$.

$$\begin{aligned} \int \frac{\sqrt{1 - x}}{\sqrt{x}} dx &= \int \frac{\cos \theta (2 \sin \theta \cos \theta d\theta)}{\sin \theta} \\ &= 2 \int \cos^2 \theta d\theta \\ &= \int (1 + \cos 2\theta) d\theta \\ &= (\theta + \sin \theta \cos \theta) + C \\ &= \arcsin \sqrt{x} + \sqrt{x} \sqrt{1 - x} + C \end{aligned}$$



38. Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $x^2 + 1 = \sec^2 \theta$.

$$\begin{aligned} \int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1} dx &= \frac{1}{4} \int \frac{4x^3 + 4x}{x^4 + 2x^2 + 1} dx + \int \frac{1}{(x^2 + 1)^2} dx \\ &= \frac{1}{4} \ln(x^4 + 2x^2 + 1) + \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\ &= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{2} \left[\ln(x^2 + 1) + \arctan x + \frac{x}{x^2 + 1} \right] + C \end{aligned}$$



40. $u = \arcsin x, \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx, dv = x dx \Rightarrow v = \frac{x^2}{2}$

$$\int x \arcsin x dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$x = \sin \theta, dx = \cos \theta d\theta, \sqrt{1-x^2} = \cos \theta$$

$$\int x \arcsin x dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta = \frac{x^2}{2} \arcsin x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right] + C = \frac{x^2}{2} \arcsin x - \frac{1}{4} [\theta - \sin \theta \cos \theta] + C$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} [\arcsin x - x\sqrt{1-x^2}] + C = \frac{1}{4} [(2x^2 - 1) \arcsin x + x\sqrt{1-x^2}] + C$$

42. Let $x - 1 = \sin \theta, dx = \cos \theta d\theta, \sqrt{1 - (x - 1)^2} = \sqrt{2x - x^2} = \cos \theta$.

$$\int \frac{x^2}{\sqrt{2x - x^2}} dx = \int \frac{x^2}{\sqrt{1 - (x - 1)^2}} dx$$

$$= \int \frac{(1 + \sin \theta)^2 (\cos \theta d\theta)}{\cos \theta}$$

$$= \int (1 + 2 \sin \theta + \sin^2 \theta) d\theta$$

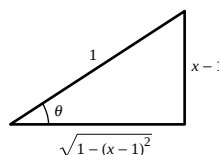
$$= \int \left(\frac{3}{2} + 2 \sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta + C$$

$$= \frac{3}{2} \theta - 2 \cos \theta - \frac{1}{2} \sin \theta \cos \theta + C$$

$$= \frac{3}{2} \arcsin(x - 1) - 2\sqrt{2x - x^2} - \frac{1}{2}(x - 1)\sqrt{2x - x^2} + C$$

$$= \frac{3}{2} \arcsin(x - 1) - \frac{1}{2} \sqrt{2x - x^2}(x + 3) + C$$



44. Let $x - 3 = 2 \sec \theta, dx = 2 \sec \theta \tan \theta d\theta, \sqrt{(x - 3)^2 - 4} = 2 \tan \theta$.

$$\int \frac{x}{\sqrt{x^2 - 6x + 5}} dx = \int \frac{x}{\sqrt{(x - 3)^2 - 4}} dx = \int \frac{(2 \sec \theta + 3)}{2 \tan \theta} (2 \sec \theta \tan \theta) d\theta$$

$$= \int (2 \sec^2 \theta + 3 \sec \theta) d\theta$$

$$= 2 \tan \theta + 3 \ln |\sec \theta + \tan \theta| + C_1$$

$$= 2 \left(\frac{\sqrt{(x - 3)^2 - 4}}{2} \right) + 3 \ln \left| \frac{x - 3}{2} + \frac{\sqrt{(x - 3)^2 - 4}}{2} \right| + C_1$$

$$= \sqrt{x^2 - 6x + 5} + 3 \ln |(x - 3) + \sqrt{x^2 - 6x + 5}| + C$$

46. Same substitution as in Exercise 45

$$\begin{aligned}
 \text{(a)} \quad \int \frac{1}{(1-t^2)^{5/2}} dt &= \int \frac{\cos \theta d\theta}{\cos^5 \theta} = \int \sec^4 \theta d\theta = \int (\tan^2 \theta + 1) \sec^2 \theta d\theta \\
 &= \frac{1}{3} \tan^3 \theta + \tan \theta + C = \frac{1}{3} \left(\frac{t}{\sqrt{1-t^2}} \right)^3 + \frac{t}{\sqrt{1-t^2}} + C \\
 \text{Thus, } \int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}} dt &= \left[\frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}} \right]_0^{\sqrt{3}/2} \\
 &= \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.
 \end{aligned}$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}} dt = \left[\frac{1}{3} \tan^3 \theta + \tan \theta \right]_0^{\pi/3} = \frac{1}{3} (\sqrt{3})^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

48. (a) Let $5x = 3 \sin \theta$, $dx = \frac{3}{5} \cos \theta d\theta$, $\sqrt{9-25x^2} = 3 \cos \theta$.

$$\begin{aligned}
 \int \sqrt{9-25x^2} dx &= \int (3 \cos \theta) \frac{3}{5} \cos \theta d\theta \\
 &= \frac{9}{5} \int \frac{1 + \cos 2\theta}{2} d\theta \\
 &= \frac{9}{10} \left[\theta + \frac{1}{2} \sin 2\theta \right] + C \\
 &= \frac{9}{10} [\theta + \sin \theta \cos \theta] + C \\
 &= \frac{9}{10} \left[\arcsin \frac{5x}{3} + \frac{5x}{3} \cdot \frac{\sqrt{9-25x^2}}{3} \right] + C \\
 \text{Thus, } \int_0^{3/5} \sqrt{9-25x^2} dx &= \frac{9}{10} \left[\arcsin \frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9} \right]_0^{3/5} = \frac{9}{10} \left[\frac{\pi}{2} \right] = \frac{9\pi}{20}.
 \end{aligned}$$

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

$$\text{Thus, } \int_0^{3/5} \sqrt{9-25x^2} dx = \left[\frac{9}{10} (\theta + \sin \theta \cos \theta) \right]_0^{\pi/2} = \frac{9}{10} \left(\frac{\pi}{2} \right) = \frac{9\pi}{20}.$$

50. (a) Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$, $\sqrt{x^2-9} = 3 \tan \theta$.

$$\begin{aligned}
 \int \frac{\sqrt{x^2-9}}{x^2} dx &= \int \frac{3 \tan \theta}{9 \sec^2 \theta} 3 \sec \theta \tan \theta d\theta \\
 &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\
 &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\
 &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\
 &= \int (\sec \theta - \cos \theta) d\theta \\
 &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \\
 &= \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| - \frac{\sqrt{x^2-9}}{x} + C
 \end{aligned}$$

—CONTINUED—

50. —CONTINUED—

$$\text{Hence, } \int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx = \left[\ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| - \frac{\sqrt{x^2 - 9}}{x} \right]_3^6 = \ln|2 + \sqrt{3}| - \frac{\sqrt{3}}{2}.$$

(b) When $x = 3$, $\theta = 0$; when $x = 6$, $\theta = \frac{\pi}{3}$.

$$\text{Hence, } \int_3^6 \frac{\sqrt{x^2 - 9}}{x^2} dx = \left[\ln|\sec \theta + \tan \theta| - \sin \theta \right]_0^{\pi/3} = \ln|2 + \sqrt{3}| - \frac{\sqrt{3}}{2}.$$

$$52. \int (x^2 + 2x + 11)^{3/2} dx = \frac{1}{4}(x + 1)(x^2 + 2x + 26)\sqrt{x^2 + 2x + 11} + \frac{75}{2} \ln|\sqrt{x^2 + 2x + 11} + (x + 1)| + C$$

$$54. \int x^2 \sqrt{x^2 - 4} dx = \frac{1}{4}x^3 \sqrt{x^2 - 4} - \frac{1}{2}x \sqrt{x^2 - 4} - 2 \ln|x + \sqrt{x^2 - 4}| + C$$

56. (a) Substitution: $u = x^2 + 1$, $du = 2x dx$

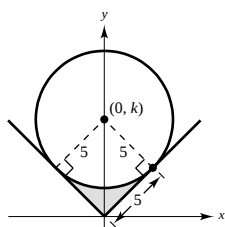
(b) Trigonometric substitution: $x = \sec \theta$

58. (a) $x^2 + (y - k)^2 = 25$

Radius of circle = 5

$$k^2 = 5^2 + 5^2 = 50$$

$$k = 5\sqrt{2}$$



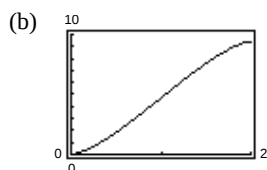
(b) Area = square - $\frac{1}{4}$ (circle)

$$= 25 - \frac{1}{4}\pi(5)^2 = 25\left(1 - \frac{\pi}{4}\right)$$

$$(c) \text{ Area} = r^2 - \frac{1}{4}\pi r^2 = r^2\left(1 - \frac{\pi}{4}\right)$$

60. (a) Place the center of the circle at $(0, 1)$; $x^2 + (y - 1)^2 = 1$. The depth d satisfies $0 \leq d \leq 2$. The volume is

$$\begin{aligned} V &= 3 \cdot 2 \int_0^d \sqrt{1 - (y - 1)^2} dy \\ &= 6 \cdot \frac{1}{2} \left[\arcsin(y - 1) + (y - 1)\sqrt{1 - (y - 1)^2} \right]_0^d \quad (\text{Theorem 7.2 (1)}) \\ &= 3[\arcsin(d - 1) + (d - 1)\sqrt{1 - (d - 1)^2} - \arcsin(-1)] \\ &= \frac{3\pi}{2} + 3 \arcsin(d - 1) + 3(d - 1)\sqrt{2d - d^2}. \end{aligned}$$



(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

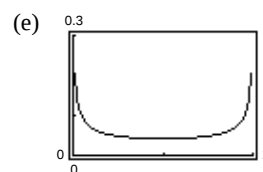
$$y = \frac{3\pi}{4}, y = \frac{3\pi}{2}, y = \frac{9\pi}{4}$$

intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

$$(d) \quad V = 6 \int_0^d \sqrt{1 - (y - 1)^2} dy$$

$$\frac{dV}{dt} = \frac{dV}{dd} \cdot \frac{dd}{dt} = 6\sqrt{1 - (d - 1)^2} \cdot d'(t) = \frac{1}{4}$$

$$\Rightarrow d'(t) = \frac{1}{24\sqrt{1 - (d - 1)^2}}$$

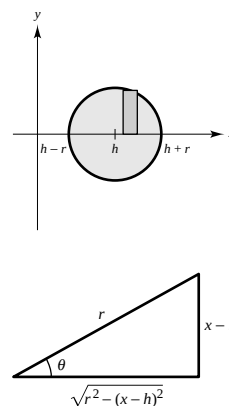


The minimum occurs at $d = 1$, which is the widest part of the tank.

62. Let $x - h = r \sin \theta$, $dx = r \cos \theta d\theta$, $\sqrt{r^2 - (x - h)^2} = r \cos \theta$.

Shell Method:

$$\begin{aligned} V &= 4\pi \int_{h-r}^{h+r} x \sqrt{r^2 - (x-h)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) r \cos \theta (r \cos \theta) d\theta = 4\pi r^2 \int_{-\pi/2}^{\pi/2} (h + r \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi r^2 \left[\frac{h}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + r \int_{-\pi/2}^{\pi/2} \sin \theta \cos^2 \theta d\theta \right] \\ &= 2\pi r^2 h \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3 \left(\frac{\cos^3 \theta}{3} \right) \right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h \end{aligned}$$



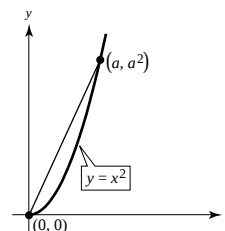
64. $y = \frac{1}{2}x^2$, $y' = x$, $1 + (y')^2 = 1 + x^2$

$$\begin{aligned} s &= \int_0^4 \sqrt{1 + x^2} dx = \left[\frac{1}{2} (x\sqrt{x^2 + 1} + \ln|x + \sqrt{x^2 + 1}|) \right]_0^4 \quad (\text{Theorem 7.2}) \\ &= \frac{1}{2} [4\sqrt{17} + \ln(4 + \sqrt{17})] \approx 9.2936 \end{aligned}$$

66. (a) Along line: $d_1 = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2}$

Along parabola: $y = x^2$, $y' = 2x$

$$\begin{aligned} d_2 &= \int_0^a \sqrt{1 + 4x^2} dx \\ &= \frac{1}{4} \left[2x\sqrt{4x^2 + 1} + \ln|2x + \sqrt{4x^2 + 1}| \right]_0^a \quad (\text{Theorem 7.2}) \\ &= \frac{1}{4} [2a\sqrt{4a^2 + 1} + \ln(2a + \sqrt{4a^2 + 1})] \end{aligned}$$



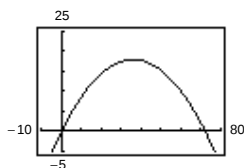
(b) For $a = 1$, $d_1 = \sqrt{2}$ and $d_2 = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}) \approx 1.4789$.

For $a = 10$, $d_1 = 10\sqrt{101} \approx 100.4988$

$d_2 \approx 101.0473$.

(c) As a increases, $d_2 - d_1 \rightarrow 0$.

68. (a)



(b) $y = 0$ for $x = 72$

(c) $y = x - \frac{x^2}{72}$, $y' = 1 - \frac{x}{36}$, $1 + (y')^2 = 1 + \left(1 - \frac{x}{36}\right)^2$

$$\begin{aligned} s &= \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} dx = -36 \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \left(-\frac{1}{36}\right) dx \\ &= -\frac{36}{2} \left[\left(1 - \frac{x}{36}\right) \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} + \ln \left| \left(1 - \frac{x}{36}\right) + \sqrt{1 + \left(1 - \frac{x}{36}\right)^2} \right| \right]_0^{72} \\ &= -18 [(-\sqrt{2} + \ln|-1 + \sqrt{2}|) - (\sqrt{2} + \ln|1 + \sqrt{2}|)] = 36\sqrt{2} + 18 \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 82.641 \end{aligned}$$

70. First find where the curves intersect.

$$y^2 = 16 - (x - 4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x - 4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x - 4)(x^2 + 4x + 32) \Rightarrow x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2 dx + \frac{1}{4}\pi(4)^2 = \left[\frac{1}{12}x^3\right]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

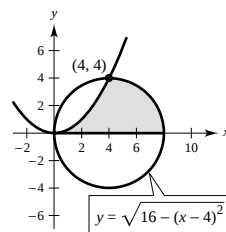
$$\begin{aligned} M_y &= \int_0^4 x \left[\frac{1}{4}x^2 \right] dx + \int_4^8 x \sqrt{16 - (x - 4)^2} dx \\ &= \left[\frac{x^4}{16} \right]_0^4 + \int_4^8 (x - 4) \sqrt{16 - (x - 4)^2} dx + \int_4^8 4 \sqrt{16 - (x - 4)^2} dx \\ &= 16 + \left[-\frac{1}{3}(16 - (x - 4)^2)^{3/2} \right]_4^8 + 2 \left[16 \arcsin \frac{x - 4}{4} + (x - 4) \sqrt{16 - (x - 4)^2} \right]_4^8 \\ &= 16 + \frac{1}{3}16^{3/2} + 2 \left[16 \left(\frac{\pi}{2} \right) \right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi \end{aligned}$$

$$\begin{aligned} M_x &= \int_0^4 \frac{1}{2} \left(\frac{1}{4}x^2 \right)^2 dx + \int_4^8 \frac{1}{2}(16 - (x - 4)^2) dx \\ &= \left[\frac{1}{32} \cdot \frac{x^5}{5} \right]_0^4 + \left[8x - \frac{(x - 4)^3}{6} \right]_4^8 \\ &= \frac{32}{5} + \left(64 - \frac{64}{6} \right) - 32 = \frac{416}{15} \end{aligned}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

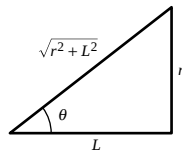
$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$$



72. Let $r = L \tan \theta$, $dr = L \sec^2 \theta d\theta$, $r^2 + L^2 = L^2 \sec^2 \theta$.

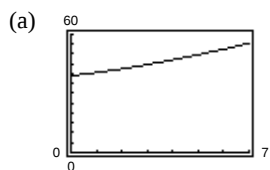
$$\begin{aligned} \frac{1}{R} \int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}} dr &= \frac{2mL}{R} \int_a^b \frac{L \sec^2 \theta d\theta}{L^3 \sec^3 \theta} \\ &= \frac{2m}{RL} \int_a^b \cos \theta d\theta \\ &= \left[\frac{2m}{RL} \sin \theta \right]_a^b \\ &= \left[\frac{2m}{RL} \frac{r}{\sqrt{r^2 + L^2}} \right]_0^R \\ &= \frac{2m}{L \sqrt{R^2 + L^2}} \end{aligned}$$



$$\begin{aligned}
 74. (a) F_{\text{inside}} &= 48 \int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2} dy \\
 &= 96 \left[0.8 \int_{-1}^{0.8} \sqrt{1 - y^2} dy - \int_{-1}^{0.8} y\sqrt{1 - y^2} dy \right] \\
 &= 96 \left[\frac{0.8}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{3} (1 - y^2)^{3/2} \right]_{-1}^{0.8} \approx 96(1.263) \approx 121.3 \text{ lbs}
 \end{aligned}$$

$$\begin{aligned}
 (b) F_{\text{outside}} &= 64 \int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2} dy \\
 &= 128 \left[0.4 \int_{-1}^{0.4} \sqrt{1 - y^2} dy - \int_{-1}^{0.4} y\sqrt{1 - y^2} dy \right] \\
 &= 128 \left[\frac{0.4}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{3} (1 - y^2)^{3/2} \right]_{-1}^{0.4} \approx 92.98
 \end{aligned}$$

$$76. S = \sqrt{1520.4 + 111.2t + 15.8t^2}$$



$$(b) S'(t) = \frac{1}{2}(1520.4 + 111.2t + 15.8t^2)^{-1/2}(111.2 + 31.6t)$$

$$S'(5) \approx 2.71$$

$$(c) \text{Average value} = \frac{1}{2} \int_{10}^{12} S(t) dt \approx 68.24$$

78. False

$$\begin{aligned}
 \int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\tan \theta}{\sec \theta} (\sec \theta \tan \theta d\theta) \\
 &= \int \tan^2 \theta d\theta
 \end{aligned}$$

80. True

$$\int_{-1}^1 x^2 \sqrt{1 - x^2} dx = 2 \int_0^1 x^2 \sqrt{1 - x^2} dx = 2 \int_0^{\pi/2} (\sin^2 \theta)(\cos \theta)(\cos \theta d\theta) = 2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta$$

Section 7.5 Partial Fractions

$$2. \frac{4x^2 + 3}{(x - 5)^3} = \frac{A}{x - 5} + \frac{B}{(x - 5)^2} + \frac{C}{(x - 5)^3}$$

$$4. \frac{x - 2}{x^2 + 4x + 3} = \frac{x - 2}{(x + 1)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x + 3}$$

$$6. \frac{2x - 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$\begin{aligned}
 8. \frac{1}{4x^2 - 9} &= \frac{1}{(2x - 3)(2x + 3)} = \frac{A}{2x - 3} + \frac{B}{2x + 3} \\
 1 &= A(2x + 3) + B(2x - 3)
 \end{aligned}$$

$$\text{When } x = \frac{3}{2}, 1 = 6A, A = \frac{1}{6}.$$

$$\text{When } x = -\frac{3}{2}, 1 = -6B, B = -\frac{1}{6}.$$

$$\begin{aligned}
 \int \frac{1}{4x^2 - 9} dx &= \frac{1}{6} \left[\int \frac{1}{2x - 3} dx - \int \frac{1}{2x + 3} dx \right] \\
 &= \frac{1}{12} [\ln|2x - 3| - \ln|2x + 3|] + C \\
 &= \frac{1}{12} \ln \left| \frac{2x - 3}{2x + 3} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 10. \int \frac{x + 1}{x^2 + 4x + 3} dx &= \int \frac{(x + 1)}{(x + 1)(x + 3)} dx \\
 &= \int \frac{1}{x + 3} dx = \ln|x + 3| + C
 \end{aligned}$$

$$12. \frac{5x^2 - 12x - 12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$5x^2 - 12x - 12 = A(x^2 - 4) + Bx(x+2) + Cx(x-2)$$

When $x = 0$, $-12 = -4A \Rightarrow A = 3$. When $x = 2$, $-16 = 8B \Rightarrow B = -2$. When $x = -2$, $32 = 8C \Rightarrow C = 4$.

$$\int \frac{5x^2 - 12x - 12}{x^3 - 4x} dx = \int \frac{3}{x} dx + \int \frac{-2}{x-2} dx + \int \frac{4}{x+2} dx = 3 \ln|x| - 2 \ln|x-2| + 4 \ln|x+2| + C$$

$$14. \frac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \frac{2x + 1}{(x+2)(x-1)} = x - 1 + \frac{A}{x+2} + \frac{B}{x-1}$$

$$2x + 1 = A(x-1) + B(x+2)$$

When $x = -2$, $-3 = -3A$, $A = 1$. When $x = 1$, $3 = 3B$, $B = 1$.

$$\begin{aligned} \int \frac{x^3 - x + 3}{x^2 + x - 2} dx &= \int \left[x - 1 + \frac{1}{x+2} + \frac{1}{x-1} \right] dx \\ &= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2 + x - 2| + C \end{aligned}$$

$$16. \frac{x+2}{x(x-4)} = \frac{A}{x-4} + \frac{B}{x}$$

$$x+2 = Ax + B(x-4)$$

When $x = 4$, $6 = 4A$, $A = \frac{3}{2}$.

When $x = 0$, $2 = -4B$, $B = -\frac{1}{2}$.

$$\begin{aligned} \int \frac{x+2}{x^2-4x} dx &= \int \left[\frac{3/2}{x-4} - \frac{1/2}{x} \right] dx \\ &= \frac{3}{2} \ln|x-4| - \frac{1}{2} \ln|x| + C \end{aligned}$$

$$18. \frac{2x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x-3 = A(x-1) + B$$

When $x = 1$, $B = -1$. When $x = 0$, $A = 2$.

$$\begin{aligned} \int \frac{2x-3}{(x-1)^2} dx &= \int \left[\frac{2}{x-1} - \frac{1}{(x-1)^2} \right] dx \\ &= 2 \ln|x-1| + \frac{1}{x-1} + C \end{aligned}$$

$$20. \frac{4x^2}{x^3 + x^2 - x - 1} = \frac{4x^2}{x^2(x+1) - (x+1)} = \frac{4x^2}{(x^2-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

When $x = -1$, $4 = -2C \Rightarrow C = -2$. When $x = 1$, $4 = 4A \Rightarrow A = 1$. When $x = 0$, $0 = 1 - B + 2 \Rightarrow B = 3$.

$$\begin{aligned} \int \frac{4x^2}{x^3 + x^2 - x - 1} dx &= \int \frac{1}{x-1} dx + \int \frac{3}{x+1} dx - \int \frac{2}{(x+1)^2} dx \\ &= \ln|x-1| + 3 \ln|x+1| + \frac{2}{(x+1)} + C \end{aligned}$$

$$22. \frac{6x}{x^3 - 8} = \frac{6x}{(x-2)(x^2 + 2x + 4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2 + 2x + 4}$$

$$6x = A(x^2 + 2x + 4) + (Bx + C)(x-2)$$

When $x = 2$, $12 = 12A \Rightarrow A = 1$. When $x = 0$, $0 = 4 - 2C \Rightarrow C = 2$. When $x = 1$, $6 = 7 + (B+2)(-1) \Rightarrow B = -1$.

$$\begin{aligned} \int \frac{6x}{x^3 - 8} dx &= \int \frac{1}{x-2} dx + \int \frac{-x+2}{x^2 + 2x + 4} dx = \int \frac{1}{x-2} dx + \int \frac{-x-1}{x^2 + 2x + 4} dx + \int \frac{3}{(x^2 + 2x + 1) + 3} dx \\ &= \ln|x-2| - \frac{1}{2} \ln|x^2 + 2x + 4| + \frac{3}{\sqrt{3}} \arctan\left(\frac{x+1}{\sqrt{3}}\right) + C \\ &= \ln|x-2| - \frac{1}{2} \ln|x^2 + 2x + 4| + \sqrt{3} \arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C \end{aligned}$$

$$24. \frac{x^2 - x + 9}{(x^2 + 9)^2} = \frac{Ax + B}{x^2 + 9} + \frac{Cx + D}{(x^2 + 9)^2}$$

$$\begin{aligned} x^2 - x + 9 &= (Ax + B)(x^2 + 9) + Cx + D \\ &= Ax^3 + Bx^2 + (9A + C)x + (9B + D) \end{aligned}$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $D = 0$, and $C = -1$.

$$\begin{aligned} \int \frac{x^2 - x + 9}{(x^2 + 9)^2} dx &= \int \frac{1}{x^2 + 9} dx - \int \frac{x}{(x^2 + 9)^2} dx \\ &= \frac{1}{3} \arctan\left(\frac{x}{3}\right) + \frac{1}{2(x^2 + 9)} + C \end{aligned}$$

$$26. \frac{x^2 - 4x + 7}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$x^2 - 4x + 7 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

When $x = -1$, $12 = 6A$. When $x = 0$, $7 = 3A + C$. When $x = 1$, $4 = 2A + 2B + 2C$. Solving these equations we have $A = 2$, $B = -1$, $C = 1$.

$$\begin{aligned} \int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3} dx &= 2 \int \frac{1}{x + 1} dx + \int \frac{-x + 1}{x^2 - 2x + 3} dx \\ &= 2 \ln|x + 1| - \frac{1}{2} \ln|x^2 - 2x + 3| + C \end{aligned}$$

$$28. \frac{x^2 + x + 3}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

$$\begin{aligned} x^2 + x + 3 &= (Ax + B)(x^2 + 3) + Cx + D \\ &= Ax^3 + Bx^2 + (3A + C)x + (3B + D) \end{aligned}$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $3A + C = 1$, $3B + D = 3$. Solving these equations we have $A = 0$, $B = 1$, $C = 1$, $D = 0$.

$$\begin{aligned} \int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9} dx &= \int \left[\frac{1}{x^2 + 3} + \frac{x}{(x^2 + 3)^2} \right] dx \\ &= \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - \frac{1}{2(x^2 + 3)} + C \end{aligned}$$

$$30. \frac{x - 1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$$

$$x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$$

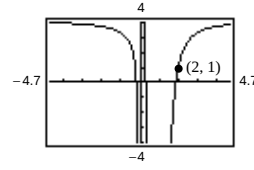
When $x = 0$, $B = -1$. When $x = -1$, $C = -2$. When $x = 1$, $0 = 2A + 2B + C$. Solving these equations we have $A = 2$, $B = -1$, $C = -2$.

$$\begin{aligned} \int_1^5 \frac{x - 1}{x^2(x + 1)} dx &= 2 \int_1^5 \frac{1}{x} dx - \int_1^5 \frac{1}{x^2} dx - 2 \int_1^5 \frac{1}{x + 1} dx \\ &= \left[2 \ln|x| + \frac{1}{x} - 2 \ln|x + 1| \right]_1^5 \\ &= \left[2 \ln \left| \frac{x}{x + 1} \right| + \frac{1}{x} \right]_1^5 \\ &= 2 \ln \frac{5}{3} - \frac{4}{5} \end{aligned}$$

$$32. \int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx = \int_0^1 dx - \int_0^1 \frac{2x + 1}{x^2 + x + 1} dx = \left[x - \ln|x^2 + x + 1| \right]_0^1 = 1 - \ln 3$$

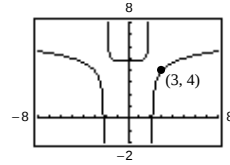
$$34. \int \frac{6x^2 + 1}{x^2(x-1)^3} dx = 3 \ln \left| \frac{x-1}{x} \right| + \frac{1}{x} + \frac{2}{x-1} - \frac{7}{2(x-1)^2} + C$$

$$(2, 1): 3 \ln \left| \frac{1}{2} \right| + \frac{1}{2} + \frac{2}{1} - \frac{7}{2} + C = 1 \Rightarrow C = 2 - 3 \ln \frac{1}{2}$$



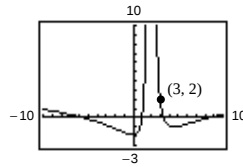
$$36. \int \frac{x^3}{(x^2 - 4)^2} dx = \frac{1}{2} \ln|x^2 - 4| - \frac{2}{x^2 - 4} + C$$

$$(3, 4): \frac{1}{2} \ln 5 - \frac{2}{5} + C = 4 \Rightarrow C = \frac{22}{5} - \frac{1}{2} \ln 5$$



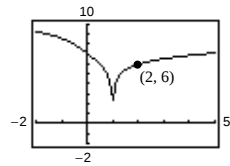
$$38. \int \frac{x(2x-9)}{x^3 - 6x^2 + 12x - 8} dx = 2 \ln|x-2| + \frac{1}{x-2} + \frac{5}{(x-2)^2} + C$$

$$(3, 2): 0 + 1 + 5 + C = 2 \Rightarrow C = -4$$



$$40. \int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1} dx = -\arctan x + \ln|x-1| + C$$

$$(2, 6): -\arctan 2 + 0 + C = 6 \Rightarrow C = 6 + \arctan 2$$



42. Let $u = \cos x$, $du = -\sin x dx$.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$. When $u = -1$, $B = -1$, $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned} \int \frac{\sin x}{\cos x + \cos^2 x} dx &= -\int \frac{1}{u(u+1)} du \\ &= \int \frac{1}{u+1} du - \int \frac{1}{u} du \\ &= \ln|u+1| - \ln|u| + C \\ &= \ln \left| \frac{u+1}{u} \right| + C \\ &= \ln \left| \frac{\cos x + 1}{\cos x} \right| + C \\ &= \ln|1 + \sec x| + C \end{aligned}$$

$$44. \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}, u = \tan x, du = \sec^2 x dx$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.

When $u = -1$, $1 = -B \Rightarrow B = -1$.

$$\begin{aligned} \int \frac{\sec^2 x dx}{\tan x(\tan x + 1)} &= \int \frac{1}{u(u+1)} du \\ &= \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du \\ &= \ln|u| - \ln|u+1| + C \\ &= \ln \left| \frac{u}{u+1} \right| + C \\ &= \ln \left| \frac{\tan x}{\tan x + 1} \right| + C \end{aligned}$$

46. Let $u = e^x$, $du = e^x dx$.

$$\frac{1}{(u^2 + 1)(u - 1)} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + 1}$$

$$1 = A(u^2 + 1) + (Bu + C)(u - 1)$$

When $u = 1$, $A = \frac{1}{2}$.

When $u = 0$, $1 = A - C$.

When $u = -1$, $1 = 2A + 2B - 2C$. Solving these equations we have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$, $u = e^x$, $du = e^x dx$.

$$\begin{aligned} \int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx &= \int \frac{1}{(u^2 + 1)(u - 1)} du \\ &= \frac{1}{2} \left(\int \frac{1}{u - 1} du - \int \frac{u + 1}{u^2 + 1} du \right) \\ &= \frac{1}{2} \left(\ln|u - 1| - \frac{1}{2} \ln|u^2 + 1| - \arctan u \right) + C \\ &= \frac{1}{4} (2 \ln|e^x - 1| - \ln|e^{2x} + 1| - 2 \arctan e^x) + C \end{aligned}$$

48. $\frac{1}{a^2 - x^2} = \frac{A}{a - x} + \frac{B}{a + x}$

$$1 = A(a + x) + B(a - x)$$

When $x = a$, $A = 1/2a$.

When $x = -a$, $B = 1/2a$.

$$\begin{aligned} \int \frac{1}{a^2 - x^2} dx &= \frac{1}{2a} \int \left(\frac{1}{a - x} + \frac{1}{a + x} \right) dx \\ &= \frac{1}{2a} (-\ln|a - x| + \ln|a + x|) + C \\ &= \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C \end{aligned}$$

50. $\frac{1}{x^2(a + bx)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{a + bx}$

$$1 = Ax(a + bx) + B(a + bx) + Cx^2$$

When $x = 0$, $1 = Ba \Rightarrow B = 1/a$.

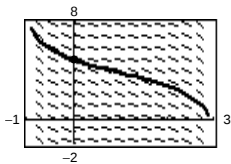
When $x = -a/b$, $1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$.

When $x = 1$, $1 = (a + b)A + (a + b)B + C \Rightarrow$

$$A = -b/a^2.$$

$$\begin{aligned} \int \frac{1}{x^2(a + bx)} dx &= \int \left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a + bx} \right) dx \\ &= -\frac{b}{a^2} \ln|x| - \frac{1}{ax} + \frac{b}{a^2} \ln|a + bx| + C \\ &= -\frac{1}{ax} + \frac{b}{a^2} \ln \left| \frac{a + bx}{x} \right| + C \\ &= -\frac{1}{ax} - \frac{b}{a^2} \ln \left| \frac{x}{a + bx} \right| + C \end{aligned}$$

52. $\frac{dy}{dx} = \frac{4}{(x^2 - 2x - 3)}, y(0) = 5$



54. (a) $\frac{N(x)}{D(x)} = \frac{A_1}{px + q} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$

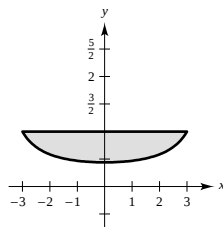
(b) $\frac{N(x)}{D(x)} = \frac{A_1 + B_1x}{(ax^2 + bx + c)} + \cdots + \frac{A_n + B_nx}{(ax^2 + bx + c)^n}$

56. $A = 2 \int_0^3 \left(1 - \frac{7}{16 - x^2} \right) dx$

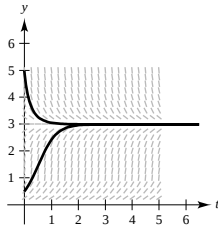
$$= 2 \int_0^3 dx - 14 \int_0^3 \frac{1}{16 - x^2} dx$$

$$= \left[2x - \frac{14}{8} \ln \left| \frac{4 + x}{4 - x} \right| \right]_0^3 \quad (\text{From Exercise 46})$$

$$= 6 - \frac{7}{4} \ln 7 \approx 2.595$$



58. (a)



(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t \rightarrow \infty} y(t) = 3$.

$$(d) \quad \frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$1 = A(L-y) + By \Rightarrow A = \frac{1}{L}, B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k dt$$

$$\frac{1}{L} \left[\int \frac{1}{y} dy + \int \frac{1}{L-y} dy \right] = \int k dt$$

$$\frac{1}{L} [\ln|y| - \ln|L-y|] = kt + C_1$$

$$\ln \left| \frac{y}{L-y} \right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

$$\text{When } t = 0, \frac{y_0}{L-y_0} = C_2 \Rightarrow \frac{y}{L-y} = \frac{y_0}{L-y_0} e^{kLt}.$$

$$\text{Solving for } y, \text{ you obtain } y = \frac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}.$$

$$60. (a) \quad V = \pi \int_0^3 \left(\frac{2x}{x^2+1} \right)^2 dx = 4\pi \int_0^3 \frac{x^2}{(x^2+1)^2} dx$$

$$= 4\pi \int_0^3 \left(\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2} \right) dx \quad (\text{partial fractions})$$

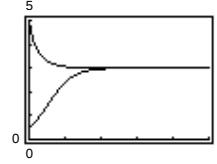
$$= 4\pi \left[\arctan x - \frac{1}{2} \left(\arctan x + \frac{x}{x^2+1} \right) \right]_0^3 \quad (\text{trigonometric substitution})$$

$$= 2\pi \left[\arctan x - \frac{x}{x^2+1} \right]_0^3 = 2\pi \left[\arctan 3 - \frac{3}{10} \right] \approx 5.963$$

(e) $k = 1, L = 3$

$$(i) y(0) = 5: y = \frac{15}{5 - 2e^{-3t}}$$

$$(ii) y(0) = \frac{1}{2}: y = \frac{3/2}{(1/2) + (5/2)e^{-3t}} = \frac{3}{1 + 5e^{-3t}}$$



$$(f) \quad \frac{dy}{dt} = ky(L-y)$$

$$\frac{d^2y}{dt^2} = k \left[y \left(\frac{-dy}{dt} \right) + (L-y) \frac{dy}{dt} \right] = 0$$

$$\Rightarrow y \frac{dy}{dt} = (L-y) \frac{dy}{dt}$$

$$\Rightarrow y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

—CONTINUED—

60. —CONTINUED—

$$(b) A = \int_0^3 \frac{2x}{x^2 + 1} dx = \left[\ln(x^2 + 1) \right]_0^3 = \ln 10$$

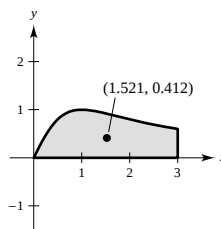
$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^3 \frac{2x^2}{x^2 + 1} dx = \frac{1}{\ln 10} \int_0^3 \left(2 - \frac{2}{x^2 + 1} \right) dx \\ &= \frac{1}{\ln 10} \left[2x - 2 \arctan x \right]_0^3 = \frac{2}{\ln 10} [3 - \arctan 3] \approx 1.521 \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{A} \left(\frac{1}{2} \right) \int_0^3 \left(\frac{2x}{x^2 + 1} \right)^2 dx = \frac{2}{\ln 10} \int_0^3 \frac{x^2}{(x^2 + 1)^2} dx \\ &= \frac{2}{\ln 10} \int_0^3 \left(\frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \right) dx \end{aligned}$$

$$= \frac{2}{\ln 10} \left[\arctan x - \frac{1}{2} \left(\arctan x + \frac{x}{x^2 + 1} \right) \right]_0^3$$

$$= \frac{2}{\ln 10} \left[\frac{1}{2} \arctan x - \frac{x}{2(x^2 + 1)} \right]_0^3 = \frac{1}{\ln 10} \left[\arctan x - \frac{x}{x^2 + 1} \right]_0^3 = \frac{1}{\ln 10} \left[\arctan 3 - \frac{3}{10} \right] \approx 0.412$$

$$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$$



(partial fractions)

(trigonometric substitution)

$$62. (a) \quad \frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x}, A = \frac{1}{z_0 - y_0}, B = -\frac{1}{z_0 - y_0} \quad (\text{Assume } y_0 \neq z_0)$$

$$\frac{1}{z_0 - y_0} \int \left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x} \right) dx = kt + C$$

$$\frac{1}{z_0 - y_0} \ln \left| \frac{z_0 - x}{y_0 - x} \right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0} \ln \frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0} \left[\ln \left| \frac{z_0 - x}{y_0 - x} \right| - \ln \left(\frac{z_0}{y_0} \right) \right] = kt$$

$$\ln \left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)} \right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0 [e^{(z_0 - y_0)kt} - 1]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

$$(b) (1) \text{ If } y_0 < z_0, \lim_{t \rightarrow \infty} x = y_0.$$

$$(2) \text{ If } y_0 > z_0, \lim_{t \rightarrow \infty} x = z_0.$$

$$(c) \text{ If } y_0 = z_0, \text{ then the original equation is}$$

$$\int \frac{1}{(y_0 - x)^2} dx = \int k dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \Rightarrow \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kt y_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kt y_0 + 1}$$

$$x = y_0 - \frac{y_0}{kt y_0 + 1}$$

$$\text{As } t \rightarrow \infty, x \rightarrow y_0 = x_0.$$

Section 7.6 Integration by Tables and Other Integration Techniques

2. By Formula 13: ($b = 2, a = -5$)

$$\begin{aligned}\frac{2}{3} \int \frac{1}{x^2(2x-5)^2} dx &= \frac{2}{3} \left(\frac{-1}{25} \right) \left[\frac{-5+4x}{x(-5+2x)} + \frac{4}{-5} \ln \left| \frac{x}{2x-5} \right| \right] + C \\ &= \frac{8}{375} \ln \left| \frac{x}{2x-5} \right| - \frac{2}{75} \frac{(4x-5)}{x(2x-5)} + C\end{aligned}$$

4. By Formula 29: ($a = 3$)

$$\frac{1}{3} \int \frac{\sqrt{x^2-9}}{x} dx = \frac{1}{3} \sqrt{x^2-9} - \operatorname{arcsec} \frac{|x|}{3} + C$$

6. By Formula 41: $\int \frac{x}{\sqrt{9-x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{3^2-(x^2)^2}} dx$
 $= \frac{1}{2} \arcsin \frac{x^2}{3} + C$

8. By Formulas 51 and 47: $\int \frac{\cos^3 \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos^3 \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx$

$$= 2 \left[\frac{\cos^2 \sqrt{x} \sin \sqrt{x}}{3} + \frac{2}{3} \int \cos \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx \right] = \frac{2}{3} \sin \sqrt{x} (\cos^2 \sqrt{x} + 2) + C$$

$$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$$

10. By Formula 71:

$$\begin{aligned}\int \frac{1}{1-\tan 5x} dx &= \frac{1}{5} \int \frac{1}{1-\tan 5x} (5) dx \\ &= \frac{1}{5} \left(\frac{1}{2} \right) (u - \ln |\cos u - \sin u|) + C \\ &= \frac{1}{10} (5x - \ln |\cos 5x - \sin 5x|) + C\end{aligned}$$

$$u = 5x, du = 5 dx$$

12. By Formula 85: ($a = -\frac{1}{2}, b = 2$)

$$\begin{aligned}\int e^{-x/2} \sin 2x dx &= \frac{e^{-x/2}}{(1/4) + 4} \left(-\frac{1}{2} \sin 2x - 2 \cos 2x \right) + C \\ &= \frac{4}{17} e^{-x/2} \left(-\frac{1}{2} \sin 2x - 2 \cos 2x \right) + C\end{aligned}$$

14. By Formulas 90 and 91: $\int (\ln x)^3 dx = x(\ln x)^3 - 3 \int (\ln x)^2 dx$

$$= x(\ln x)^3 - 3x[2 - 2 \ln x + (\ln x)^2] + C$$

$$= x[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6] + C$$

16. (a) By Formula 89: $\int x^4 \ln x dx = \frac{x^5}{5^2} [-1 + (4+1) \ln x] + C = \frac{-x^5}{25} + \frac{1}{5} x^5 \ln x + C$

(b) Integration by parts: $u = \ln x, du = \frac{1}{x} dx, dv = x^4 dx, v = \frac{x^5}{5}$

$$\int x^4 \ln x dx = \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \frac{1}{x} dx = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C$$

18. (a) By Formula 24: $a = \sqrt{75}$, $x = u$, and

$$\begin{aligned}\int \frac{1}{x^2 - 75} dx &= \frac{1}{2\sqrt{75}} \ln \left| \frac{x - \sqrt{75}}{x + \sqrt{75}} \right| + C \\ &= \frac{\sqrt{3}}{30} \ln \left| \frac{x - \sqrt{75}}{x + \sqrt{75}} \right| + C\end{aligned}$$

(b) Partial fractions:

$$\begin{aligned}\frac{1}{x^2 - 75} &= \frac{A}{x - \sqrt{75}} + \frac{B}{x + \sqrt{75}} \\ 1 &= A(x + \sqrt{75}) + B(x - \sqrt{75})\end{aligned}$$

$$x = \sqrt{75}: 1 = 2A\sqrt{75} \Rightarrow A = \frac{1}{2\sqrt{75}} = \frac{1}{10\sqrt{3}} = \frac{\sqrt{3}}{30}$$

$$x = -\sqrt{75}: 1 = -2B\sqrt{75} \Rightarrow B = -\frac{\sqrt{3}}{30}$$

$$\begin{aligned}\int \frac{1}{x^2 - 75} dx &= \int \left[\frac{\sqrt{3}/30}{x - \sqrt{75}} - \frac{\sqrt{3}/30}{x + \sqrt{75}} \right] dx \\ &= \frac{\sqrt{3}}{30} \ln \left| \frac{x - \sqrt{75}}{x + \sqrt{75}} \right| + C\end{aligned}$$

$$20. \text{ By Formula 21: } \int \frac{x}{\sqrt{1+x}} dx = -\frac{2}{3}(2-x)\sqrt{1+x} + C$$

$$\begin{aligned}22. \text{ By Formula 79: } \int \operatorname{arcsec} 2x \, dx &= \frac{1}{2} [2x \operatorname{arcsec} 2x - \ln|2x + \sqrt{4x^2 - 1}|] + C \\ u &= 2x, du = 2 \, dx\end{aligned}$$

$$24. \text{ By Formula 52: } \int x \sin x \, dx = \sin x - x \cos x + C$$

$$26. \text{ By Formula 7: } \int \frac{x^2}{(3x-5)^2} dx = \frac{1}{27} \left(3x - \frac{25}{3x-5} + 10 \ln|3x-5| \right) + C$$

$$28. \text{ By Formula 14: } \int \frac{1}{x^2 + 2x + 2} dx = \frac{2}{\sqrt{4}} \arctan\left(\frac{2x+2}{2}\right) + C = \arctan(x+1) + C$$

30. By Formula 56:

$$\begin{aligned}\int \frac{\theta^2}{1 - \sin \theta^3} d\theta &= \frac{1}{3} \int \frac{1}{1 - \sin \theta^3} 3\theta^2 d\theta \\ &= \frac{1}{3} (\tan \theta^3 + \sec \theta^3) + C\end{aligned}$$

32. By Formula 71:

$$\begin{aligned}\int \frac{e^x}{1 - \tan e^x} dx &= \frac{1}{2} (e^x - \ln|\cos e^x - \sin e^x|) + C \\ u &= e^x, du = e^x dx\end{aligned}$$

$$\begin{aligned}34. \text{ By Formula 23: } \int \frac{1}{t[1 + (\ln t)^2]} dt &= \int \frac{1}{1 + (\ln t)^2} \left(\frac{1}{t}\right) dt = \arctan(\ln t) + C \\ u &= \ln t, du = \frac{1}{t} dt\end{aligned}$$

$$36. \text{ By Formula 26: } \int \sqrt{3+x^2} \, dx = \frac{1}{2} (x\sqrt{x^2+3} + 3 \ln|x + \sqrt{x^2+3}|) + C$$

$$\begin{aligned}38. \text{ By Formula 27: } \int x^2 \sqrt{2 + (3x)^2} \, dx &= \frac{1}{27} \int (3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2} 3 \, dx \\ &= \frac{1}{8(27)} [3x(18x^2 + 2)\sqrt{2 + 9x^2} - 4 \ln|3x + \sqrt{2 + 9x^2}|] + C\end{aligned}$$

$$\begin{aligned}
 40. \text{ By Formula 77: } \int \sqrt{x} \arctan(x^{3/2}) dx &= \frac{2}{3} \int \arctan(x^{3/2}) \left(\frac{3}{2} \sqrt{x} \right) dx \\
 &= \frac{2}{3} [x^{3/2} \arctan(x^{3/2}) - \ln \sqrt{1+x^3}] + C
 \end{aligned}$$

$$\begin{aligned}
 42. \text{ By Formula 45: } \int \frac{e^x}{(1-e^{2x})^{3/2}} dx &= \frac{e^x}{\sqrt{1-e^{2x}}} + C \\
 u &= e^x, du = e^x dx
 \end{aligned}$$

$$\begin{aligned}
 44. \text{ By Formula 27:} \\
 \int (2x-3)^2 \sqrt{(2x-3)^2+4} dx &= \frac{1}{2} \int (2x-3)^2 \sqrt{(2x-3)^2+4} (2) dx \\
 &= \frac{1}{8} (2x-3) [(2x-3)^2+2] \sqrt{(2x-3)^2+4} - \ln|2x-3+\sqrt{(2x-3)^2+4}| + C \\
 u &= 2x-3, du = 2 dx
 \end{aligned}$$

$$\begin{aligned}
 46. \text{ By Formula 31: } \int \frac{\cos x}{\sqrt{\sin^2 x + 1}} dx &= \ln|\sin x + \sqrt{\sin^2 x + 1}| + C \\
 u &= \sin x, du = \cos x dx
 \end{aligned}$$

$$\begin{aligned}
 48. \int \sqrt{\frac{3-x}{3+x}} dx &= \int \frac{3-x}{\sqrt{9-x^2}} dx \\
 &= 3 \int \frac{1}{\sqrt{9-x^2}} dx + \int \frac{-x}{\sqrt{9-x^2}} dx \\
 &= 3 \arcsin \frac{x}{3} + \sqrt{9-x^2} + C
 \end{aligned}$$

50. By Formula 67:

$$\begin{aligned}
 \int \tan^3 \theta d\theta &= \frac{\tan^2 \theta}{2} - \int \tan \theta d\theta \\
 &= \frac{\tan^2 \theta}{2} + \ln|\cos x| + C
 \end{aligned}$$

$$52. \text{ Integration by parts: } w = u^n, dw = nu^{n-1} du, dv = \frac{du}{\sqrt{a+bu}}, v = \frac{2}{b} \sqrt{a+bu}$$

$$\begin{aligned}
 \int \frac{u^n}{\sqrt{a+bu}} du &= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2n}{b} \int u^{n-1} \sqrt{a+bu} du \\
 &= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2n}{b} \int u^{n-1} \sqrt{a+bu} \cdot \frac{\sqrt{a+bu}}{\sqrt{a+bu}} du \\
 &= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2n}{b} \int \frac{au^{n-1} + bu^n}{\sqrt{a+bu}} du \\
 &= \frac{2u^n}{b} \sqrt{a+bu} - \frac{2na}{b} \int \frac{u^{n-1}}{\sqrt{a+bu}} du - 2n \int \frac{u^n}{\sqrt{a+bu}} du
 \end{aligned}$$

$$\text{Therefore, } (2n+1) \int \frac{u^n}{\sqrt{a+bu}} du = \frac{2}{b} \left[u^n \sqrt{a+bu} - na \int \frac{u^{n-1}}{\sqrt{a+bu}} du \right] \text{ and}$$

$$\int \frac{u^n}{\sqrt{a+bu}} = \frac{2}{(2n+1)b} \left[u^n \sqrt{a+bu} - na \int \frac{u^{n-1}}{\sqrt{a+bu}} du \right].$$

$$54. \int u^n (\cos u) du = u^n \sin u - n \int u^{n-1} (\sin u) du$$

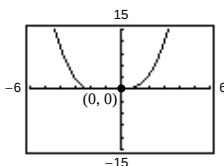
$$w = u^n, dv = \cos u du, dw = nu^{n-1} du, v = \sin u$$

$$56. \int (\ln u)^n du = u(\ln u)^n - \int n(\ln u)^{n-1} \left(\frac{1}{u}\right) u du = u(\ln u)^n - n \int (\ln u)^{n-1} du$$

$$w = (\ln u)^n, dv = du, dw = n(\ln u)^{n-1} \left(\frac{1}{u}\right) du, v = u$$

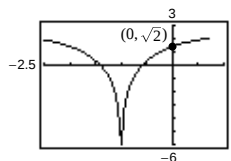
$$58. \int x\sqrt{x^2 + 2x} dx = \frac{1}{6} [2(x^2 + 2x)^{3/2} - 3(x+1)\sqrt{x^2 + 2x} + 3\ln|x+1 + \sqrt{x^2 + 2x}|] + C$$

$$(0, 0): \frac{1}{6} [3\ln|1|] + C = 0 \Rightarrow C = 0$$



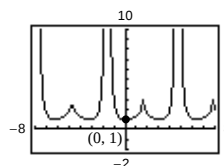
$$60. \int \frac{\sqrt{2-2x-x^2}}{x+1} dx = \sqrt{2-2x-x^2} - \sqrt{3} \ln \left| \frac{\sqrt{3} + \sqrt{2-2x-x^2}}{x+1} \right| + C$$

$$(0, \sqrt{2}): \sqrt{2} - \sqrt{3} \ln(\sqrt{3} + \sqrt{2}) + C = \sqrt{2} \Rightarrow C = \sqrt{3} \ln(\sqrt{3} + \sqrt{2})$$



$$62. \int \frac{\sin \theta}{(\cos \theta)(1 + \sin \theta)} d\theta = \frac{1}{2} \left[\frac{-\sin \theta}{1 + \sin \theta} + \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \right] + C$$

$$(0, 1): C = 1 \Rightarrow y = \frac{1}{2} \left[\frac{-\sin \theta}{1 + \sin \theta} + \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \right] + 1$$



$$64. \int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta = - \int \frac{-\sin \theta}{1 + (\cos \theta)^2} d\theta = -\arctan(\cos \theta) + C$$

$$\begin{aligned} 66. \int_0^{\pi/2} \frac{1}{3 - 2\cos \theta} d\theta &= \int_0^1 \left[\frac{\frac{2u}{1+u^2}}{3 - \frac{2(1-u^2)}{1+u^2}} \right] du \\ &= 2 \int_0^1 \frac{1}{5u^2 + 1} du \\ &= \left[\frac{2}{\sqrt{5}} \arctan(\sqrt{5}u) \right]_0^1 \\ &= \frac{2}{\sqrt{5}} \arctan \sqrt{5} \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned} 68. \int \frac{\cos \theta}{1 + \cos \theta} d\theta &= \int \frac{\cos \theta(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)} d\theta \\ &= \int \frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} d\theta \\ &= \int (\csc \theta \cot \theta - \cot^2 \theta) d\theta \\ &= \int (\csc \theta \cot \theta - (\csc^2 \theta - 1)) d\theta \\ &= -\csc \theta + \cot \theta + \theta + C \end{aligned}$$

$$\begin{aligned} 70. \int \frac{1}{\sec \theta - \tan \theta} d\theta &= \int \frac{1}{(1/\cos \theta) - (\sin \theta/\cos \theta)} d\theta \\ &= - \int \frac{-\cos \theta}{1 - \sin \theta} d\theta \\ &= -\ln|1 - \sin \theta| + C \\ u = 1 - \sin \theta, du &= -\cos \theta d\theta \end{aligned}$$

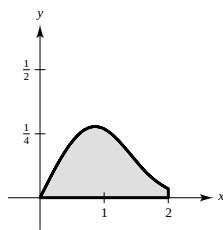
$$72. A = \int_0^2 \frac{x}{1 + e^{x^2}} dx$$

$$= \frac{1}{2} \int_0^2 \frac{2x dx}{1 + e^{x^2}}$$

$$= \frac{1}{2} \left[x^2 - \ln(1 + e^{x^2}) \right]_0^2$$

$$= \frac{1}{2} \left[4 - \ln(1 + e^4) \right] + \frac{1}{2} \ln 2$$

$$\approx 0.337 \text{ square units}$$



$$74. \text{ Log Rule: } \int \frac{1}{u} du, u = e^x + 1$$

76. Integration by parts

78. Formula 16 with $u = e^{2x}$

80. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formula 50, 54.

$$82. W = \int_0^5 \frac{500x}{\sqrt{26 - x^2}} dx$$

$$= -250 \int_0^5 (26 - x^2)^{-1/2} (-2x) dx$$

$$= \left[-500 \sqrt{26 - x^2} \right]_0^5$$

$$= 500(\sqrt{26} - 1)$$

$$\approx 2049.51 \text{ ft} \cdot \text{lbs}$$

$$84. \frac{1}{2-0} \int_0^2 \frac{5000}{1 + e^{4.8-1.9t}} dt = \frac{2500}{-1.9} \int_0^2 \frac{-1.9 dt}{1 + e^{4.8-1.9t}}$$

$$= -\frac{2500}{1.9} \left[(4.8 - 1.9t) - \ln(1 + e^{4.8-1.9t}) \right]_0^2$$

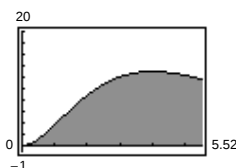
$$= -\frac{2500}{1.9} [(1 - \ln(1 + e)) - (4.8 - \ln(1 + e^{4.8}))]$$

$$= \frac{2500}{1.9} \left[3.8 + \ln\left(\frac{1 + e}{1 + e^{4.8}}\right) \right] \approx 401.4$$

$$86. (a) \int_0^k 6x^2 e^{-x/2} dx = 50$$

By trial and error, $k = 5.51897$.

$$(b) \int_0^{5.51897} 6x^2 e^{-x/2} dx$$

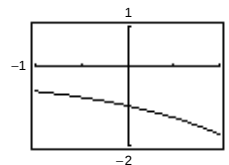


88. True

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

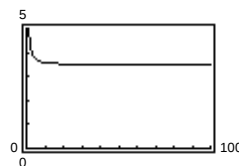
2. $\lim_{x \rightarrow 0} \frac{1 - e^x}{x} \approx -1$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.00005	-1.005	-1.0517



4. $\lim_{x \rightarrow \infty} \frac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641$ (exact: $\frac{6}{\sqrt{3}}$)

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



6. (a) $\lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(2x - 3)(x + 1)}{x + 1} = \lim_{x \rightarrow -1} (2x - 3) = -5$

(b) $\lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(d/dx)[2x^2 - x - 3]}{(d/dx)[x + 1]} = \lim_{x \rightarrow -1} \frac{4x - 1}{1} = -5$

8. (a) $\lim_{x \rightarrow 0} \frac{\sin 4x}{2x} = \lim_{x \rightarrow 0} 2 \left(\frac{\sin 4x}{4x} \right) = 2(1) = 2$

(b) $\lim_{x \rightarrow 0} \frac{\sin 4x}{2x} = \lim_{x \rightarrow 0} \frac{(d/dx)[\sin 4x]}{(d/dx)[2x]} = \lim_{x \rightarrow 0} \frac{4 \cos 4x}{2} = 2$

10. (a) $\lim_{x \rightarrow \infty} \frac{2x + 1}{4x^2 + x} = \lim_{x \rightarrow \infty} \frac{(2/x) + (1/x^2)}{4 + (1/x)} = \frac{0}{4} = 0$

(b) $\lim_{x \rightarrow \infty} \frac{2x + 1}{4x^2 + x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[2x + 1]}{(d/dx)[4x^2 + x]} = \lim_{x \rightarrow \infty} \frac{2}{8x + 1} = 0$

12. $\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{x + 1} = \lim_{x \rightarrow -1} \frac{2x - 1}{1} = -3$

14. $\lim_{x \rightarrow 2^-} \frac{\sqrt{4 - x^2}}{x - 2} = \lim_{x \rightarrow 2^-} \frac{-x/\sqrt{4 - x^2}}{1} = \lim_{x \rightarrow 2^-} \frac{-x}{\sqrt{4 - x^2}} = -\infty$

16. $\lim_{x \rightarrow 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{3x^2} = \lim_{x \rightarrow 0^+} \frac{e^x}{6x} = \infty$

18. $\lim_{x \rightarrow 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{2 \ln x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{2/x}{2x} = \lim_{x \rightarrow 1} \frac{1}{x^2} = 1$

20. $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}$

22. $\lim_{x \rightarrow 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x \rightarrow 1} \frac{1/(1 + x^2)}{1} = \frac{1}{2}$

24. $\lim_{x \rightarrow \infty} \frac{x - 1}{x^2 + 2x + 3} = \lim_{x \rightarrow \infty} \frac{1}{2x + 2} = 0$

26. $\lim_{x \rightarrow \infty} \frac{x^3}{x + 1} = \lim_{x \rightarrow \infty} \frac{3x^2}{1} = \infty$

28. $\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$

$$30. \lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{1 + (1/x)^2}} = \infty$$

$$32. \lim_{x \rightarrow \infty} \frac{\sin x}{x - \pi} = 0$$

Note: Use the Squeeze Theorem for $x > \pi$.

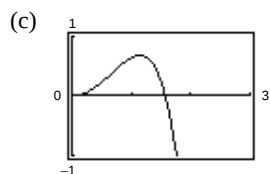
$$-\frac{1}{x - \pi} \leq \frac{\sin x}{x - \pi} \leq \frac{1}{x - \pi}$$

$$34. \lim_{x \rightarrow \infty} \frac{\ln x^4}{x^3} = \lim_{x \rightarrow \infty} \frac{4 \ln x}{x^3} = \lim_{x \rightarrow \infty} \frac{4/x}{3x^2} \\ = \lim_{x \rightarrow \infty} \frac{4}{3x^3} = 0$$

$$36. \lim_{x \rightarrow \infty} \frac{e^{x/2}}{x} = \lim_{x \rightarrow \infty} \frac{(1/2)e^{x/2}}{1} = \infty$$

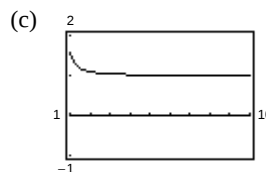
$$38. (a) \lim_{x \rightarrow 0^+} x^3 \cot x = (0)(\infty)$$

$$(b) \lim_{x \rightarrow 0^+} x^3 \cot x = \lim_{x \rightarrow 0^+} \frac{x^3}{\tan x} = \lim_{x \rightarrow 0^+} \frac{3x^2}{\sec^2 x} = 0$$



$$40. (a) \lim_{x \rightarrow \infty} \left(x \tan \frac{1}{x} \right) = (\infty)(0)$$

$$(b) \lim_{x \rightarrow \infty} x \tan \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\tan(1/x)}{1/x} \\ = \lim_{x \rightarrow \infty} \frac{-(1/x^2) \sec^2(1/x)}{-(1/x^2)} \\ = \lim_{x \rightarrow \infty} \sec^2\left(\frac{1}{x}\right) = 1$$

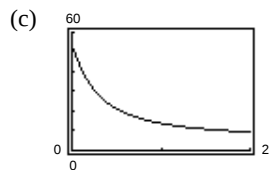


$$42. (a) \lim_{x \rightarrow 0^+} (e^x + x)^{2/x} = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow 0^+} (e^x + x)^{2/x}.$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{2 \ln(e^x + x)}{x} \\ = \lim_{x \rightarrow 0^+} \frac{2(e^x + 1)/(e^x + x)}{1} = 4$$

Thus, $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$.



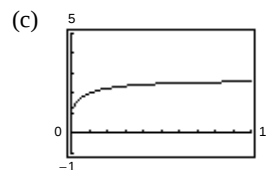
$$44. (a) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x.$$

$$\ln y = \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{1}{x} \right) \right] = \lim_{x \rightarrow \infty} \frac{\ln[1 + (1/x)]}{1/x} \\ = \lim_{x \rightarrow \infty} \frac{\left[\frac{(-1/x^2)}{1 + (1/x)} \right]}{(-1/x^2)} = \lim_{x \rightarrow \infty} \frac{1}{1 + (1/x)} = 1$$

Thus, $\ln y = 1 \Rightarrow y = e^1 = e$. Therefore,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e.$$



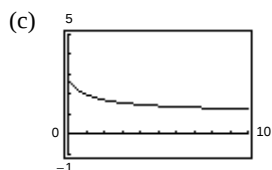
46. (a) $\lim_{x \rightarrow \infty} (1+x)^{1/x} = \infty^0$

(b) Let $y = \lim_{x \rightarrow \infty} (1+x)^{1/x}$.

$$\begin{aligned}\ln y &= \lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow \infty} \left(\frac{1/(1+x)}{1} \right) = 0\end{aligned}$$

Thus, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x \rightarrow \infty} (1+x)^{1/x} = 1$.

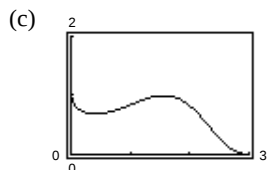


50. (a) $\lim_{x \rightarrow 0^+} \left[\cos\left(\frac{\pi}{2} - x\right) \right]^x = 0^0$

(b) Let $y = \lim_{x \rightarrow 0^+} \left[\cos\left(\frac{\pi}{2} - x\right) \right]^x$.

$$\begin{aligned}\ln y &= \lim_{x \rightarrow 0^+} x \ln \left[\cos\left(\frac{\pi}{2} - x\right) \right] \\ &= 0 \cdot 0 = 0\end{aligned}$$

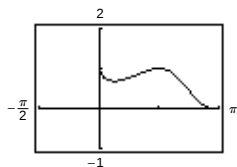
Hence, $\lim_{x \rightarrow 0^+} \left[\cos\left(\frac{\pi}{2} - x\right) \right]^x = 1$.



54. (a) $\lim_{x \rightarrow 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right) = \infty - \infty$

(b) $\lim_{x \rightarrow 0^+} \left(\frac{10}{x} - \frac{3}{x^2} \right) = \lim_{x \rightarrow 0^+} \left(\frac{10x - 3}{x^2} \right) = -\infty$

56. (a)



(b) Let $y = (\sin x)^x$, then $\ln y = x \ln(\sin x)$.

$$\lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{\cos x / \sin x}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{-x^2}{\tan x} = \lim_{x \rightarrow 0^+} \frac{-2x}{\sec^2 x} = 0$$

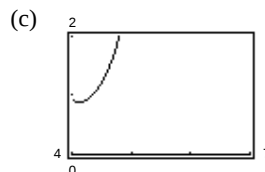
Therefore, since $\ln y = 0$, $y = 1$ and $\lim_{x \rightarrow 0^+} (\sin x)^x = 1$.

48. (a) $\lim_{x \rightarrow 4^+} [3(x-4)]^{x-4} = 0^0$

(b) Let $y = \lim_{x \rightarrow 4^+} [3(x-4)]^{x-4}$.

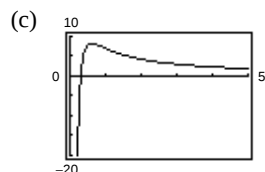
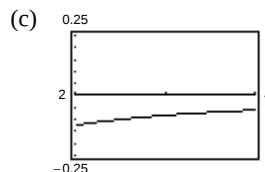
$$\begin{aligned}\ln y &= \lim_{x \rightarrow 4^+} (x-4) \ln[3(x-4)] \\ &= \lim_{x \rightarrow 4^+} \frac{\ln[3(x-4)]}{1/(x-4)} \\ &= \lim_{x \rightarrow 4^+} \frac{1/(x-4)}{-1/(x-4)^2} \\ &= \lim_{x \rightarrow 4^+} [-(x-4)] = 0\end{aligned}$$

Hence, $\lim_{x \rightarrow 4^+} [3(x-4)]^{x-4} = 1$.

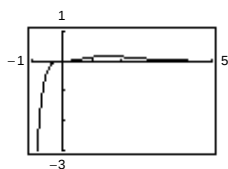


52. (a) $\lim_{x \rightarrow 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x-1}}{x^2 - 4} \right) = \infty - \infty$

$$\begin{aligned}(b) \lim_{x \rightarrow 2^+} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x-1}}{x^2 - 4} \right) &= \lim_{x \rightarrow 2^+} \frac{1 - \sqrt{x-1}}{x^2 - 4} \\ &= \lim_{x \rightarrow 2^+} \frac{-1/(2\sqrt{x-1})}{2x} \\ &= \lim_{x \rightarrow 2^+} \frac{-1}{4x\sqrt{x-1}} = \frac{-1}{8}\end{aligned}$$



58. (a)



$$(b) \lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2e^{2x}} = \lim_{x \rightarrow \infty} \frac{6x}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{6}{8e^{2x}} = 0$$

60. See Theorem 7.4.

 62. Let $f(x) = x + 25$ and $g(x) = x$.

$$64. \lim_{x \rightarrow \infty} \frac{x^3}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2e^{2x}} = \lim_{x \rightarrow \infty} \frac{6x}{4e^{2x}} = \lim_{x \rightarrow \infty} \frac{6}{8e^{2x}} = 0$$

$$\begin{aligned} 66. \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x^3} &= \lim_{x \rightarrow \infty} \frac{(2 \ln x)/x}{3x^2} \\ &= \lim_{x \rightarrow \infty} \frac{2 \ln x}{3x^3} \\ &= \lim_{x \rightarrow \infty} \frac{2/x}{9x^2} = \lim_{x \rightarrow \infty} \frac{2}{9x^3} = 0 \end{aligned}$$

$$\begin{aligned} 68. \lim_{x \rightarrow \infty} \frac{x^m}{e^{nx}} &= \lim_{x \rightarrow \infty} \frac{mx^{m-1}}{ne^{nx}} \\ &= \lim_{x \rightarrow \infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}} \\ &= \cdots = \lim_{x \rightarrow \infty} \frac{m!}{n^me^{nx}} = 0 \end{aligned}$$

70.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

 72. $y = x^x, x > 0$

$$\lim_{x \rightarrow \infty} x^x = \infty \text{ and } \lim_{x \rightarrow 0^+} x^x = 1$$

No horizontal asymptotes

$$\ln y = x \ln x$$

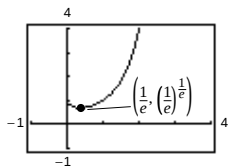
$$\frac{1}{y} \frac{dy}{dx} = x \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = x^x(1 + \ln x) = 0$$

 Critical number: $x = e^{-1}$

 Intervals: $(0, e^{-1})$ $(e^{-1}, 0)$

 Sign of dy/dx : $-$ $+$
 $y = f(x)$: Decreasing Increasing

 Relative minimum: $(e^{-1}, (e^{-1})^{e^{-1}}) = \left(\frac{1}{e}, \left(\frac{1}{e} \right)^{1/e} \right)$


$$74. y = \frac{\ln x}{x}$$

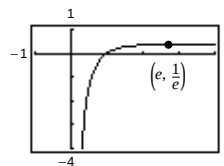
 Horizontal asymptote: $y = 0$ (See Exercise 29)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

 Critical number: $x = e$

 Intervals: $(0, e)$ (e, ∞)

 Sign of dy/dx : $+$ $-$
 $y = f(x)$: Increasing Decreasing

 Relative maximum: $\left(e, \frac{1}{e} \right)$


$$76. \lim_{x \rightarrow \infty} \frac{\sin \pi x - 1}{x} = 0 \quad (\text{Numerator is bounded})$$

 Limit is not of the form $0/0$ or ∞/∞ .

L'Hôpital's Rule does not apply.

$$78. \lim_{x \rightarrow \infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$$

 Limit is not of the form $0/0$ or ∞/∞ .

L'Hôpital's Rule does not apply.

80. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt \ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n \rightarrow \infty} \left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}} \right] = \lim_{n \rightarrow \infty} \left[\frac{-\frac{r}{n^2} \left(\frac{1}{1 + (r/n)} \right)}{-\left(\frac{1}{n^2 t} \right)} \right] = \lim_{n \rightarrow \infty} \left[rt \left(\frac{1}{1 + \frac{r}{n}} \right) \right] = rt$$

Since $\lim_{n \rightarrow \infty} \ln A = \ln P + rt$, we have $\lim_{n \rightarrow \infty} A = e^{(\ln P + rt)} = e^{\ln P} e^{rt} = P e^{rt}$. Alternatively,

$$\lim_{n \rightarrow \infty} A = \lim_{n \rightarrow \infty} P \left(1 + \frac{r}{n}\right)^{nt} = \lim_{n \rightarrow \infty} P \left[\left(1 + \frac{r}{n}\right)^{n/r} \right]^{rt} = P e^{rt}.$$

82. Let N be a fixed value for n . Then

$$\lim_{x \rightarrow \infty} \frac{x^{N-1}}{e^x} = \lim_{x \rightarrow \infty} \frac{(N-1)x^{N-2}}{e^x} = \lim_{x \rightarrow \infty} \frac{(N-1)(N-2)x^{N-3}}{e^x} = \dots = \lim_{x \rightarrow \infty} \left[\frac{(N-1)!}{e^x} \right] = 0. \quad (\text{See Exercise 68})$$

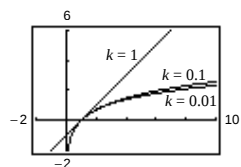
84. $f(x) = \frac{x^k - 1}{k}$

$k = 1, \quad f(x) = x - 1$

$k = 0.1, \quad f(x) = \frac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$

$k = 0.01, \quad f(x) = \frac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$

$$\lim_{k \rightarrow 0^+} \frac{x^k - 1}{k} = \lim_{k \rightarrow 0^+} \frac{x^k (\ln x)}{1} = \ln x$$



86. $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

88. $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

90. False. If $y = e^x/x^2$, then

$$y' = \frac{x^2 e^x - 2x e^x}{x^4} = \frac{x e^x (x - 2)}{x^4} = \frac{e^x (x - 2)}{x^3}.$$

92. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x \rightarrow \infty} \frac{x}{x+1} = 1, \text{ but } \lim_{x \rightarrow \infty} [x - (x+1)] = -1.$$

$$94. g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

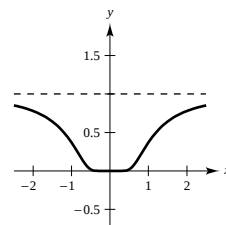
$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$$

Let $y = \frac{e^{-1/x^2}}{x}$, then $\ln y = \ln\left(\frac{e^{-1/x^2}}{x}\right) = -\frac{1}{x^2} - \ln x = \frac{-1 - x^2 \ln x}{x^2}$. Since

$$\lim_{x \rightarrow 0} x^2 \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x^2} = \lim_{x \rightarrow 0} \frac{1/x}{-2/x^3} = \lim_{x \rightarrow 0} \left(-\frac{x^2}{2}\right) = 0$$

we have $\lim_{x \rightarrow 0} \left(\frac{-1 - x^2 \ln x}{x^2}\right) = -\infty$. Thus, $\lim_{x \rightarrow 0} y = e^{-\infty} = 0 \Rightarrow g'(0) = 0$.

Note: The graph appears to support this conclusion—the tangent line is horizontal at $(0, 0)$.



$$96. \lim_{x \rightarrow a} f(x)^{g(x)}$$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$$

As $x \rightarrow a$, $\ln y \Rightarrow \infty$, and hence $y = \infty$. Thus,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \infty.$$

$$98. \lim_{x \rightarrow 0^+} x^{\ln 2 / (1 + \ln x)}$$

Let $y = x^{\ln 2 / (1 + \ln x)}$, then:

$$\ln y = \frac{\ln 2}{1 + \ln x} \cdot \ln x = \frac{(\ln 2)(\ln x)}{1 + \ln x}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln y &= \lim_{x \rightarrow 0^+} \frac{(\ln 2)(\ln x)}{1 + \ln x} = \lim_{x \rightarrow 0^+} \frac{(\ln 2)/x}{1/x} \\ &= \lim_{x \rightarrow 0^+} (\ln 2) = \ln 2 \end{aligned}$$

Thus, $\lim_{x \rightarrow 0^+} y = e^{\ln 2} = 2$.

Section 7.8 Improper Integrals

2. Infinite discontinuity at $x = 3$.

$$\begin{aligned} \int_3^4 \frac{1}{(x-3)^{3/2}} dx &= \lim_{b \rightarrow 3^+} \int_b^4 (x-3)^{-3/2} dx \\ &= \lim_{b \rightarrow 3^+} \left[-2(x-3)^{-1/2} \right]_b^4 \\ &= \lim_{b \rightarrow 3^+} \left[-2 + \frac{2}{\sqrt{b-3}} \right] = \infty \end{aligned}$$

Diverges

4. Infinite discontinuity at $x = 1$.

$$\begin{aligned} \int_0^2 \frac{1}{(x-1)^{2/3}} dx &= \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^{2/3}} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{b \rightarrow 1^-} \left[3\sqrt[3]{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[3\sqrt[3]{x-1} \right]_c^2 = (0+3) + (3-0) = 6 \end{aligned}$$

Converges

6. Infinite limit of integration.

$$\begin{aligned}\int_{-\infty}^0 e^{2x} dx &= \lim_{b \rightarrow -\infty} \int_b^0 e^{2x} dx \\ &= \lim_{b \rightarrow -\infty} \left[\frac{1}{2} e^{2x} \right]_b^0 = \frac{1}{2} - 0 = \frac{1}{2}\end{aligned}$$

Converges

8. $\int_0^{\infty} e^{-x} dx \neq 0$. You need to evaluate the limit.

$$\begin{aligned}\lim_{b \rightarrow \infty} \int_0^b e^{-x} dx &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-e^{-b} + 1 \right] = 1\end{aligned}$$

$$\begin{aligned}10. \int_1^{\infty} \frac{5}{x^3} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{5}{x^3} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{5}{2} x^{-2} \right]_1^b = \frac{5}{2}\end{aligned}$$

$$\begin{aligned}12. \int_1^{\infty} \frac{4}{\sqrt[4]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b 4x^{-1/4} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{16}{3} x^{3/4} \right]_1^b = \infty \quad \text{Diverges}\end{aligned}$$

$$14. \int_0^{\infty} x e^{-x/2} dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x/2} dx = \lim_{b \rightarrow \infty} \left[e^{-x/2} (-2x - 4) \right]_0^b = \lim_{b \rightarrow \infty} e^{-b/2} (-2b - 4) + 4 = 4$$

$$16. \int_0^{\infty} (x-1)e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b (x-1)e^{-x} dx = \lim_{b \rightarrow \infty} \left[-xe^{-x} \right]_0^b = \lim_{b \rightarrow \infty} \left(\frac{-b}{e^b} + 0 \right) = 0 \text{ by L'Hôpital's Rule.}$$

$$\begin{aligned}18. \int_0^{\infty} e^{-ax} \sin bx dx &= \lim_{c \rightarrow \infty} \left[\frac{e^{-ax} (-a \sin bx - b \cos bx)}{a^2 + b^2} \right]_0^c \\ &= 0 - \frac{-b}{a^2 + b^2} = \frac{b}{a^2 + b^2}\end{aligned}$$

$$\begin{aligned}20. \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{(\ln x)^2}{2} \right]_1^b = \infty \quad \text{Diverges}\end{aligned}$$

$$\begin{aligned}22. \int_0^{\infty} \frac{x^3}{(x^2 + 1)^2} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{x}{x^2 + 1} dx - \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2 + 1)^2} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} \right]_0^b \\ &= \infty - \frac{1}{2}\end{aligned}$$

Diverges

$$\begin{aligned}24. \int_0^{\infty} \frac{e^x}{1 + e^x} dx &= \lim_{b \rightarrow \infty} \left[\ln(1 + e^x) \right]_0^b = \infty - \ln 2 \\ &\text{Diverges}\end{aligned}$$

$$\begin{aligned}26. \int_0^{\infty} \sin \frac{x}{2} dx &= \lim_{b \rightarrow \infty} \left[-2 \cos \frac{x}{2} \right]_0^b \\ &\text{Diverges since } \cos \frac{x}{2} \text{ does not approach a limit as } x \rightarrow \infty.\end{aligned}$$

$$\begin{aligned}28. \int_0^4 \frac{8}{x} dx &= \lim_{b \rightarrow 0^+} \int_b^4 \frac{8}{x} dx = \lim_{b \rightarrow 0^+} \left[8 \ln x \right]_b^4 = \infty \\ &\text{Diverges}\end{aligned}$$

$$\begin{aligned}30. \int_0^6 \frac{4}{\sqrt{6-x}} dx &= \lim_{b \rightarrow 6^-} \int_0^b 4(6-x)^{-1/2} dx \\ &= \lim_{b \rightarrow 6^-} \left[-8(6-x)^{1/2} \right]_0^b \\ &= -8(0) + 8\sqrt{6} \\ &= 8\sqrt{6}\end{aligned}$$

$$\begin{aligned}32. \int_0^e \ln x^2 dx &= \lim_{b \rightarrow 0^+} \int_0^b 2 \ln x dx \\ &= \lim_{b \rightarrow 0^+} \left[2x \ln x - 2x \right]_b^e \\ &= \lim_{b \rightarrow 0^+} [(2e - 2e) - (2b \ln b - 2b)] \\ &= 0\end{aligned}$$

$$34. \int_0^{\pi/2} \sec \theta d\theta = \lim_{b \rightarrow (\pi/2)} \left[\ln |\sec \theta + \tan \theta| \right]_0^b = \infty,$$

Diverges

$$36. \int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{b \rightarrow 2^-} \left[\arcsin \left(\frac{x}{2} \right) \right]_0^b = \frac{\pi}{2}$$

$$38. \int_0^2 \frac{1}{4-x^2} dx = \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{4} \left(\frac{1}{2+x} + \frac{1}{2-x} \right) dx = \lim_{b \rightarrow 2^-} \left[\frac{1}{4} \ln \left| \frac{2+x}{2-x} \right| \right]_0^b = \infty - 0$$

Diverges

$$\begin{aligned} 40. \int_1^3 \frac{2}{(x-2)^{8/3}} dx &= \int_1^2 2(x-2)^{-8/3} dx + \int_2^3 2(x-2)^{-8/3} dx \\ &= \lim_{b \rightarrow 2^-} \int_1^b 2(x-2)^{-8/3} dx + \lim_{c \rightarrow 2^+} \int_c^3 2(x-2)^{-8/3} dx \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{6}{5} (x-2)^{-5/3} \right]_1^b + \lim_{c \rightarrow 2^+} \left[-\frac{6}{5} (x-2)^{-5/3} \right]_c^3 = \infty \end{aligned}$$

Diverges

$$42. \int \frac{1}{x \ln x} dx = \ln |\ln |x|| + C$$

Thus,

$$\begin{aligned} \int_1^\infty \frac{1}{x \ln x} dx &= \int_1^e \frac{1}{x \ln x} dx + \int_e^\infty \frac{1}{x \ln x} dx \\ &= \lim_{b \rightarrow 1^+} \left[\ln(\ln x) \right]_1^b + \lim_{c \rightarrow \infty} \left[\ln(\ln x) \right]_e^c. \end{aligned}$$

Diverges

$$44. \text{ If } p = 1, \int_0^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \ln x \Big|_a^1 = \lim_{a \rightarrow 0^+} -\ln a = \infty.$$

Diverges. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \left[\frac{x^{1-p}}{1-p} \right]_a^1 = \lim_{a \rightarrow 0^+} \left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right].$$

This converges to $\frac{1}{1-p}$ if $1-p > 0$ or $p < 1$.

$$46. (a) \text{ Assume } \int_a^\infty g(x) dx = L \text{ (converges).}$$

Since $0 \leq f(x) \leq g(x)$ on $[a, \infty)$, $0 \leq \int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx = L$ and $\int_a^\infty f(x) dx$ converges.

$$(b) \int_a^\infty g(x) dx \text{ diverges, because otherwise, by part (a), if } \int_a^\infty g(x) dx \text{ converges, then so does } \int_a^\infty f(x) dx.$$

$$48. \int_0^1 \frac{1}{\sqrt[3]{x}} dx = \frac{1}{1-(1/3)} = \frac{3}{2} \text{ converges.}$$

(See Exercise 44, $p = \frac{1}{3}$.)

$$50. \int_0^\infty x^4 e^{-x} dx \text{ converges.}$$

(See Exercise 45.)

$$52. \text{ Since } \frac{1}{\sqrt{x-1}} \geq \frac{1}{x} \text{ on } [2, \infty) \text{ and } \int_2^\infty \frac{1}{x} dx \text{ diverges by Exercise 43, } \int_2^\infty \frac{1}{\sqrt{x-1}} dx \text{ diverges.}$$

$$54. \text{ Since } \frac{1}{\sqrt{x}(1+x)} \leq \frac{1}{x^{3/2}} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x^{3/2}} dx \text{ converges by Exercise 43, } \int_1^\infty \frac{1}{\sqrt{x}(1+x)} dx \text{ converges.}$$

$$56. \frac{1}{\sqrt{x} \ln x} \geq \frac{1}{x} \text{ since } \sqrt{x} \ln x < x \text{ on } [2, \infty). \text{ Since } \int_2^\infty \frac{1}{x} dx \text{ diverges by Exercise 43, } \int_2^\infty \frac{1}{\sqrt{x} \ln x} dx \text{ diverges.}$$

58. See the definitions, pages 540, 543.

60. Answers will vary.

$$(a) \int_{-\infty}^\infty \frac{e^x}{1+e^{2x}} dx$$

Converges (Example 4)

$$(b) \int_{-\infty}^\infty x dx$$

Diverges

62. $f(t) = t$

$$\begin{aligned} F(s) &= \int_0^\infty t e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{1}{s^2} (-st - 1) e^{-st} \right]_0^b \\ &= \frac{1}{s^2}, s > 0 \end{aligned}$$

64. $f(t) = e^{at}$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{t(a-s)} dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} e^{t(a-s)} \right]_0^b \\
 &= 0 - \frac{1}{a-s} = \frac{1}{s-a}, s > a
 \end{aligned}$$

66. $f(t) = \sin at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \sin at dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^b \\
 &= 0 + \frac{a}{s^2 + a^2} = \frac{a}{s^2 + a^2}, s > 0
 \end{aligned}$$

68. $f(t) = \sinh at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \sinh at dt = \int_0^{\infty} e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} [e^{t(-s+a)} - e^{t(-s-a)}] dt \\
 &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\frac{1}{(-s+a)} e^{t(-s+a)} - \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] \\
 &= \frac{-1}{2} \left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)} \right] = \frac{a}{s^2 - a^2}, s > |a|
 \end{aligned}$$

70. (a) $A = \int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = 1$

(b) **Disk:**

$$V = \pi \int_1^{\infty} \frac{1}{x^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{\pi}{3x^3} \right]_1^b = \frac{\pi}{3}$$

(c) **Shell:**

$$V = 2\pi \int_1^{\infty} x \left(\frac{1}{x^2} \right) dx = \lim_{b \rightarrow \infty} \left[2\pi(\ln x) \right]_1^b = \infty$$

Diverges

72. $(x-2)^2 + y^2 = 1$

$2(x-2) + 2yy' = 0$

$$y' = \frac{-(x-2)}{y}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + [(x-2)^2/y^2]} = \frac{1}{y} \text{ (Assume } y > 0.)$$

$$\begin{aligned}
 S &= 4\pi \int_1^3 \frac{x}{y} dx = 4\pi \int_1^3 \frac{x}{\sqrt{1 - (x-2)^2}} dx = 4\pi \int_1^3 \left[\frac{x-2}{\sqrt{1 - (x-2)^2}} + \frac{2}{\sqrt{1 - (x-2)^2}} \right] dx \\
 &= \lim_{\substack{a \rightarrow 1^+ \\ b \rightarrow 3^-}} \left\{ 4\pi \left[-\sqrt{1 - (x-2)^2} + 2 \arcsin(x-2) \right]_a^b \right\} = 4\pi[0 + 2 \arcsin(1) - 2 \arcsin(-1)] = 8\pi^2
 \end{aligned}$$

74. (a) $F(x) = \frac{K}{x^2}, 5 = \frac{K}{(4000)^2}, K = 80,000,000$

$$W = \int_{4000}^{\infty} \frac{80,000,000}{x^2} dx = \lim_{b \rightarrow \infty} \left[\frac{-80,000,000}{x} \right]_{4000}^b = 20,000 \text{ mi-ton}$$

$$(b) \quad \frac{W}{2} = 10,000 = \left[\frac{-80,000,000}{x} \right]_{4000}^b = \frac{-80,000,000}{b} + 20,000$$

$$\frac{80,000,000}{b} = 10,000$$

$$b = 8000$$

Therefore, 4000 miles *above* the earth's surface.

$$76. (a) \int_{-\infty}^{\infty} \frac{2}{5} e^{-2t/5} dt = \int_0^{\infty} \frac{2}{5} e^{-2t/5} dt = \lim_{b \rightarrow \infty} \left[-e^{-2t/5} \right]_0^b = 1$$

$$(b) \int_0^4 \frac{2}{5} e^{-2t/5} dt = \left[-e^{-2t/5} \right]_0^4 = -e^{-8/5} + 1 \\ \approx 0.7981 = 79.81\%$$

$$(c) \int_0^{\infty} t \left[\frac{2}{5} e^{-2t/5} \right] dt = \lim_{b \rightarrow \infty} \left[-te^{2t/5} - \frac{5}{2} e^{-2t/5} \right]_0^b = \frac{5}{2}$$

$$78. (a) C = 650,000 + \int_0^5 25,000(1 + 0.08t)e^{-0.06t} dt \\ = 650,000 + 25,000 \left[-\frac{1}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^5 \approx \$778,512.58$$

$$(b) C = 650,000 + \int_0^{10} 25,000(1 + 0.08t)e^{-0.06t} dt \\ = 650,000 + 25,000 \left[-\frac{1}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^{10} \approx \$905,718.14$$

$$(c) C = 650,000 + \int_0^{\infty} 25,000(1 + 0.08t)e^{-0.06t} dt \\ = 650,000 + 25,000 \lim_{b \rightarrow \infty} \left[-\frac{t}{0.06} e^{-0.06t} - 0.08 \left(\frac{t}{0.06} e^{-0.06t} + \frac{1}{(0.06)^2} e^{-0.06t} \right) \right]_0^b \approx \$1,622,222.22$$

$$80. (a) \int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \left[\ln|x| \right]_1^b = \infty$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1$$

$$\int_1^{\infty} \frac{1}{x^n} dx \text{ will converge if } n > 1 \text{ and will diverge if } n \leq 1.$$

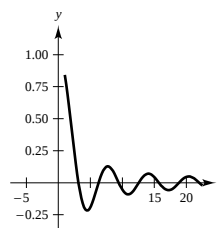
$$(c) \text{ Let } dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$$

$$\int_1^{\infty} \frac{\sin x}{x} dx = \lim_{b \rightarrow \infty} \left[-\frac{\cos x}{x} \right]_1^b - \int_1^{\infty} \frac{\cos x}{x^2} dx \\ = \cos 1 - \int_1^{\infty} \frac{\cos x}{x^2} dx$$

Converges

(b) It would appear to converge.



82. (a) Yes, the integral is not defined at $x = \pi/2$.

(c) As $n \rightarrow \infty$, the integral approaches $4(\pi/4) = \pi$.

$$(d) I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n} dx$$

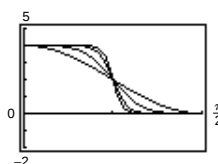
$$I_2 \approx 3.14159$$

$$I_4 \approx 3.14159$$

$$I_8 \approx 3.14159$$

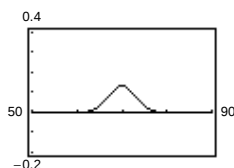
$$I_{12} \approx 3.14159$$

(b)



84. (a) $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x) dx \approx 1.0$$



(b) $P(72 \leq x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \leq x \leq 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$$P(70 \leq x < \infty) = 0.5$$

and

$$0.5 = P(70 \leq x < \infty)$$

$$= P(70 \leq x \leq 72) + P(72 \leq x < \infty).$$

86. False. This is equivalent to Exercise 85.

88. True

Review Exercises for Chapter 7

$$\begin{aligned} 2. \int x e^{x^2-1} dx &= \frac{1}{2} \int e^{x^2-1} (2x) dx \\ &= \frac{1}{2} e^{x^2-1} + C \end{aligned}$$

$$\begin{aligned} 4. \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int (1-x^2)^{-1/2} (-2x) dx \\ &= -\frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + C \\ &= -\sqrt{1-x^2} + C \end{aligned}$$

$$\begin{aligned} 6. \int 2x \sqrt{2x-3} dx &= \int (u^4 + 3u^2) du = \frac{u^5}{5} + u^3 + C \\ &= \frac{2(2x-3)^{3/2}}{5} (x+1) + C \\ u = \sqrt{2x-3}, x &= \frac{u^2+3}{2}, dx = u du \end{aligned}$$

$$\begin{aligned} 8. \frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} &= 1 + \frac{x}{(x^2+1)^2} \\ \int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2} dx &= \int dx + \frac{1}{2} \int \frac{2x}{(x^2+1)^2} dx \\ &= x - \frac{1}{2(x^2+1)} + C \end{aligned}$$

$$10. \int (x^2-1)e^x dx = (x^2-1)e^x - 2 \int x e^x dx = (x^2-1)e^x - 2x e^x + 2 \int e^x dx = e^x(x^2-2x+1) + 1$$

$$(1) dv = e^x dx \Rightarrow v = e^x$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$(2) dv = e^x dx \Rightarrow v = e^x$$

$$u = x \Rightarrow du = dx$$

$$12. u = \arctan 2x, du = \frac{2}{1+4x^2} dx, dv = dx, v = x$$

$$\begin{aligned} \int \arctan 2x dx &= x \arctan 2x - \int \frac{2x}{1+4x^2} dx \\ &= x \arctan 2x - \frac{1}{4} \ln(1+4x^2) + C \end{aligned}$$

$$\begin{aligned} 14. \int \ln \sqrt{x^2-1} dx &= \frac{1}{2} \int \ln(x^2-1) dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int \frac{x^2}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int dx - \int \frac{1}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - x - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2-1) \Rightarrow du = \frac{2x}{x^2-1} dx$$

$$\begin{aligned}
 16. \int e^x \arctan(e^x) dx &= e^x \arctan(e^x) - \int \frac{e^{2x}}{1 + e^{2x}} dx \\
 &= e^x \arctan(e^x) - \frac{1}{2} \ln(1 + e^{2x}) + C
 \end{aligned}$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$u = \arctan e^x \Rightarrow du = \frac{e^x}{1 + e^{2x}} dx$$

$$18. \int \sin^2 \frac{\pi x}{2} dx = \int \frac{1}{2} (1 - \cos \pi x) dx = \frac{1}{2} \left[x - \frac{1}{\pi} \sin \pi x \right] + C = \frac{1}{2\pi} [\pi x - \sin \pi x] + C$$

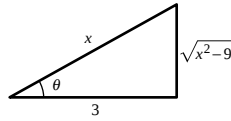
$$20. \int \tan \theta \sec^4 \theta d\theta = \int (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta = \frac{1}{4} \tan^4 \theta + \frac{1}{2} \tan^2 \theta + C_1$$

or

$$\int \tan \theta \sec^4 \theta d\theta = \int \sec^3 \theta (\sec \theta \tan \theta) d\theta = \frac{1}{4} \sec^4 \theta + C_2$$

$$\begin{aligned}
 22. \int \cos 2\theta (\sin \theta + \cos \theta)^2 d\theta &= \int (\cos^2 \theta - \sin^2 \theta) (\sin \theta + \cos \theta)^2 d\theta \\
 &= \int (\sin \theta + \cos \theta)^3 (\cos \theta - \sin \theta) d\theta = \frac{1}{4} (\sin \theta + \cos \theta)^4 + C
 \end{aligned}$$

$$\begin{aligned}
 24. \int \frac{\sqrt{x^2 - 9}}{x} dx &= \int \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta d\theta) \\
 &= 3 \int \tan^2 \theta d\theta \\
 &= 3 \int (\sec^2 \theta - 1) d\theta \\
 &= 3(\tan \theta - \theta) + C \\
 &= \sqrt{x^2 - 9} - 3 \operatorname{arcsec} \left(\frac{x}{3} \right) + C
 \end{aligned}$$



$$x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta, \sqrt{x^2 - 9} = 3 \tan \theta$$

$$\begin{aligned}
 26. \int \sqrt{9 - 4x^2} dx &= \frac{1}{2} \int \sqrt{9 - (2x)^2} (2) dx \\
 &= \frac{1}{2} \cdot \frac{1}{2} \left[9 \arcsin \frac{2x}{3} + 2x \sqrt{9 - 4x^2} \right] + C \\
 &= \frac{9}{4} \arcsin \frac{2x}{3} + \frac{x}{2} \sqrt{9 - 4x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \int \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta &= \frac{-1}{\sqrt{2}} \int \frac{1}{1 + 2 \cos^2 \theta} (-\sqrt{2} \sin \theta) d\theta \\
 &= \frac{-1}{\sqrt{2}} \arctan(\sqrt{2} \cos \theta) + C
 \end{aligned}$$

$$u = \sqrt{2} \cos \theta, du = -\sqrt{2} \sin \theta d\theta$$

$$\begin{aligned}
 30. \text{ (a) } \int x\sqrt{4+x} \, dx &= 64 \int \tan^3 \theta \sec^3 \theta \, d\theta \\
 &= 64 \int (\sec^4 \theta - \sec^2 \theta) \sec \theta \tan \theta \, d\theta \\
 &= \frac{64 \sec^3 \theta}{15} (3 \sec^3 \theta - 5) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$x = 4 \tan^2 \theta, \, dx = 8 \tan \theta \sec^2 \theta \, d\theta,$$

$$\sqrt{4+x} = 2 \sec \theta$$

$$\begin{aligned}
 \text{(c) } \int x\sqrt{4+x} \, dx &= \int (u^{3/2} - 4u^{1/2}) \, du \\
 &= \frac{2u^{3/2}}{15} (3u-20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u = 4+x, \, du = dx$$

$$\begin{aligned}
 \text{(b) } \int x\sqrt{4+x} \, dx &= 2 \int (u^4 - 4u^2) \, du \\
 &= \frac{2u^3}{15} (3u^2 - 20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u^2 = 4+x, \, dx = 2u \, du$$

$$\begin{aligned}
 \text{(d) } \int x\sqrt{4+x} \, dx &= \frac{2x}{3} (4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} \, dx \\
 &= \frac{2x}{3} (4+x)^{3/2} - \frac{4}{15} (4+x)^{5/2} + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$dv = \sqrt{4+x} \, dx \Rightarrow v = \frac{2}{3} (4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$32. \quad \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$

$$\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} \, dx = \int \left(2x - 3 + \frac{4}{x} - \frac{3}{x-1} \right) \, dx = x^2 - 3x + 4 \ln|x| - 3 \ln|x-1| + C$$

$$34. \quad \frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$4x-2 = 3A(x-1) + 3B$$

$$\text{Let } x = 1: 2 = 3B \Rightarrow B = \frac{2}{3}$$

$$\text{Let } x = 2: 6 = 3A + 3B \Rightarrow A = \frac{4}{3}$$

$$\int \frac{4x-2}{3(x-1)^2} \, dx = \frac{4}{3} \int \frac{1}{x-1} \, dx + \frac{2}{3} \int \frac{1}{(x-1)^2} \, dx = \frac{4}{3} \ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3} \left(2 \ln|x-1| - \frac{1}{x-1} \right) + C$$

$$\begin{aligned}
 36. \quad \int \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} \, d\theta &= \int \frac{1}{u(u-1)} \, du = \int \frac{1}{u-1} \, du - \int \frac{1}{u} \, du \\
 &= \ln|u-1| - \ln|u| + C = \ln \left| \frac{\tan \theta - 1}{\tan \theta} \right| + C = \ln|1 - \cot \theta| + C
 \end{aligned}$$

$$u = \tan \theta, \, du = \sec^2 \theta \, d\theta$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

$$\text{Let } u = 0: 1 = -A \Rightarrow A = -1$$

$$\text{Let } u = 1: 1 = B$$

$$\begin{aligned}
 38. \int \frac{x}{\sqrt{2+3x}} dx &= \frac{-2(4-3x)}{27} \sqrt{2+3x} + C \quad (\text{Formula 21}) & 40. \int \frac{x}{1+e^{x^2}} dx &= \frac{1}{2} \int \frac{1}{1+e^u} du & (u = x^2) \\
 &= \frac{6x-8}{27} \sqrt{2+3x} + C & &= \frac{1}{2} [u - \ln(1+e^u)] + C & (\text{Formula 84}) \\
 & & &= \frac{1}{2} [x^2 - \ln(1+e^{x^2})] + C
 \end{aligned}$$

$$\begin{aligned}
 42. \int \frac{3}{2x\sqrt{9x^2-1}} dx &= \frac{3}{2} \int \frac{1}{3x\sqrt{(3x)^2-1}} 3 dx \quad (u = 3x) \\
 &= \frac{3}{2} \operatorname{arcsec}|3x| + C \quad (\text{Formula 33})
 \end{aligned}$$

$$\begin{aligned}
 44. \int \frac{1}{1+\tan \pi x} dx &= \frac{1}{\pi} \int \frac{1}{1+\tan \pi x} (\pi) dx & (u = \pi x) \\
 &= \frac{1}{\pi} \frac{1}{2} [\pi x + \ln|\cos \pi x + \sin \pi x|] + C \quad (\text{Formula 71})
 \end{aligned}$$

$$\begin{aligned}
 46. \int \tan^n x dx &= \int \tan^{n-2} x (\sec^2 x - 1) dx & 48. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx &= \sqrt{2} \int \csc \sqrt{2x} \left(\frac{1}{\sqrt{2x}} \right) dx \\
 &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx & &= -\sqrt{2} \ln|\csc \sqrt{2x} + \cot \sqrt{2x}| + C \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx & u = \sqrt{2x}, du &= \frac{1}{\sqrt{2x}} dx
 \end{aligned}$$

$$\begin{aligned}
 50. \int \sqrt{1+\sqrt{x}} dx &= \int u(4u^3-4u) du = \int (4u^4-4u^2) du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15} (1+\sqrt{x})^{3/2} (3\sqrt{x}-2) + C \\
 u &= \sqrt{1+\sqrt{x}}, x = u^4 - 2u^2 + 1, dx = (4u^3 - 4u) du
 \end{aligned}$$

$$\begin{aligned}
 52. \frac{3x^3+4x}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \\
 3x^3+4x &= (Ax+B)(x^2+1) + Cx+D \\
 &= Ax^3+Bx^2+(A+C)x+(B+D) \\
 A=3, B=0, A+C=4 &\Rightarrow C=1, \\
 B+D=0 &\Rightarrow D=0
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{3x^3+4x}{(x^2+1)^2} dx &= 3 \int \frac{x}{x^2+1} dx + \int \frac{x}{(x^2+1)^2} dx \\
 &= \frac{3}{2} \ln(x^2+1) - \frac{1}{2(x^2+1)} + C
 \end{aligned}$$

$$\begin{aligned}
 54. \int (\sin \theta + \cos \theta)^2 d\theta &= \int (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \\
 &= \int (1 + \sin 2\theta) d\theta = \theta - \frac{1}{2} \cos 2\theta + C = \frac{1}{2} (2\theta - \cos 2\theta) + C
 \end{aligned}$$

$$\begin{aligned}
 56. \quad y &= \int \frac{\sqrt{4-x^2}}{2x} dx = \int \frac{2 \cos \theta (2 \cos \theta) d\theta}{4 \sin \theta} \\
 &= \int (\csc \theta - \sin \theta) d\theta \\
 &= [-\ln|\csc \theta + \cos \theta| + \cos \theta] + C \\
 &= -\ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + \frac{\sqrt{4-x^2}}{2} + C
 \end{aligned}$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 58. \quad y &= \int \sqrt{1-\cos \theta} d\theta = \int \frac{\sin \theta}{\sqrt{1+\cos \theta}} d\theta = -\int (1+\cos \theta)^{-1/2} (-\sin \theta) d\theta = -2\sqrt{1+\cos \theta} + C \\
 u &= 1 + \cos \theta, du = -\sin \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \int_0^1 \frac{x}{(x-2)(x-4)} dx &= \left[2 \ln|x-4| - \ln|x-2| \right]_0^1 \\
 &= 2 \ln 3 - 2 \ln 4 + \ln 2 \\
 &= \ln \frac{9}{8} \approx 0.118
 \end{aligned}$$

$$62. \quad \int_0^2 x e^{3x} dx = \left[\frac{e^{3x}}{9} (3x-1) \right]_0^2 = \frac{1}{9} (5e^6 + 1) \approx 224.238$$

$$64. \quad \int_0^3 \frac{x}{\sqrt{1+x}} dx = \left[\frac{-2(2-x)}{3} \sqrt{1+x} \right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

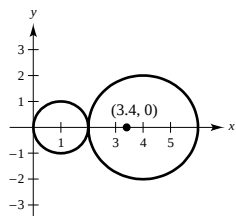
$$\begin{aligned}
 66. \quad A &= \int_0^4 \frac{1}{25-x^2} dx \\
 &= \left[-\frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| \right]_0^4 = -\frac{1}{10} \ln \frac{1}{9} = \frac{1}{10} \ln 9 \approx 0.220
 \end{aligned}$$

68. By symmetry, $\bar{y} = 0$.

$$A = \pi + 4\pi = 5\pi$$

$$\begin{aligned}
 \bar{x} &= \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi} \\
 &= \frac{17\pi}{5\pi} = 3.4
 \end{aligned}$$

$$(\bar{x}, \bar{y}) = (3.4, 0)$$



$$70. \quad s = \int_0^\pi \sqrt{1 + \sin^2 2x} dx \approx 3.82$$

$$72. \quad \lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{2\pi \cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$74. \quad \lim_{x \rightarrow \infty} x e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2x e^{x^2}} = 0$$

$$76. \quad y = \lim_{x \rightarrow 1^+} (x-1)^{\ln x}$$

$$\ln y = \lim_{x \rightarrow 1^+} [(\ln x) \ln(x-1)]$$

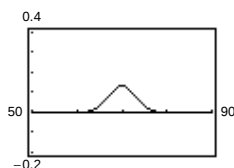
$$\begin{aligned}
 &= \lim_{x \rightarrow 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x} \right) - 1} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-\ln^2 x}{\frac{x-1}{x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}} \right]
 \end{aligned}$$

$$= \lim_{x \rightarrow 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.

84. (a) $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x) dx \approx 1.0$$



(b) $P(72 \leq x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \leq x \leq 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$$P(70 \leq x < \infty) = 0.5$$

and

$$0.5 = P(70 \leq x < \infty)$$

$$= P(70 \leq x \leq 72) + P(72 \leq x < \infty).$$

86. False. This is equivalent to Exercise 85.

88. True

Review Exercises for Chapter 7

$$\begin{aligned} 2. \int x e^{x^2-1} dx &= \frac{1}{2} \int e^{x^2-1} (2x) dx \\ &= \frac{1}{2} e^{x^2-1} + C \end{aligned}$$

$$\begin{aligned} 4. \int \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int (1-x^2)^{-1/2} (-2x) dx \\ &= -\frac{1}{2} \frac{(1-x^2)^{1/2}}{1/2} + C \\ &= -\sqrt{1-x^2} + C \end{aligned}$$

$$\begin{aligned} 6. \int 2x \sqrt{2x-3} dx &= \int (u^4 + 3u^2) du = \frac{u^5}{5} + u^3 + C \\ &= \frac{2(2x-3)^{3/2}}{5} (x+1) + C \\ u = \sqrt{2x-3}, x &= \frac{u^2+3}{2}, dx = u du \end{aligned}$$

$$\begin{aligned} 8. \frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} &= 1 + \frac{x}{(x^2+1)^2} \\ \int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2} dx &= \int dx + \frac{1}{2} \int \frac{2x}{(x^2+1)^2} dx \\ &= x - \frac{1}{2(x^2+1)} + C \end{aligned}$$

$$10. \int (x^2 - 1)e^x dx = (x^2 - 1)e^x - 2 \int x e^x dx = (x^2 - 1)e^x - 2x e^x + 2 \int e^x dx = e^x(x^2 - 2x + 1) + 1$$

$$(1) dv = e^x dx \Rightarrow v = e^x$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$(2) dv = e^x dx \Rightarrow v = e^x$$

$$u = x \Rightarrow du = dx$$

$$12. u = \arctan 2x, du = \frac{2}{1+4x^2} dx, dv = dx, v = x$$

$$\begin{aligned} \int \arctan 2x dx &= x \arctan 2x - \int \frac{2x}{1+4x^2} dx \\ &= x \arctan 2x - \frac{1}{4} \ln(1+4x^2) + C \end{aligned}$$

$$\begin{aligned} 14. \int \ln \sqrt{x^2-1} dx &= \frac{1}{2} \int \ln(x^2-1) dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int \frac{x^2}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - \int dx - \int \frac{1}{x^2-1} dx \\ &= \frac{1}{2} x \ln|x^2-1| - x - \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2-1) \Rightarrow du = \frac{2x}{x^2-1} dx$$

$$\begin{aligned}
 16. \int e^x \arctan(e^x) dx &= e^x \arctan(e^x) - \int \frac{e^{2x}}{1 + e^{2x}} dx \\
 &= e^x \arctan(e^x) - \frac{1}{2} \ln(1 + e^{2x}) + C
 \end{aligned}$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$u = \arctan e^x \Rightarrow du = \frac{e^x}{1 + e^{2x}} dx$$

$$18. \int \sin^2 \frac{\pi x}{2} dx = \int \frac{1}{2} (1 - \cos \pi x) dx = \frac{1}{2} \left[x - \frac{1}{\pi} \sin \pi x \right] + C = \frac{1}{2\pi} [\pi x - \sin \pi x] + C$$

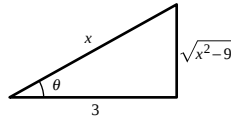
$$20. \int \tan \theta \sec^4 \theta d\theta = \int (\tan^3 \theta + \tan \theta) \sec^2 \theta d\theta = \frac{1}{4} \tan^4 \theta + \frac{1}{2} \tan^2 \theta + C_1$$

or

$$\int \tan \theta \sec^4 \theta d\theta = \int \sec^3 \theta (\sec \theta \tan \theta) d\theta = \frac{1}{4} \sec^4 \theta + C_2$$

$$\begin{aligned}
 22. \int \cos 2\theta (\sin \theta + \cos \theta)^2 d\theta &= \int (\cos^2 \theta - \sin^2 \theta) (\sin \theta + \cos \theta)^2 d\theta \\
 &= \int (\sin \theta + \cos \theta)^3 (\cos \theta - \sin \theta) d\theta = \frac{1}{4} (\sin \theta + \cos \theta)^4 + C
 \end{aligned}$$

$$\begin{aligned}
 24. \int \frac{\sqrt{x^2 - 9}}{x} dx &= \int \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta d\theta) \\
 &= 3 \int \tan^2 \theta d\theta \\
 &= 3 \int (\sec^2 \theta - 1) d\theta \\
 &= 3(\tan \theta - \theta) + C \\
 &= \sqrt{x^2 - 9} - 3 \operatorname{arcsec} \left(\frac{x}{3} \right) + C
 \end{aligned}$$



$$x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta, \sqrt{x^2 - 9} = 3 \tan \theta$$

$$\begin{aligned}
 26. \int \sqrt{9 - 4x^2} dx &= \frac{1}{2} \int \sqrt{9 - (2x)^2} (2) dx \\
 &= \frac{1}{2} \cdot \frac{1}{2} \left[9 \arcsin \frac{2x}{3} + 2x \sqrt{9 - 4x^2} \right] + C \\
 &= \frac{9}{4} \arcsin \frac{2x}{3} + \frac{x}{2} \sqrt{9 - 4x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \int \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta &= \frac{-1}{\sqrt{2}} \int \frac{1}{1 + 2 \cos^2 \theta} (-\sqrt{2} \sin \theta) d\theta \\
 &= \frac{-1}{\sqrt{2}} \arctan(\sqrt{2} \cos \theta) + C
 \end{aligned}$$

$$u = \sqrt{2} \cos \theta, du = -\sqrt{2} \sin \theta d\theta$$

$$\begin{aligned}
 30. \text{ (a) } \int x\sqrt{4+x} \, dx &= 64 \int \tan^3 \theta \sec^3 \theta \, d\theta \\
 &= 64 \int (\sec^4 \theta - \sec^2 \theta) \sec \theta \tan \theta \, d\theta \\
 &= \frac{64 \sec^3 \theta}{15} (3 \sec^3 \theta - 5) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$x = 4 \tan^2 \theta, \, dx = 8 \tan \theta \sec^2 \theta \, d\theta,$$

$$\sqrt{4+x} = 2 \sec \theta$$

$$\begin{aligned}
 \text{(c) } \int x\sqrt{4+x} \, dx &= \int (u^{3/2} - 4u^{1/2}) \, du \\
 &= \frac{2u^{3/2}}{15} (3u-20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u = 4+x, \, du = dx$$

$$\begin{aligned}
 \text{(b) } \int x\sqrt{4+x} \, dx &= 2 \int (u^4 - 4u^2) \, du \\
 &= \frac{2u^3}{15} (3u^2 - 20) + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$u^2 = 4+x, \, dx = 2u \, du$$

$$\begin{aligned}
 \text{(d) } \int x\sqrt{4+x} \, dx &= \frac{2x}{3} (4+x)^{3/2} - \frac{2}{3} \int (4+x)^{3/2} \, dx \\
 &= \frac{2x}{3} (4+x)^{3/2} - \frac{4}{15} (4+x)^{5/2} + C \\
 &= \frac{2(4+x)^{3/2}}{15} (3x-8) + C
 \end{aligned}$$

$$dv = \sqrt{4+x} \, dx \Rightarrow v = \frac{2}{3} (4+x)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$32. \quad \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$

$$\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} \, dx = \int \left(2x - 3 + \frac{4}{x} - \frac{3}{x-1} \right) \, dx = x^2 - 3x + 4 \ln|x| - 3 \ln|x-1| + C$$

$$34. \quad \frac{4x-2}{3(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$4x-2 = 3A(x-1) + 3B$$

$$\text{Let } x = 1: 2 = 3B \Rightarrow B = \frac{2}{3}$$

$$\text{Let } x = 2: 6 = 3A + 3B \Rightarrow A = \frac{4}{3}$$

$$\int \frac{4x-2}{3(x-1)^2} \, dx = \frac{4}{3} \int \frac{1}{x-1} \, dx + \frac{2}{3} \int \frac{1}{(x-1)^2} \, dx = \frac{4}{3} \ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3} \left(2 \ln|x-1| - \frac{1}{x-1} \right) + C$$

$$\begin{aligned}
 36. \quad \int \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} \, d\theta &= \int \frac{1}{u(u-1)} \, du = \int \frac{1}{u-1} \, du - \int \frac{1}{u} \, du \\
 &= \ln|u-1| - \ln|u| + C = \ln \left| \frac{\tan \theta - 1}{\tan \theta} \right| + C = \ln|1 - \cot \theta| + C
 \end{aligned}$$

$$u = \tan \theta, \, du = \sec^2 \theta \, d\theta$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

$$\text{Let } u = 0: 1 = -A \Rightarrow A = -1$$

$$\text{Let } u = 1: 1 = B$$

$$\begin{aligned}
 38. \int \frac{x}{\sqrt{2+3x}} dx &= \frac{-2(4-3x)}{27} \sqrt{2+3x} + C \quad (\text{Formula 21}) & 40. \int \frac{x}{1+e^{x^2}} dx &= \frac{1}{2} \int \frac{1}{1+e^u} du & (u = x^2) \\
 &= \frac{6x-8}{27} \sqrt{2+3x} + C & &= \frac{1}{2} [u - \ln(1+e^u)] + C & (\text{Formula 84}) \\
 & & &= \frac{1}{2} [x^2 - \ln(1+e^{x^2})] + C
 \end{aligned}$$

$$\begin{aligned}
 42. \int \frac{3}{2x\sqrt{9x^2-1}} dx &= \frac{3}{2} \int \frac{1}{3x\sqrt{(3x)^2-1}} 3 dx \quad (u = 3x) \\
 &= \frac{3}{2} \operatorname{arcsec}|3x| + C \quad (\text{Formula 33})
 \end{aligned}$$

$$\begin{aligned}
 44. \int \frac{1}{1+\tan \pi x} dx &= \frac{1}{\pi} \int \frac{1}{1+\tan \pi x} (\pi) dx & (u = \pi x) \\
 &= \frac{1}{\pi} \frac{1}{2} [\pi x + \ln|\cos \pi x + \sin \pi x|] + C \quad (\text{Formula 71})
 \end{aligned}$$

$$\begin{aligned}
 46. \int \tan^n x dx &= \int \tan^{n-2} x (\sec^2 x - 1) dx & 48. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx &= \sqrt{2} \int \csc \sqrt{2x} \left(\frac{1}{\sqrt{2x}} \right) dx \\
 &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx & &= -\sqrt{2} \ln|\csc \sqrt{2x} + \cot \sqrt{2x}| + C \\
 &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx & u = \sqrt{2x}, du &= \frac{1}{\sqrt{2x}} dx
 \end{aligned}$$

$$\begin{aligned}
 50. \int \sqrt{1+\sqrt{x}} dx &= \int u(4u^3-4u) du = \int (4u^4-4u^2) du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15} (1+\sqrt{x})^{3/2} (3\sqrt{x}-2) + C \\
 u &= \sqrt{1+\sqrt{x}}, x = u^4 - 2u^2 + 1, dx = (4u^3 - 4u) du
 \end{aligned}$$

$$\begin{aligned}
 52. \frac{3x^3+4x}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \\
 3x^3+4x &= (Ax+B)(x^2+1) + Cx+D \\
 &= Ax^3+Bx^2+(A+C)x+(B+D) \\
 A=3, B=0, A+C=4 &\Rightarrow C=1, \\
 B+D=0 &\Rightarrow D=0
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{3x^3+4x}{(x^2+1)^2} dx &= 3 \int \frac{x}{x^2+1} dx + \int \frac{x}{(x^2+1)^2} dx \\
 &= \frac{3}{2} \ln(x^2+1) - \frac{1}{2(x^2+1)} + C
 \end{aligned}$$

$$\begin{aligned}
 54. \int (\sin \theta + \cos \theta)^2 d\theta &= \int (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta \\
 &= \int (1 + \sin 2\theta) d\theta = \theta - \frac{1}{2} \cos 2\theta + C = \frac{1}{2} (2\theta - \cos 2\theta) + C
 \end{aligned}$$

$$\begin{aligned}
 56. \quad y &= \int \frac{\sqrt{4-x^2}}{2x} dx = \int \frac{2 \cos \theta (2 \cos \theta) d\theta}{4 \sin \theta} \\
 &= \int (\csc \theta - \sin \theta) d\theta \\
 &= [-\ln|\csc \theta + \cos \theta| + \cos \theta] + C \\
 &= -\ln \left| \frac{2 + \sqrt{4-x^2}}{x} \right| + \frac{\sqrt{4-x^2}}{2} + C
 \end{aligned}$$

$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 58. \quad y &= \int \sqrt{1-\cos \theta} d\theta = \int \frac{\sin \theta}{\sqrt{1+\cos \theta}} d\theta = -\int (1+\cos \theta)^{-1/2} (-\sin \theta) d\theta = -2\sqrt{1+\cos \theta} + C \\
 u &= 1 + \cos \theta, du = -\sin \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \int_0^1 \frac{x}{(x-2)(x-4)} dx &= \left[2 \ln|x-4| - \ln|x-2| \right]_0^1 \\
 &= 2 \ln 3 - 2 \ln 4 + \ln 2 \\
 &= \ln \frac{9}{8} \approx 0.118
 \end{aligned}$$

$$62. \quad \int_0^2 x e^{3x} dx = \left[\frac{e^{3x}}{9} (3x-1) \right]_0^2 = \frac{1}{9} (5e^6 + 1) \approx 224.238$$

$$64. \quad \int_0^3 \frac{x}{\sqrt{1+x}} dx = \left[\frac{-2(2-x)}{3} \sqrt{1+x} \right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

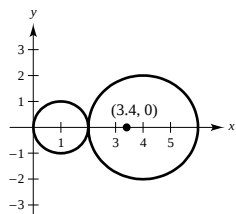
$$\begin{aligned}
 66. \quad A &= \int_0^4 \frac{1}{25-x^2} dx \\
 &= \left[-\frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| \right]_0^4 = -\frac{1}{10} \ln \frac{1}{9} = \frac{1}{10} \ln 9 \approx 0.220
 \end{aligned}$$

68. By symmetry, $\bar{y} = 0$.

$$A = \pi + 4\pi = 5\pi$$

$$\begin{aligned}
 \bar{x} &= \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi} \\
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$$(\bar{x}, \bar{y}) = (3.4, 0)$$



$$70. \quad s = \int_0^\pi \sqrt{1 + \sin^2 2x} dx \approx 3.82$$

$$72. \quad \lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{2\pi \cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$74. \quad \lim_{x \rightarrow \infty} x e^{-x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2x e^{x^2}} = 0$$

$$76. \quad y = \lim_{x \rightarrow 1^+} (x-1)^{\ln x}$$

$$\ln y = \lim_{x \rightarrow 1^+} [(\ln x) \ln(x-1)]$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x} \right) - 1} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-\ln^2 x}{\frac{x-1}{x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}} \right]
 \end{aligned}$$

$$= \lim_{x \rightarrow 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.

$$\begin{aligned}
 78. \lim_{x \rightarrow 1^+} \left(\frac{2}{\ln x} - \frac{2}{x-1} \right) &= \lim_{x \rightarrow 1^+} \left[\frac{2x - 2 - 2 \ln x}{(\ln x)(x-1)} \right] \\
 &= \lim_{x \rightarrow 1^+} \left[\frac{2 - (2/x)}{(x-1)(1/x) + \ln x} \right] \\
 &= \lim_{x \rightarrow 1^+} \frac{2x - 2}{(x-1) + x \ln x} = \lim_{x \rightarrow 1^+} \frac{2}{1 + 1 + \ln x} = 1
 \end{aligned}$$

$$80. \int_0^1 \frac{6}{x-1} dx = \lim_{b \rightarrow 1^-} \left[6 \ln |x-1| \right]_0^b = -\infty$$

Diverges

$$82. \int_0^\infty \frac{e^{-1/x}}{x^2} dx = \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \left[e^{-1/x} \right]_a^b = 1 - 0 = 1$$

$$\begin{aligned}
 84. V &= \pi \int_0^\infty (xe^{-x})^2 dx \\
 &= \pi \int_0^\infty x^2 e^{-2x} dx \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{\pi e^{-2x}}{4} (2x^2 + 2x + 1) \right]_0^b = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 86. \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}} \right] dx &< \int_2^\infty \frac{1}{x^5 - 1} dx < \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}} \right] dx \\
 \lim_{b \rightarrow \infty} \left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}} \right]_2^b &< \int_2^\infty \frac{1}{x^5 - 1} dx < \lim_{b \rightarrow \infty} \left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}} \right]_2^b \\
 0.015846 &< \int_2^\infty \frac{1}{x^5 - 1} dx < 0.015851
 \end{aligned}$$

Problem Solving for Chapter 7

$$\begin{aligned}
 2. (a) \int_0^1 \ln x dx &= \lim_{b \rightarrow 0^+} \left[x \ln x - x \right]_b^1 \\
 &= (-1) - \lim_{b \rightarrow 0^+} (b \ln b - b) = -1
 \end{aligned}$$

$$\text{Note: } \lim_{b \rightarrow 0^+} b \ln b = \lim_{b \rightarrow 0^+} \frac{\ln b}{b^{-1}} = \lim_{b \rightarrow 0^+} \frac{1/b}{-1/b^2} = 0$$

$$\begin{aligned}
 \int_0^1 (\ln x)^2 dx &= \lim_{b \rightarrow 0^+} \left[x(\ln x)^2 - 2x \ln x + 2x \right]_b^1 \\
 &= 2 - \lim_{b \rightarrow 0^+} (b(\ln b)^2 - 2b \ln b + 2b) = 2
 \end{aligned}$$

(b) Note first that $\lim_{b \rightarrow 0^+} b(\ln b)^n = 0$ (Mathematical induction).

$$\text{Also, } \int (\ln x)^{n+1} dx = x(\ln x)^{n+1} - (n+1) \int (\ln x)^n dx.$$

$$\text{Assume } \int_0^1 (\ln x)^n dx = (-1)^n n!.$$

$$\begin{aligned}
 \text{Then, } \int_0^1 (\ln x)^{n+1} dx &= \lim_{b \rightarrow 0^+} \left[x(\ln x)^{n+1} \right]_b^1 - (n+1) \int_0^1 (\ln x)^n dx \\
 &= 0 - (n+1)(-1)^n n! = (-1)^{n+1} (n+1)!.
 \end{aligned}$$

$$4. \quad \lim_{x \rightarrow \infty} \left(\frac{x-c}{x+c} \right)^x = \frac{1}{4}$$

$$\lim_{x \rightarrow \infty} x \ln \left(\frac{x-c}{x+c} \right) = \ln \frac{1}{4}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x-c) - \ln(x+c)}{1/x} = -\ln 4$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x-c} - \frac{1}{x+c}}{-\frac{1}{x^2}} = -\ln 4$$

$$\lim_{x \rightarrow \infty} \frac{2c}{(x-c)(x+c)} (-x^2) = -\ln 4$$

$$\lim_{x \rightarrow \infty} \frac{2cx^2}{x^2 - c^2} = \ln 4$$

$$2c = \ln 4$$

$$2x = 2 \ln 2$$

$$c = \ln 2$$

$$6. \quad \sin \theta = BD, \cos \theta = OD$$

$$\text{Area } \triangle DAB = \frac{1}{2}(DA)(BD) = \frac{1}{2}(1 - \cos \theta) \sin \theta$$

$$\text{Shaded area} = \frac{\theta}{2} - \frac{1}{2}(1)(BD) = \frac{\theta}{2} - \frac{1}{2} \sin \theta$$

$$R = \frac{\triangle DAB}{\text{Shaded area}} = \frac{1/2(1 - \cos \theta) \sin \theta}{1/2(\theta - \sin \theta)}$$

$$\lim_{\theta \rightarrow 0^+} R = \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta) \sin \theta}{\theta - \sin \theta} = \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta) \cos \theta + \sin^2 \theta}{1 - \cos \theta}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{(1 - \cos \theta)(-\sin \theta) + \cos \theta \sin \theta + 2 \sin \theta \cos \theta}{\sin \theta}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - 4 \cos \theta \sin \theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{4 \cos \theta - 1}{1} = 3$$

$$8. \quad u = \tan \frac{x}{2}, \cos x = \frac{1-u^2}{1+u^2}, 2 + \cos x = 2 + \frac{1-u^2}{1+u^2} = \frac{3+u^2}{1+u^2}$$

$$dx = \frac{2 du}{1+u^2}$$

$$\int_0^{\pi/2} \frac{1}{2 + \cos x} dx = \int_0^1 \left(\frac{1+u^2}{3+u^2} \right) \left(\frac{2}{1+u^2} \right) du$$

$$= \int_0^1 \frac{2}{3+u^2} du$$

$$= \left[2 \frac{1}{\sqrt{3}} \arctan \left(\frac{u}{\sqrt{3}} \right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \arctan \left(\frac{1}{\sqrt{3}} \right)$$

$$= \frac{2}{\sqrt{3}} \frac{\pi}{6} = \frac{\pi\sqrt{3}}{9} \approx 0.6046$$

10. Let $u = cx$, $du = c dx$.

$$\int_0^b e^{-c^2 x^2} dx = \int_0^{cb} e^{-u^2} \frac{du}{c} = \frac{1}{c} \int_0^{cb} e^{-u^2} du$$

$$\text{As } b \rightarrow \infty, cb \rightarrow \infty. \text{ Hence, } \int_0^\infty e^{-c^2 x^2} dx = \frac{1}{c} \int_0^\infty e^{-u^2} du.$$

$\bar{x} = 0$ by symmetry.

$$\begin{aligned} \bar{y} &= \frac{M_x}{m} = \frac{2 \int_0^\infty \frac{(e^{-c^2 x^2})}{2} dx}{2 \int_0^\infty e^{-c^2 x^2} dx} \\ &= \frac{1}{2} \frac{\int_0^\infty e^{-2c^2 x^2} dx}{\int_0^\infty e^{-c^2 x^2} dx} \\ &= \frac{1}{2} \frac{\frac{1}{\sqrt{2}c} \int_0^\infty e^{-x^2} dx}{\frac{1}{c} \int_0^\infty e^{-x^2} dx} \\ &= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \end{aligned}$$

$$\text{Thus, } (\bar{x}, \bar{y}) = \left(0, \frac{\sqrt{2}}{4}\right).$$

12. (a) Let $y = f^{-1}(x)$, $f(y) = x$, $dx = f'(y) dy$.

$$\int f^{-1}(x) dx = \int y f'(y) dy$$

$$= y f(y) - \int f(y) dy$$

$$= x f^{-1}(x) - \int f(y) dy$$

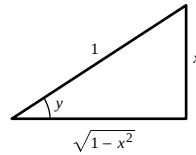
$$\left[\begin{array}{l} u = y, du = dy \\ dv = f'(y) dy, v = f(y) \end{array} \right]$$

- (b) $f^{-1}(x) = \arcsin x = y$, $f(y) = \sin y$

$$\int \arcsin x dx = x \arcsin x - \int \sin y dy$$

$$= x \arcsin x + \cos y + C$$

$$= x \arcsin x + \sqrt{1-x^2} + C$$



- (c) $f(x) = e^x$, $f^{-1}(x) = \ln x = y$ $x = 1 \Leftrightarrow y = 0$; $x = e \Leftrightarrow y = 1$

$$\int_1^e \ln x dx = \left[x \ln x \right]_1^e - \int_0^1 e^y dy$$

$$= e - \left[e^y \right]_0^1$$

$$= e - (e - 1) = 1$$

14. (a) Let $x = \frac{\pi}{2} - u$, $dx = -du$.

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx = \int_{\pi/2}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)} (-du) \\ &= \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u} du \end{aligned}$$

Hence,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx \\ &= \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad I &= \int_{\pi/2}^0 \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)} (-du) \\ &= \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u} du \end{aligned}$$

$$\text{Thus, } 2I = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

$$16. \frac{N(x)}{D(x)} = \frac{P_1}{x - c_1} + \frac{P_2}{x - c_2} + \cdots + \frac{P_n}{x - c_n}$$

$$N(x) = P_1(x - c_2)(x - c_3) \cdots (x - c_n) + P_2(x - c_1)(x - c_3) \cdots (x - c_n) + \cdots + P_n(x - c_1)(x - c_2) \cdots (x - c_{n-1})$$

$$\text{Let } x = c_1: N(c_1) = P_1(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)$$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)}$$

$$\text{Let } x = c_2: N(c_2) = P_2(c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)$$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)}$$

$\vdots$

$\vdots$

$$\text{Let } x = c_n: N(c_n) = P_n(c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1})$$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1})}$$

If $D(x) = (x - c_1)(x - c_2)(x - c_3) \cdots (x - c_n)$, then by the Product Rule

$$D'(x) = (x - c_2)(x - c_3) \cdots (x - c_n) + (x - c_1)(x - c_3) \cdots (x - c_n) + \cdots + (x - c_1)(x - c_2)(x - c_3) \cdots (x - c_{n-1})$$

and

$$D'(c_1) = (c_1 - c_2)(c_1 - c_3) \cdots (c_1 - c_n)$$

$$D'(c_2) = (c_2 - c_1)(c_2 - c_3) \cdots (c_2 - c_n)$$

$\vdots$

$$D'(c_n) = (c_n - c_1)(c_n - c_2) \cdots (c_n - c_{n-1}).$$

Thus, $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \dots, n$.

$$\begin{aligned}
 18. \quad s(t) &= \int \left[-32t + 12,000 \ln \frac{50,000}{50,000 - 400t} \right] dt \\
 &= -16t^2 + 12,000 \int [\ln 50,000 - \ln(50,000 - 400t)] dt \\
 &= 16t^2 + 12,000t \ln 50,000 - 12,000 \left[t \ln(50,000 - 400t) - \int \frac{-400t}{50,000 - 400t} dt \right] \\
 &= -16t^2 + 12,000t \ln \frac{50,000}{50,000 - 400t} + 12,000t \int \left[1 - \frac{50,000}{50,000 - 400t} \right] dt \\
 &= -16t^2 + 12,000t \ln \frac{50,000}{50,000 - 400t} + 12,000t + 1,500,000 \ln(50,000 - 400t) + C \\
 s(0) &= 1,500,000 \ln 50,000 + C = 0 \\
 C &= -1,500,000 \ln 50,000 \\
 s(t) &= -16t^2 + 12,000t \left[1 + \ln \frac{50,000}{50,000 - 400t} \right] + 1,500,000 \ln \frac{50,000 - 400t}{50,000}
 \end{aligned}$$

When $t = 100$, $s(100) \approx 557,168.626$ feet

$$\begin{aligned}
 20. \quad \text{Let } u &= (x - a)(x - b), \quad du = [(x - a) + (x - b)] dx, \quad dv = f'(x) dx, \quad v = f(x). \\
 \int_a^b (x - a)(x - b) dx &= \left[(x - a)(x - b) f'(x) \right]_a^b - \int_a^b [(x - a) + (x - b)] f'(x) dx \\
 &= - \int_a^b (2x - a - b) f'(x) dx \quad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x) dx \end{pmatrix} \\
 &= \left[-(2x - a - b) f(x) \right]_a^b + \int_a^b 2f(x) dx \\
 &= 2 \int_a^b f(x) dx
 \end{aligned}$$

C H A P T E R 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1	Basic Integration Rules	50
Section 7.2	Integration by Parts	55
Section 7.3	Trigonometric Integrals	65
Section 7.4	Trigonometric Substitution	74
Section 7.5	Partial Fractions	84
Section 7.6	Integration by Tables and Other Integration Techniques . .	90
Section 7.7	Indeterminate Forms and L'Hôpital's Rule	96
Section 7.8	Improper Integrals	102
	Review Exercises	108
	Problem Solving	114

CHAPTER 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Odd-Numbered Exercises

$$1. (a) \frac{d}{dx}[2\sqrt{x^2+1} + C] = 2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2+1}}$$

$$(b) \frac{d}{dx}[\sqrt{x^2+1} + C] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

$$(c) \frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2+1}}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{x}{\sqrt{x^2+1}} dx \text{ matches (b).}$$

$$3. (a) \frac{d}{dx}[\ln\sqrt{x^2+1} + C] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$$

$$(b) \frac{d}{dx}\left[\frac{2x}{(x^2+1)^2} + C\right] = \frac{(x^2+1)^2(2) - (2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$$(c) \frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{1}{x^2+1} dx \text{ matches (c).}$$

$$5. \int (3x-2)^4 dx$$

$$u = 3x-2, du = 3 dx, n = 4$$

$$\text{Use } \int u^n du.$$

$$7. \int \frac{1}{\sqrt{x}(1-2\sqrt{x})} dx$$

$$u = 1-2\sqrt{x}, du = -\frac{1}{\sqrt{x}} dx$$

$$\text{Use } \int \frac{du}{u}.$$

$$9. \int \frac{3}{\sqrt{1-t^2}} dt$$

$$u = t, du = dt, a = 1$$

$$\text{Use } \int \frac{du}{\sqrt{a^2-u^2}}$$

$$11. \int t \sin t^2 dt$$

$$u = t^2, du = 2t dt$$

$$\text{Use } \int \sin u du.$$

$$13. \int \cos x e^{\sin x} dx$$

$$u = \sin x, du = \cos x dx$$

$$\text{Use } \int e^u du.$$

15. Let
- $u = -2x + 5$
- ,
- $du = -2 dx$
- .

$$\begin{aligned}\int (-2x + 5)^{3/2} dx &= -\frac{1}{2} \int (-2x + 5)^{3/2} (-2) dx \\ &= -\frac{1}{5} (-2x + 5)^{5/2} + C\end{aligned}$$

17. Let
- $u = z - 4$
- ,
- $du = dz$

$$\begin{aligned}\int \frac{5}{(z-4)^5} dz &= 5 \int (z-4)^{-5} dz = 5 \frac{(z-4)^{-4}}{-4} + C \\ &= \frac{-5}{4(z-4)^4} + C\end{aligned}$$

19. Let
- $u = t^3 - 1$
- ,
- $du = 3t^2 dt$
- .

$$\begin{aligned}\int t^2 \sqrt[3]{t^3 - 1} dt &= \frac{1}{3} \int (t^3 - 1)^{1/3} (3t^2) dt \\ &= \frac{1}{3} \frac{(t^3 - 1)^{4/3}}{4/3} + C \\ &= \frac{(t^3 - 1)^{4/3}}{4} + C\end{aligned}$$

$$\begin{aligned}21. \int \left[v + \frac{1}{(3v-1)^3} \right] dv &= \int v dv + \frac{1}{3} \int (3v-1)^{-3} (3) dv \\ &= \frac{1}{2} v^2 - \frac{1}{6(3v-1)^2} + C\end{aligned}$$

23. Let
- $u = -t^3 + 9t + 1$
- ,
- $du = (-3t^2 + 9) dt = -3(t^2 - 3) dt$
- .

$$\int \frac{t^2 - 3}{-t^3 + 9t + 1} dt = -\frac{1}{3} \int \frac{-3(t^2 - 3)}{-t^3 + 9t + 1} dt = -\frac{1}{3} \ln |-t^3 + 9t + 1| + C$$

$$\begin{aligned}25. \int \frac{x^2}{x-1} dx &= \int (x+1) dx + \int \frac{1}{x-1} dx \\ &= \frac{1}{2} x^2 + x + \ln|x-1| + C\end{aligned}$$

27. Let
- $u = 1 + e^x$
- ,
- $du = e^x dx$
- .

$$\int \frac{e^x}{1 + e^x} dx = \ln(1 + e^x) + C$$

$$29. \int (1 + 2x^2)^2 dx = \int (4x^4 + 4x^2 + 1) dx = \frac{4}{5} x^5 + \frac{4}{3} x^3 + x + C = \frac{x}{15} (12x^4 + 20x^2 + 15) + C$$

31. Let
- $u = 2\pi x^2$
- ,
- $du = 4\pi x dx$
- .

$$\begin{aligned}\int x(\cos 2\pi x^2) dx &= \frac{1}{4\pi} \int (\cos 2\pi x^2)(4\pi x) dx \\ &= \frac{1}{4\pi} \sin 2\pi x^2 + C\end{aligned}$$

33. Let
- $u = \pi x$
- ,
- $du = \pi dx$
- .

$$\int \csc(\pi x) \cot(\pi x) dx = \frac{1}{\pi} \int \csc(\pi x) \cot(\pi x) \pi dx = -\frac{1}{\pi} \csc(\pi x) + C$$

35. Let
- $u = 5x$
- ,
- $du = 5 dx$
- .

$$\int e^{5x} dx = \frac{1}{5} \int e^{5x} (5) dx = \frac{1}{5} e^{5x} + C$$

37. Let
- $u = 1 + e^x$
- ,
- $du = e^x dx$
- .

$$\begin{aligned}\int \frac{2}{e^{-x} + 1} dx &= 2 \int \left(\frac{1}{e^{-x} + 1} \right) \left(\frac{e^x}{e^x} \right) dx \\ &= 2 \int \frac{e^x}{1 + e^x} dx = 2 \ln(1 + e^x) + C\end{aligned}$$

$$39. \int \frac{\ln x^2}{x} dx = 2 \int (\ln x) \frac{1}{x} dx = 2 \frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$$

$$41. \int \frac{1 + \sin x}{\cos x} dx = \int (\sec x + \tan x) dx = \ln|\sec x + \tan x| + \ln|\sec x| + C = \ln|\sec x(\sec x + \tan x)| + C$$

$$43. \frac{1}{\cos \theta - 1} = \frac{1}{\cos \theta - 1} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos \theta + 1}{\cos^2 \theta - 1} = \frac{\cos \theta + 1}{-\sin^2 \theta}$$

$$= -\csc \theta \cdot \cot \theta - \csc^2 \theta$$

$$\int \frac{1}{\cos \theta - 1} d\theta = \int (-\csc \theta \cot \theta - \csc^2 \theta) d\theta$$

$$= \csc \theta + \cot \theta + C$$

$$= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} + C$$

$$= \frac{1 + \cos \theta}{\sin \theta} + C$$

$$45. \int \frac{3z + 2}{z^2 + 9} dz = \frac{3}{2} \int \frac{2z}{z^2 + 9} dz + 2 \int \frac{dz}{z^2 + 9}$$

$$= \frac{3}{2} \ln(z^2 + 9) + \frac{2}{3} \arctan\left(\frac{z}{3}\right) + C$$

$$47. \text{ Let } u = 2t - 1, du = 2 dt.$$

$$\int \frac{-1}{\sqrt{1 - (2t - 1)^2}} dt = -\frac{1}{2} \int \frac{2}{\sqrt{1 - (2t - 1)^2}} dt$$

$$= -\frac{1}{2} \arcsin(2t - 1) + C$$

$$49. \text{ Let } u = \cos\left(\frac{2}{t}\right), du = \frac{2 \sin(2/t)}{t^2} dt.$$

$$\int \frac{\tan(2/t)}{t^2} dt = \frac{1}{2} \int \frac{1}{\cos(2/t)} \left[\frac{2 \sin(2/t)}{t^2} \right] dt$$

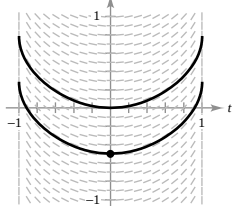
$$= \frac{1}{2} \ln \left| \cos\left(\frac{2}{t}\right) \right| + C$$

$$51. \int \frac{3}{\sqrt{6x - x^2}} dx = 3 \int \frac{1}{\sqrt{9 - (x - 3)^2}} dx = 3 \arcsin\left(\frac{x - 3}{3}\right) + C$$

$$53. \int \frac{4}{4x^2 + 4x + 65} dx = \int \frac{1}{[x + (1/2)]^2 + 16} dx = \frac{1}{4} \arctan\left[\frac{x + (1/2)}{4}\right] + C = \frac{1}{4} \arctan\left(\frac{2x + 1}{8}\right) + C$$

$$55. \frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}, \left(0, -\frac{1}{2}\right)$$

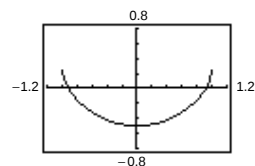
(a)

(b) $u = t^2, du = 2t dt$

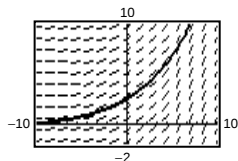
$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{2t}{\sqrt{1 - (t^2)^2}} dt = \frac{1}{2} \arcsin t^2 + C$$

$$\left(0, -\frac{1}{2}\right): -\frac{1}{2} = \frac{1}{2} \arcsin 0 + C \Rightarrow C = -\frac{1}{2}$$

$$s = \frac{1}{2} \arcsin t^2 - \frac{1}{2}$$



57.



$$y = 3e^{0.2x}$$

$$61. \frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$$

$$\text{Let } u = \tan x, du = \sec^2 x \, dx.$$

$$y = \int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right) + C$$

$$65. \text{ Let } u = -x^2, du = -2x \, dx.$$

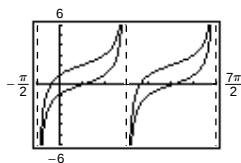
$$\begin{aligned} \int_0^1 x e^{-x^2} dx &= -\frac{1}{2} \int_0^1 e^{-x^2} (-2x) dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-1}) \approx 0.316 \end{aligned}$$

$$69. \text{ Let } u = 3x, du = 3 \, dx.$$

$$\begin{aligned} \int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} dx &= \frac{1}{3} \int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2} dx \\ &= \left[\frac{1}{6} \arctan\left(\frac{3x}{2}\right) \right]_0^{2/\sqrt{3}} \\ &= \frac{\pi}{18} \approx 0.175 \end{aligned}$$

$$73. \int \frac{1}{1 + \sin \theta} d\theta = \tan \theta - \sec \theta + C \left(\text{or } \frac{-2}{1 + \tan(\theta/2)} \right)$$

The antiderivatives are vertical translations of each other.



$$77. \text{ Log Rule: } \int \frac{du}{u} = \ln|u| + C, u = x^2 + 1.$$

$$\begin{aligned} 59. y &= \int (1 + e^x)^2 dx = \int (e^{2x} + 2e^x + 1) dx \\ &= \frac{1}{2} e^{2x} + 2e^x + x + C \end{aligned}$$

$$63. \text{ Let } u = 2x, du = 2 \, dx.$$

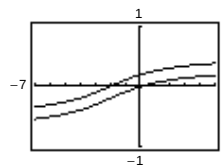
$$\begin{aligned} \int_0^{\pi/4} \cos 2x \, dx &= \frac{1}{2} \int_0^{\pi/4} \cos 2x (2) \, dx \\ &= \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{1}{2} \end{aligned}$$

$$67. \text{ Let } u = x^2 + 9, du = 2x \, dx.$$

$$\begin{aligned} \int_0^4 \frac{2x}{\sqrt{x^2 + 9}} dx &= \int_0^4 (x^2 + 9)^{-1/2} (2x) \, dx \\ &= \left[2\sqrt{x^2 + 9} \right]_0^4 = 4 \end{aligned}$$

$$71. \int \frac{1}{x^2 + 4x + 13} dx = \frac{1}{3} \arctan\left(\frac{x+2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



$$75. \text{ Power Rule: } \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1.$$

$$u = x^2 + 1, n = 3$$

79. The are equivalent because

$$e^{x+C_1} = e^x \cdot e^{C_1} = Ce^x, C = e^{C_1}$$

81. $\sin x + \cos x = a \sin(x + b)$

$$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$$

$$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$$

Equate coefficients of like terms to obtain the following.

$$1 = a \cos b \quad \text{and} \quad 1 = a \sin b$$

Thus, $a = 1/\cos b$. Now, substitute for a in $1 = a \sin b$.

$$1 = \left(\frac{1}{\cos b}\right) \sin b$$

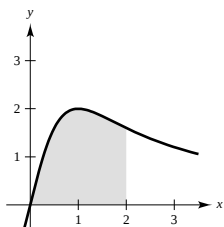
$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

$$\text{Since } b = \frac{\pi}{4}, a = \frac{1}{\cos(\pi/4)} = \sqrt{2}. \text{ Thus, } \sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right).$$

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2} \sin(x + (\pi/4))} = \frac{1}{\sqrt{2}} \int \csc\left(x + \frac{\pi}{4}\right) dx = -\frac{1}{\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right) \right| + C$$

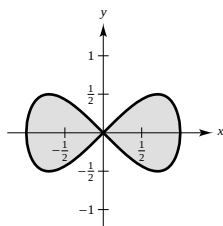
83. $\int_0^2 \frac{4x}{x^2 + 1} dx \approx 3$

Matches (a).



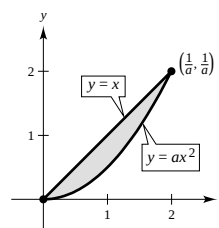
85. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} A &= 4 \int_0^1 x \sqrt{1 - x^2} dx \\ &= -2 \int_0^1 (1 - x^2)^{1/2} (-2x) dx \\ &= \left[-\frac{4}{3} (1 - x^2)^{3/2} \right]_0^1 = \frac{4}{3} \end{aligned}$$



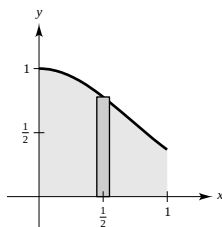
$$\begin{aligned} 87. \int_0^{1/a} (x - ax^2) dx &= \left[\frac{1}{2}x^2 - \frac{a}{3}x^3 \right]_0^{1/a} \\ &= \frac{1}{6a^2} \end{aligned}$$

$$\text{Let } \frac{1}{6a^2} = \frac{2}{3}, 12a^2 = 3, a = \frac{1}{2}.$$

89. (a) **Shell Method:**

$$\text{Let } u = -x^2, du = -2x dx.$$

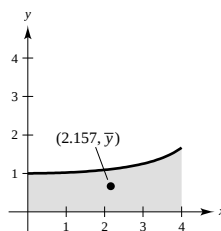
$$\begin{aligned} V &= 2\pi \int_0^1 x e^{-x^2} dx \\ &= -\pi \int_0^1 e^{-x^2} (-2x) dx \\ &= \left[-\pi e^{-x^2} \right]_0^1 \\ &= \pi(1 - e^{-1}) \approx 1.986 \end{aligned}$$

(b) **Shell Method:**

$$\begin{aligned} V &= 2\pi \int_0^b x e^{-x^2} dx \\ &= \left[-\pi e^{-x^2} \right]_0^b \\ &= \pi(1 - e^{-b^2}) = \frac{4}{3} \\ e^{-b^2} &= \frac{3\pi - 4}{3\pi} \\ b &= \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \\ &\approx 0.743 \end{aligned}$$

$$91. A = \int_0^4 \frac{5}{\sqrt{25-x^2}} dx = \left[5 \arcsin \frac{x}{5} \right]_0^4 = 5 \arcsin \frac{4}{5}$$

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^4 x \left(\frac{5}{\sqrt{25-x^2}} \right) dx \\ &= \frac{1}{5 \arcsin(4/5)} \left(-\frac{5}{2} \right) \int_0^4 (25-x^2)^{-1/2} (-2x) dx \\ &= \frac{1}{5 \arcsin(4/5)} (-5) \left[(25-x^2)^{1/2} \right]_0^4 \\ &= -\frac{1}{\arcsin(4/5)} [3-5] \\ &= \frac{2}{\arcsin(4/5)} \approx 2.157 \end{aligned}$$



$$93. \quad y = \tan(\pi x)$$

$$y' = \pi \sec^2(\pi x)$$

$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$

$$\begin{aligned} s &= \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)} dx \\ &\approx 1.0320 \end{aligned}$$

Section 7.2 Integration by Parts

$$1. \frac{d}{dx} [\sin x - x \cos x] = \cos x - (-x \sin x + \cos x) = x \sin x. \text{ Matches (b)}$$

$$3. \frac{d}{dx} [x^2 e^x - 2x e^x + 2e^x] = x^2 e^x + 2x e^x - 2x e^x - 2e^x + 2e^x = x^2 e^x. \text{ Matches (c)}$$

$$5. \int x e^{2x} dx$$

$$u = x, dv = e^{2x} dx$$

$$7. \int (\ln x)^2 dx$$

$$u = (\ln x)^2, dv = dx$$

$$9. \int x \sec^2 x dx$$

$$u = x, dv = \sec^2 x dx$$

$$11. dv = e^{-2x} dx \Rightarrow v = \int e^{-2x} dx = -\frac{1}{2} e^{-2x}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x e^{-2x} dx &= -\frac{1}{2} x e^{-2x} - \int -\frac{1}{2} e^{-2x} dx \\ &= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C = \frac{-1}{4e^{2x}} (2x + 1) + C \end{aligned}$$

13. Use integration by parts three times.

$$(1) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (2) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (3) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^3 \Rightarrow du = 3x^2 dx \quad u = x^2 \Rightarrow du = 2x dx \quad u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6 \int x e^x dx \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C \end{aligned}$$

$$15. \int x^2 e^{x^3} dx = \frac{1}{3} \int e^{x^3} (3x^2) dx = \frac{1}{3} e^{x^3} + C$$

$$17. \, dv = t \, dt \Rightarrow v = \int t \, dt = \frac{t^2}{2}$$

$$u = \ln(t+1) \Rightarrow du = \frac{1}{t+1} dt$$

$$\begin{aligned} \int t \ln(t+1) dt &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \frac{t^2}{t+1} dt \\ &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \left(t - 1 + \frac{1}{t+1} \right) dt \\ &= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \left[\frac{t^2}{2} - t + \ln(t+1) \right] + C \\ &= \frac{1}{4} [2(t^2 - 1) \ln|t+1| - t^2 + 2t] + C \end{aligned}$$

$$19. \text{ Let } u = \ln x, \, du = \frac{1}{x} dx.$$

$$\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \left(\frac{1}{x} \right) dx = \frac{(\ln x)^3}{3} + C$$

$$21. \, dv = \frac{1}{(2x+1)^2} dx \Rightarrow v = \int (2x+1)^{-2} dx = -\frac{1}{2(2x+1)}$$

$$u = xe^{2x} \Rightarrow du = (2xe^{2x} + e^{2x}) dx = e^{2x}(2x+1) dx$$

$$\begin{aligned} \int \frac{xe^{2x}}{(2x+1)^2} dx &= -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2} dx \\ &= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C \\ &= \frac{e^{2x}}{4(2x+1)} + C \end{aligned}$$

23. Use integration by parts twice.

$$(1) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int (x^2 - 1)e^x dx &= \int x^2 e^x dx - \int e^x dx = x^2 e^x - 2 \int x e^x dx - e^x \\ &= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] - e^x = x^2 e^x - 2x e^x + e^x + C = (x-1)^2 e^x + C \end{aligned}$$

$$25. dv = \sqrt{x-1} dx \Rightarrow v = \int (x-1)^{1/2} dx = \frac{2}{3}(x-1)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{x-1} dx &= \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3} \int (x-1)^{3/2} dx \\ &= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C \\ &= \frac{2(x-1)^{3/2}}{15}(3x+2) + C \end{aligned}$$

$$27. dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x$$

$$u = x \Rightarrow du = dx$$

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

29. Use integration by parts three times.

$$(1) u = x^3, du = 3x^2, dv = \sin x dx, v = -\cos x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3 \int x^2 \cos x dx$$

$$(2) u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3 \left[x^2 \sin x - 2 \int x \sin x dx \right] \\ &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx \end{aligned}$$

$$(3) u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3x^2 \sin x - 6 \left[-x \cos x + \int \cos x dx \right] \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \end{aligned}$$

$$31. u = t, du = dt, dv = \csc t \cot t dt, v = -\csc t$$

$$\begin{aligned} \int t \csc t \cot t dt &= -t \csc t + \int \csc t dt \\ &= -t \csc t - \ln |\csc t + \cot t| + C \end{aligned}$$

$$33. dv = dx \Rightarrow v = \int dx = x$$

$$u = \arctan x \Rightarrow du = \frac{1}{1+x^2} dx$$

$$\begin{aligned} \int \arctan x dx &= x \arctan x - \int \frac{x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

35. Use integration by parts twice.

$$(1) dv = e^{2x} dx \Rightarrow v = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$\int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left(\frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x dx \right)$$

$$\frac{5}{4} \int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x$$

$$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$$

$$(2) dv = e^{2x} dx \Rightarrow v = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

37. $y' = xe^{x^2}$

$$y = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

39. Use integration by parts twice.

$$(1) \quad dv = \frac{1}{\sqrt{2+3t}} dt \Rightarrow v = \int (2+3t)^{-1/2} dt = \frac{2}{3}\sqrt{2+3t}$$

$$u = t^2 \Rightarrow du = 2t dt$$

$$(2) \quad dv = \sqrt{2+3t} dt \Rightarrow v = \int (2+3t)^{1/2} dt = \frac{2}{9}(2+3t)^{3/2}$$

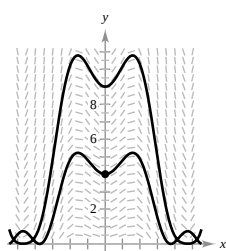
$$u = t \Rightarrow du = dt$$

$$\begin{aligned} y &= \int \frac{t^2}{\sqrt{2+3t}} dt = \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3} \int t\sqrt{2+3t} dt \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3} \left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9} \int (2+3t)^{3/2} dt \right] \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C \\ &= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C \end{aligned}$$

41. $(\cos y)y' = 2x$

$$\begin{aligned} \int \cos y dy &= \int 2x dx \\ \sin y &= x^2 + C \end{aligned}$$

43. (a)



(b) $\frac{dy}{dx} = x\sqrt{y} \cos x, (0, 4)$

$$\int \frac{dy}{\sqrt{y}} = \int x \cos x dx$$

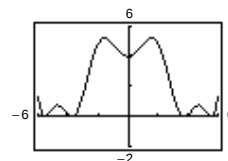
$$\int y^{-1/2} dy = \int x \cos x dx \quad (u = x, du = dx, dv = \cos x dx, v = \sin x)$$

$$2y^{1/2} = x \sin x - \int \sin x dx$$

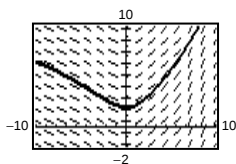
$$= x \sin x + \cos x + C$$

$$(0, 4): 2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$$

$$2\sqrt{y} = x \sin x + \cos x + 3$$



45. $\frac{dy}{dx} = \frac{x}{y} e^{x/8}, y(0) = 2$



47. $u = x, du = dx, dv = e^{-x/2} dx, v = -2e^{-x/2}$

$$\int x e^{-x/2} dx = -2x e^{-x/2} + \int 2e^{-x/2} dx = -2x e^{-x/2} - 4e^{-x/2} + C$$

$$\begin{aligned} \text{Thus, } \int_0^4 x e^{-x/2} dx &= \left[-2x e^{-x/2} - 4e^{-x/2} \right]_0^4 \\ &= -8e^{-2} - 4e^{-2} + 4 \\ &= -12e^{-2} + 4 \approx 2.376. \end{aligned}$$

49. See Exercise 27.

$$\int_0^{\pi/2} x \cos x dx = \left[x \sin x + \cos x \right]_0^{\pi/2} = \frac{\pi}{2} - 1$$

51. $u = \arccos x, du = -\frac{1}{\sqrt{1-x^2}} dx, dv = dx, v = x$

$$\int \arccos x dx = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx = x \arccos x - \sqrt{1-x^2} + C$$

$$\begin{aligned} \text{Thus, } \int_0^{1/2} \arccos x dx &= \left[x \arccos x - \sqrt{1-x^2} \right]_0^{1/2} \\ &= \frac{1}{2} \arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1 \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658. \end{aligned}$$

53. Use integration by parts twice.

$$(1) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$2 \int e^x \sin x dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

$$\text{Thus, } \int_0^1 e^x \sin x dx = \left[\frac{e^x}{2} (\sin x - \cos x) \right]_0^1 = \frac{e}{2} (\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$$

$$(2) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

55. $dv = x^2 dx, v = \frac{x^3}{3}, u = \ln x, du = \frac{1}{x} dx$

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \left(\frac{1}{x} \right) dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$\text{Hence, } \int_1^2 x^2 \ln x dx = \left[\frac{x^3}{3} \ln x - \frac{1}{9} x^3 \right]_1^2$$

$$= \frac{8}{3} \ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3} \ln 2 - \frac{7}{9} \approx 1.071.$$

57. $dv = x dx$, $v = \frac{x^2}{2}$, $u = \operatorname{arcsec} x$, $du = \frac{1}{x\sqrt{x^2-1}} dx$

$$\begin{aligned}\int x \operatorname{arcsec} x dx &= \frac{x^2}{2} \operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2-1}} dx \\ &= \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{4} \int \frac{2x}{\sqrt{x^2-1}} dx \\ &= \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2-1} + C\end{aligned}$$

Hence,

$$\begin{aligned}\int_2^4 x \operatorname{arcsec} x dx &= \left[\frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2-1} \right]_2^4 \\ &= \left(8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} \right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \\ &= 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \\ &\approx 7.380.\end{aligned}$$

59. $\int x^2 e^{2x} dx = x^2 \left(\frac{1}{2} e^{2x} \right) - (2x) \left(\frac{1}{4} e^{2x} \right) + 2 \left(\frac{1}{8} e^{2x} \right) + C$

$$\begin{aligned}&= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C \\ &= \frac{1}{4} e^{2x} (2x^2 - 2x + 1) + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2} e^{2x}$
+	2	$\frac{1}{4} e^{2x}$
−	0	$\frac{1}{8} e^{2x}$

61. $\int x^3 \sin x dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x \cos x - 6 \sin x + C$

$$\begin{aligned}&= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \\ &= (3x^2 - 6) \sin x - (x^3 - 6x) \cos x + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\sin x$
−	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
−	6	$\cos x$
+	0	$\sin x$

63. $\int x \sec^2 x dx = x \tan x + \ln|\cos x| + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x	$\sec^2 x$
−	1	$\tan x$
+	0	$-\ln \cos x $

65. Integration by parts is based on the product rule.

67. No. Substitution.

69. Yes. $u = x^2$, $dv = e^{2x} dx$

71. Yes. Let $u = x$ and $du = \frac{1}{\sqrt{x+1}} dx$.

(Substitution also works. Let $u = \sqrt{x+1}$)

73. $\int t^3 e^{-4t} dt = -\frac{e^{-4t}}{128} (32t^3 + 24t^2 + 12t + 3) + C$

75. $\int_0^{\pi/2} e^{-2x} \sin 3x dx = \left[\frac{e^{-2x}(-2 \sin 3x - 3 \cos 3x)}{13} \right]_0^{\pi/2} = \frac{1}{13} (2e^{-\pi} + 3) \approx 0.2374$

$$77. (a) \, dv = \sqrt{2x-3} \, dx \Rightarrow v = \int (2x-3)^{1/2} dx = \frac{1}{3}(2x-3)^{3/2}$$

$$u = 2x \quad \Rightarrow \quad du = 2 \, dx$$

$$\begin{aligned} \int 2x\sqrt{2x-3} \, dx &= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{3} \int (2x-3)^{3/2} dx \\ &= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{15}(2x-3)^{5/2} + C \\ &= \frac{2}{15}(2x-3)^{3/2}(3x+3) + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C \end{aligned}$$

$$(b) \, u = 2x-3 \Rightarrow x = \frac{u+3}{2} \text{ and } dx = \frac{1}{2} du$$

$$\begin{aligned} \int 2x\sqrt{2x-3} \, dx &= \int 2\left(\frac{u+3}{2}\right) u^{1/2} \left(\frac{1}{2}\right) du = \frac{1}{2} \int (u^{3/2} + 3u^{1/2}) du = \frac{1}{2} \left[\frac{2}{5} u^{5/2} + 2u^{3/2} \right] + C \\ &= \frac{1}{5} u^{3/2}(u+5) + C = \frac{1}{5} (2x-3)^{3/2} [(2x-3)+5] + C = \frac{2}{5} (2x-3)^{3/2}(x+1) + C \end{aligned}$$

$$79. (a) \, dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \int (4+x^2)^{-1/2} x \, dx = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - 2 \int x \sqrt{4+x^2} dx \\ &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{1}{3} \sqrt{4+x^2} (x^2-8) + C \end{aligned}$$

$$(b) \, u = 4+x^2 \Rightarrow x^2 = u-4 \text{ and } 2x \, dx = du \Rightarrow x \, dx = \frac{1}{2} du$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{x^2}{\sqrt{4+x^2}} x \, dx = \int \frac{u-4}{\sqrt{u}} \frac{1}{2} du \\ &= \frac{1}{2} \int (u^{1/2} - 4u^{-1/2}) du = \frac{1}{2} \left(\frac{2}{3} u^{3/2} - 8u^{1/2} \right) + C \\ &= \frac{1}{3} u^{1/2}(u-12) + C = \frac{1}{3} \sqrt{4+x^2} [(4+x^2)-12] + C = \frac{1}{3} \sqrt{4+x^2} (x^2-8) + C \end{aligned}$$

$$81. \, n=0: \int \ln x \, dx = x(\ln x - 1) + C$$

$$n=1: \int x \ln x \, dx = \frac{x^2}{4}(2 \ln x - 1) + C$$

$$n=2: \int x^2 \ln x \, dx = \frac{x^3}{9}(3 \ln x - 1) + C$$

$$n=3: \int x^3 \ln x \, dx = \frac{x^4}{16}(4 \ln x - 1) + C$$

$$n=4: \int x^4 \ln x \, dx = \frac{x^5}{25}(5 \ln x - 1) + C$$

$$\text{In general, } \int x^n \ln x \, dx = \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C. \text{ (See Exercise 85.)}$$

$$83. dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = x^n \Rightarrow du = nx^{n-1} \, dx$$

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$85. dv = x^n \, dx \Rightarrow v = \frac{x^{n+1}}{n+1}$$

$$u = \ln x \Rightarrow du = \frac{1}{x} \, dx$$

$$\begin{aligned} \int x^n \ln x \, dx &= \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n}{n+1} \, dx \\ &= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C \\ &= \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C \end{aligned}$$

87. Use integration by parts twice.

$$(1) dv = e^{ax} \, dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \sin bx \Rightarrow du = b \cos bx \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx$$

$$= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \left[\frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx \, dx \right] = \frac{e^{ax} \sin bx}{a} - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx$$

$$\text{Therefore, } \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$$

$$(2) dv = e^{ax} \, dx \Rightarrow v = \frac{1}{a} e^{ax}$$

$$u = \cos bx \Rightarrow du = -b \sin bx \, dx$$

89. $n = 3$ (Use formula in Exercise 85.)

$$\int x^3 \ln x \, dx = \frac{x^4}{16} [4 \ln x - 1] + C$$

91. $a = 2, b = 3$ (Use formula in Exercise 88.)

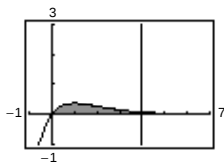
$$\int e^{2x} \cos 3x \, dx = \frac{e^{2x}(2 \cos 3x + 3 \sin 3x)}{13} + C$$

$$93. dv = e^{-x} \, dx \Rightarrow v = -e^{-x}$$

$$u = x \Rightarrow du = dx$$

$$A = \int_0^4 x e^{-x} \, dx = \left[-x e^{-x} \right]_0^4 + \int_0^4 e^{-x} \, dx = \frac{-4}{e^4} - \left[e^{-x} \right]_0^4$$

$$= 1 - \frac{5}{e^4} \approx 0.908$$

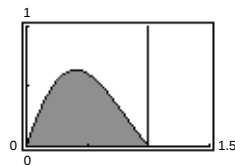


$$95. A = \int_0^1 e^{-x} \sin(\pi x) \, dx$$

$$= \left[\frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2} \right]_0^1$$

$$= \frac{1}{1 + \pi^2} \left(\frac{\pi}{e} + \pi \right) = \frac{\pi}{1 + \pi^2} \left(\frac{1}{e} + 1 \right)$$

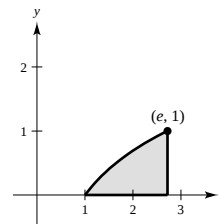
$$\approx 0.395 \text{ (See Exercise 87.)}$$



97. (a) $A = \int_1^e \ln x \, dx = \left[-x + x \ln x \right]_1^e = 1$ (See Exercise 4.)

(b) $R(x) = \ln x, r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^e (\ln x)^2 \, dx \\ &= \pi \left[x(\ln x)^2 - 2x \ln x + 2x \right]_1^e \quad (\text{Use integration by parts twice, see Exercise 7.}) \\ &= \pi(e - 2) \approx 2.257 \end{aligned}$$



(c) $p(x) = x, h(x) = \ln x$

$$\begin{aligned} V &= 2\pi \int_1^e x \ln x \, dx = 2\pi \left[\frac{x^2}{4}(-1 + 2 \ln x) \right]_1^e \\ &= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \quad (\text{See Exercise 85.}) \end{aligned}$$

(d) $\bar{x} = \frac{\int_1^e x \ln x \, dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$

$$\bar{y} = \frac{\frac{1}{2} \int_1^e (\ln x)^2 \, dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2} \right) \approx (2.097, 0.359)$$

99. Average value $= \frac{1}{\pi} \int_0^\pi e^{-4t}(\cos 2t + 5 \sin 2t) \, dt$

$$\begin{aligned} &= \frac{1}{\pi} \left[e^{-4t} \left(\frac{-4 \cos 2t + 2 \sin 2t}{20} \right) + 5e^{-4t} \left(\frac{-4 \sin 2t - 2 \cos 2t}{20} \right) \right]_0^\pi \quad (\text{From Exercises 87 and 88}) \\ &= \frac{7}{10\pi} (1 - e^{-4\pi}) \approx 0.223 \end{aligned}$$

101. $c(t) = 100,000 + 4000t, r = 5\%, t_1 = 10$

$$P = \int_0^{10} (100,000 + 4000t)e^{-0.05t} \, dt = 4000 \int_0^{10} (25 + t)e^{-0.05t} \, dt$$

Let $u = 25 + t, dv = e^{-0.05t} \, dt, du = dt, v = -\frac{100}{5}e^{-0.05t}$

$$\begin{aligned} P &= 4000 \left\{ \left[(25 + t) \left(-\frac{100}{5} e^{-0.05t} \right) \right]_0^{10} + \frac{100}{5} \int_0^{10} e^{-0.05t} \, dt \right\} \\ &= 4000 \left\{ \left[(25 + t) \left(-\frac{100}{5} e^{-0.05t} \right) \right]_0^{10} - \left[\frac{10,000}{25} e^{-0.05t} \right]_0^{10} \right\} \approx \$931,265 \end{aligned}$$

103. $\int_{-\pi}^{\pi} x \sin nx \, dx = \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_{-\pi}^{\pi}$

$$= -\frac{\pi}{n} \cos \pi n - \frac{\pi}{n} \cos(-\pi n)$$

$$= -\frac{2\pi}{n} \cos \pi n$$

$$= \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$$

105. Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$\begin{aligned} I_1 &= \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx = \left[\frac{-2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_0^1 \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

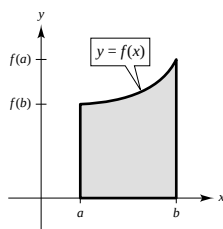
Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$\begin{aligned} I_2 &= \int_1^2 (-x + 2) \sin\left(\frac{n\pi}{2}x\right) dx = \left[\frac{-2(-x + 2)}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_1^2 \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$h(I_1 + I_2) = b_n = h \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right] = \frac{8h}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

107. Shell Method:

$$\begin{aligned} V &= 2\pi \int_a^b x f(x) dx \\ dv &= x dx \Rightarrow v = \frac{x^2}{2} \\ u &= f(x) \Rightarrow du = f'(x) dx \\ V &= 2\pi \left[\frac{x^2}{2} f(x) - \int \frac{x^2}{2} f'(x) dx \right]_a^b \\ &= \pi \left[(b^2 f(b) - a^2 f(a)) - \int_a^b x^2 f'(x) dx \right] \end{aligned}$$



Disk Method:

$$\begin{aligned} V &= \pi \int_0^{f(a)} (b^2 - a^2) dy + \pi \int_{f(a)}^{f(b)} [b^2 - [f^{-1}(y)]^2] dy \\ &= \pi(b^2 - a^2) f(a) + \pi b^2 (f(b) - f(a)) - \pi \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \\ &= \pi \left[(b^2 f(b) - a^2 f(a)) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \right] \end{aligned}$$

Since $x = f^{-1}(y)$, we have $f(x) = y$ and $f'(x)dx = dy$. When $y = f(a)$, $x = a$. When $y = f(b)$, $x = b$. Thus,

$$\int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy = \int_a^b x^2 f'(x) dx$$

and the volumes are the same.

109. $f'(x) = xe^{-x}$

(a) $f(x) = \int xe^{-x} dx = -xe^{-x} - e^{-x} + C$

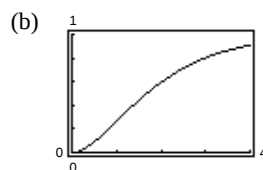
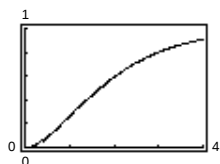
(Parts: $u = x, dv = e^{-x} dx$)

$f(0) = 0 = -1 + C \Rightarrow C = 1$

$f(x) = -xe^{-x} - e^{-x} + 1$

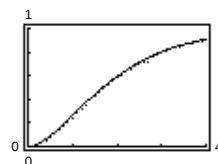
(c) You obtain the points

n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	2.378×10^{-3}
3	0.15	0.0069
4	0.20	0.0134
$\vdots$	$\vdots$	$\vdots$
80	4.0	0.9064



(d) You obtain the points

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
$\vdots$	$\vdots$	$\vdots$
40	4.0	0.9039



(e) $f(4) = 0.9084$

The approximations are tangent line approximations. The results in (c) are better because Δx is smaller.

Section 7.3 Trigonometric Integrals

1. $f(x) = \sin^4 x + \cos^4 x$

$$\begin{aligned}
 \text{(a) } \sin^4 x + \cos^4 x &= \left(\frac{1 - \cos 2x}{2} \right)^2 + \left(\frac{1 + \cos 2x}{2} \right)^2 \\
 &= \frac{1}{4} [1 - 2 \cos 2x + \cos^2 2x + 1 + 2 \cos 2x + \cos^2 2x] \\
 &= \frac{1}{4} \left[2 + 2 \frac{1 + \cos 4x}{2} \right] \\
 &= \frac{1}{4} [3 + \cos 4x]
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \sin^4 x + \cos^4 x &= (\sin^2 x)^2 + \cos^4 x \\
 &= (1 - \cos^2 x)^2 + \cos^4 x \\
 &= 1 - 2 \cos^2 x + 2 \cos^4 x
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } \sin^4 x + \cos^4 x &= \sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x - 2 \sin^2 x \cos^2 x \\
 &= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x \\
 &= 1 - 2 \sin^2 x \cos^2 x
 \end{aligned}$$

—CONTINUED—

1. —CONTINUED—

$$\begin{aligned}
 \text{(d) } 1 - 2 \sin^2 x \cos^2 x &= 1 - (2 \sin x \cos x)(\sin x \cos x) \\
 &= 1 - (\sin 2x) \left(\frac{1}{2} \sin 2x \right) \\
 &= 1 - \frac{1}{2} \sin^2(2x)
 \end{aligned}$$

(e) Four ways. There is often more than one way to rewrite a trigonometric expression.

3. Let $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}
 \int \cos^3 x \sin x \, dx &= -\int \cos^3 x (-\sin x) \, dx \\
 &= -\frac{1}{4} \cos^4 x + C
 \end{aligned}$$

5. Let $u = \sin 2x$, $du = 2 \cos 2x \, dx$.

$$\begin{aligned}
 \int \sin^5 2x \cos 2x \, dx &= \frac{1}{2} \int \sin^5 2x (2 \cos 2x) \, dx \\
 &= \frac{1}{12} \sin^6 2x + C
 \end{aligned}$$

7. Let $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}
 \int \sin^5 x \cos^2 x \, dx &= \int \sin x (1 - \cos^2 x)^2 \cos^2 x \, dx \\
 &= -\int (\cos^2 x - 2 \cos^4 x + \cos^6 x)(-\sin x) \, dx = \frac{-1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C
 \end{aligned}$$

$$\begin{aligned}
 \text{9. } \int \cos^3 \theta \sqrt{\sin \theta} \, d\theta &= \int \cos \theta (1 - \sin^2 \theta)(\sin \theta)^{1/2} \, d\theta \\
 &= \int [(\sin \theta)^{1/2} - (\sin \theta)^{5/2}] \cos \theta \, d\theta \\
 &= \frac{2}{3} (\sin \theta)^{3/2} - \frac{2}{7} (\sin \theta)^{7/2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{11. } \int \cos^2 3x \, dx &= \int \frac{1 + \cos 6x}{2} \, dx \\
 &= \frac{1}{2} \left(x + \frac{1}{6} \sin 6x \right) + C \\
 &= \frac{1}{12} (6x + \sin 6x) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{13. } \int \sin^2 \alpha \cdot \cos^2 \alpha \, d\alpha &= \int \frac{1 - \cos 2\alpha}{2} \cdot \frac{1 + \cos 2\alpha}{2} \, d\alpha \\
 &= \frac{1}{4} \int (1 - \cos^2 2\alpha) \, d\alpha \\
 &= \frac{1}{4} \int \left(1 - \frac{1 + \cos 4\alpha}{2} \right) \, d\alpha \\
 &= \frac{1}{8} \int (1 - \cos 4\alpha) \, d\alpha \\
 &= \frac{1}{8} \left[\alpha - \frac{1}{4} \sin 4\alpha \right] + C \\
 &= \frac{1}{32} [4\alpha - \sin 4\alpha] + C
 \end{aligned}$$

15. Integration by parts.

$$dv = \sin^2 x \, dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sin^2 x \, dx &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \int (2x - \sin 2x) \, dx \\ &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \left(x^2 + \frac{1}{2} \cos 2x \right) + C = \frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + C \end{aligned}$$

17. Let $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int_0^{\pi/2} \cos^3 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x) \cos x \, dx \\ &= \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{2}{3} \end{aligned}$$

19. Let $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int_0^{\pi/2} \cos^7 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x)^3 \cos x \, dx = \int_0^{\pi/2} (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cos x \, dx \\ &= \left[\sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x \right]_0^{\pi/2} = \frac{16}{35} \end{aligned}$$

$$21. \int \sec(3x) \, dx = \frac{1}{3} \ln |\sec 3x + \tan 3x| + C$$

$$\begin{aligned} 23. \int \sec^4 5x \, dx &= \int (1 + \tan^2 5x) \sec^2 5x \, dx \\ &= \frac{1}{5} \left(\tan 5x + \frac{\tan^3 5x}{3} \right) + C \\ &= \frac{\tan 5x}{15} (3 + \tan^2 5x) + C \end{aligned}$$

$$25. dv = \sec^2 \pi x \, dx \Rightarrow v = \frac{1}{\pi} \tan \pi x$$

$$u = \sec \pi x \Rightarrow du = \pi \sec \pi x \tan \pi x \, dx$$

$$\int \sec^3 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x (\sec^2 \pi x - 1) \, dx$$

$$2 \int \sec^3 \pi x \, dx = \frac{1}{\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C_1$$

$$\int \sec^3 \pi x \, dx = \frac{1}{2\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C$$

$$\begin{aligned} 27. \int \tan^5 \frac{x}{4} \, dx &= \int \left(\sec^2 \frac{x}{4} - 1 \right) \tan^3 \frac{x}{4} \, dx \\ &= \int \tan^3 \frac{x}{4} \sec^2 \frac{x}{4} \, dx - \int \tan^3 \frac{x}{4} \, dx \\ &= \tan^4 \frac{x}{4} - \int \left(\sec^2 \frac{x}{4} - 1 \right) \tan \frac{x}{4} \, dx \\ &= \tan^4 \frac{x}{4} - 2 \tan^2 \frac{x}{4} - 4 \ln \left| \cos \frac{x}{4} \right| + C \end{aligned}$$

$$29. u = \tan x, du = \sec^2 x \, dx$$

$$\int \sec^2 x \tan x \, dx = \frac{1}{2} \tan^2 x + C$$

$$31. \int \tan^2 x \sec^2 x \, dx = \frac{\tan^3 x}{3} + C$$

$$33. \int \sec^6 4x \tan 4x \, dx = \frac{1}{4} \int \sec^5 4x (4 \sec 4x \tan 4x) \, dx \\ = \frac{\sec^6 4x}{24} + C$$

$$35. \text{ Let } u = \sec x, du = \sec x \tan x \, dx.$$

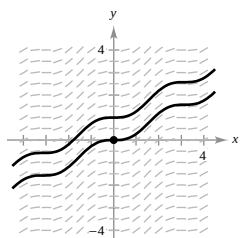
$$\int \sec^3 x \tan x \, dx = \int \sec^2 x (\sec x \tan x) \, dx \\ = \frac{1}{3} \sec^3 x + C$$

$$37. \int \frac{\tan^2 x}{\sec x} \, dx = \int \frac{(\sec^2 x - 1)}{\sec x} \, dx \\ = \int (\sec x - \cos x) \, dx \\ = \ln|\sec x + \tan x| - \sin x + C$$

$$39. r = \int \sin^4(\pi\theta) \, d\theta = \frac{1}{4} \int [1 - \cos(2\pi\theta)]^2 \, d\theta \\ = \frac{1}{4} \int [1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)] \, d\theta \\ = \frac{1}{4} \int \left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2} \right] \, d\theta \\ = \frac{1}{4} \left[\theta - \frac{1}{\pi} \sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi} \sin(4\pi\theta) \right] + C \\ = \frac{1}{32\pi} [12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C$$

$$41. y = \int \tan^3 3x \sec 3x \, dx \\ = \int (\sec^2 3x - 1) \sec 3x \tan 3x \, dx \\ = \frac{1}{3} \int \sec^2 3x (3 \sec 3x \tan 3x) \, dx - \frac{1}{3} \int 3 \sec 3x \tan 3x \, dx \\ = \frac{1}{9} \sec^3 3x - \frac{1}{3} \sec 3x + C$$

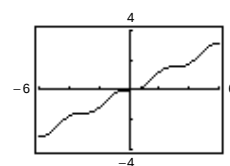
43. (a)



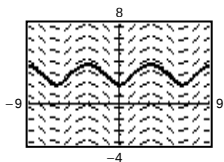
$$(b) \frac{dy}{dx} = \sin^2 x, (0, 0)$$

$$y = \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \\ = \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

$$(0, 0): 0 = C, y = \frac{1}{2}x - \frac{\sin 2x}{4}$$



$$45. \frac{dy}{dx} = \frac{3 \sin x}{y}, y(0) = 2$$



$$47. \int \sin 3x \cos 2x \, dx = \frac{1}{2} \int (\sin 5x + \sin x) \, dx \\ = \frac{-1}{2} \left(\frac{1}{5} \cos 5x + \cos x \right) + C \\ = \frac{-1}{10} (\cos 5x + 5 \cos x) + C$$

$$49. \int \sin \theta \sin 3\theta \, d\theta = \frac{1}{2} \int (\cos 2\theta - \cos 4\theta) \, d\theta \\ = \frac{1}{2} \left(\frac{1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta \right) + C \\ = \frac{1}{8} (2 \sin 2\theta - \sin 4\theta) + C$$

$$51. \int \cot^3 2x \, dx = \int (\csc^2 2x - 1) \cot 2x \, dx \\ = -\frac{1}{2} \int \cot 2x (-2 \csc^2 2x) \, dx - \frac{1}{2} \int \frac{2 \cos 2x}{\sin 2x} \, dx \\ = -\frac{1}{4} \cot^2 2x - \frac{1}{2} \ln|\sin 2x| + C \\ = \frac{1}{4} (\ln|\csc^2 2x| - \cot^2 2x) + C$$

53. Let $u = \cot \theta$, $du = -\csc^2 \theta d\theta$.

$$\begin{aligned}\int \csc^4 \theta d\theta &= \int \csc^2 \theta (1 + \cot^2 \theta) d\theta \\ &= \int \csc^2 \theta d\theta + \int \csc^2 \theta \cot^2 \theta d\theta \\ &= -\cot \theta - \frac{1}{3} \cot^3 \theta + C\end{aligned}$$

$$\begin{aligned}57. \int \frac{1}{\sec x \tan x} dx &= \int \frac{\cos^2 x}{\sin x} dx = \int \frac{1 - \sin^2 x}{\sin x} dx \\ &= \int (\csc x - \sin x) dx \\ &= \ln|\csc x - \cot x| + \cos x + C\end{aligned}$$

$$\begin{aligned}59. \int (\tan^4 t - \sec^4 t) dt &= \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t) dt \quad (\tan^2 t - \sec^2 t = -1) \\ &= -\int (\tan^2 t + \sec^2 t) dt = -\int (2 \sec^2 t - 1) dt = -2 \tan t + t + C\end{aligned}$$

$$\begin{aligned}61. \int_{-\pi}^{\pi} \sin^2 x dx &= 2 \int_0^{\pi} \frac{1 - \cos 2x}{2} dx \\ &= \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \pi\end{aligned}$$

$$\begin{aligned}63. \int_0^{\pi/4} \tan^3 x dx &= \int_0^{\pi/4} (\sec^2 x - 1) \tan x dx \\ &= \int_0^{\pi/4} \sec^2 x \tan x dx - \int_0^{\pi/4} \frac{\sin x}{\cos x} dx \\ &= \left[\frac{1}{2} \tan^2 x + \ln|\cos x| \right]_0^{\pi/4} \\ &= \frac{1}{2}(1 - \ln 2)\end{aligned}$$

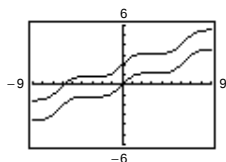
65. Let $u = 1 + \sin t$, $du = \cos t dt$.

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t} dt = \left[\ln|1 + \sin t| \right]_0^{\pi/2} = \ln 2$$

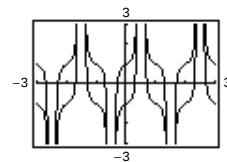
67. Let $u = \sin x$, $du = \cos x dx$.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos^3 x dx &= 2 \int_0^{\pi/2} (1 - \sin^2 x) \cos x dx \\ &= 2 \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{4}{3}\end{aligned}$$

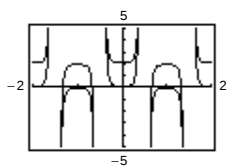
$$69. \int \cos^4 \frac{x}{2} dx = \frac{1}{16} [6x + 8 \sin x + \sin 2x] + C$$



$$71. \int \sec^5 \pi x dx = \frac{1}{4\pi} \left\{ \sec^3 \pi x \tan \pi x + \frac{3}{2} [\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|] \right\} + C$$



$$73. \int \sec^5 \pi x \tan \pi x \, dx = \frac{1}{5\pi} \sec^5 \pi x + C$$



$$75. \int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta = \frac{1}{2} \left[\sin \theta - \frac{1}{5} \sin 5\theta \right]_0^{\pi/4} = \frac{3\sqrt{2}}{10}$$

$$77. \int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{4} \left[\frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x \right]_0^{\pi/2} = \frac{3\pi}{16}$$

79. (a) Save one sine factor and convert the remaining sine factors to cosine. Then expand and integrate.
 (b) Save one cosine factor and convert the remaining cosine factors to sine. Then expand and integrate.
 (c) Make repeated use of the power reducing formula to convert the integrand to odd powers of the cosine.

81. (a) Let $u = \tan 3x$, $du = 3 \sec^2 3x \, dx$.

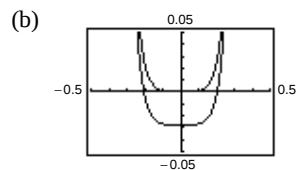
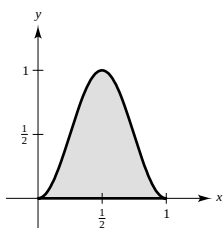
$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x \, dx \\ &= \frac{1}{3} \int (\tan^2 3x + 1) \tan^3 3x (3 \sec^2 3x) \, dx \\ &= \frac{1}{3} \int (\tan^5 3x + \tan^3 3x) (3 \sec^2 3x) \, dx \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1 \end{aligned}$$

Or let $u = \sec 3x$, $du = 3 \sec 3x \tan 3x \, dx$.

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x \, dx \\ &= \frac{1}{3} \int \sec^3 3x (\sec^2 3x - 1) (3 \sec 3x \tan 3x) \, dx \\ &= \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C \end{aligned}$$

$$\begin{aligned} \text{(c) } \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C &= \frac{(1 + \tan^2 3x)^3}{18} - \frac{(1 + \tan^2 3x)^2}{12} + C \\ &= \frac{1}{18} \tan^6 3x + \frac{1}{6} \tan^4 3x + \frac{1}{6} \tan^2 3x + \frac{1}{18} - \frac{1}{12} \tan^4 3x - \frac{1}{6} \tan^2 3x - \frac{1}{12} + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12} \right) + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2 \end{aligned}$$

$$\begin{aligned} 83. A &= \int_0^1 \sin^2(\pi x) \, dx \\ &= \int_0^1 \frac{1 - \cos(2\pi x)}{2} \, dx \\ &= \left[\frac{x}{2} - \frac{1}{4\pi} \sin(2\pi x) \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$



$$85. (a) V = \pi \int_0^{\pi} \sin^2 x \, dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx = \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi^2}{2}$$

$$(b) A = \int_0^{\pi} \sin x \, dx = \left[-\cos x \right]_0^{\pi} = 1 + 1 = 2$$

Let $u = x$, $dv = \sin x \, dx$, $du = dx$, $v = -\cos x$.

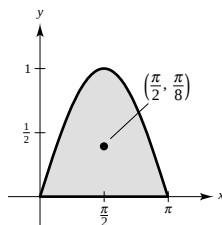
$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx = \frac{1}{2} \left[\left[-x \cos x \right]_0^{\pi} + \int_0^{\pi} \cos x \, dx \right] = \frac{1}{2} \left[-x \cos x + \sin x \right]_0^{\pi} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{1}{8} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8} \right)$$



$$87. dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = \sin^{n-1} x \Rightarrow du = (n-1) \sin^{n-2} x \cos x \, dx$$

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$\text{Therefore, } n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

$$89. \text{ Let } u = \sin^{n-1} x, du = (n-1) \sin^{n-2} x \cos x \, dx, dv = \cos^m x \sin x \, dx, v = \frac{-\cos^{m+1} x}{m+1}.$$

$$\int \cos^m x \sin^n x \, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^{m+2} x \, dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x (1 - \sin^2 x) \, dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx - \frac{n-1}{m+1} \int \sin^n x \cos^m x \, dx$$

$$\frac{m+n}{m+1} \int \cos^m x \sin^n x \, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx$$

$$\int \cos^m x \sin^n x \, dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$$

$$\begin{aligned}
91. \int \sin^5 x \, dx &= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \int \sin^3 x \, dx \\
&= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \left[-\frac{\sin^2 x \cos x}{3} + \frac{2}{3} \int \sin x \, dx \right] \\
&= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C \\
&= -\frac{\cos x}{15} [3 \sin^4 x + 4 \sin^2 x + 8] + C
\end{aligned}$$

$$\begin{aligned}
93. \int \sec^4\left(\frac{2\pi x}{5}\right) dx &= \frac{5}{2\pi} \int \sec^4\left(\frac{2\pi x}{5}\right) \frac{2\pi}{5} dx \\
&= \frac{5}{2\pi} \left[\frac{1}{3} \sec^2\left(\frac{2\pi x}{5}\right) \tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3} \int \sec^2\left(\frac{2\pi x}{5}\right) \frac{2\pi}{5} dx \right] \\
&= \frac{5}{6\pi} \left[\sec^2\left(\frac{2\pi x}{5}\right) \tan\left(\frac{2\pi x}{5}\right) + 2 \tan\left(\frac{2\pi x}{5}\right) \right] + C \\
&= \frac{5}{6\pi} \tan\left(\frac{2\pi x}{5}\right) \left[\sec^2\left(\frac{2\pi x}{5}\right) + 2 \right] + C
\end{aligned}$$

95. (a) $f(t) = a_0 + a_1 \cos \frac{\pi t}{6} + b_1 \sin \frac{\pi t}{6}$ where:

$$a_0 = \frac{1}{12} \int_0^{12} f(t) \, dt$$

$$a_1 = \frac{1}{6} \int_0^{12} f(t) \cos \frac{\pi t}{6} \, dt$$

$$b_1 = \frac{1}{6} \int_0^{12} f(t) \sin \frac{\pi t}{6} \, dt$$

$$\begin{aligned}
a_0 &\approx \frac{12-0}{3(12)^2} [30.9 + 4(32.2) + 2(41.1) + 4(53.7) + 2(64.6) + 4(74.0) + 2(78.2) + 4(77.0) + 2(71.0) + \\
&\quad 4(60.1) + 2(47.1) + 4(35.7) + 30.9] \approx 55.46
\end{aligned}$$

$$\begin{aligned}
a_1 &\approx \frac{12-0}{6(3)(12)} \left[30.9 \cos 0 + 4 \left(32.2 \cos \frac{\pi}{6} \right) + 2 \left(41.1 \cos \frac{\pi}{3} \right) + 4 \left(53.7 \cos \frac{\pi}{2} \right) + 2 \left(64.6 \cos \frac{2\pi}{3} \right) + \right. \\
&\quad \left. 4 \left(74.0 \cos \frac{5\pi}{6} \right) + 2(78.2 \cos \pi) + 4 \left(77.0 \cos \frac{7\pi}{6} \right) + 2 \left(71.0 \cos \frac{4\pi}{3} \right) + \right. \\
&\quad \left. 4 \left(60.1 \cos \frac{3\pi}{2} \right) + 2 \left(47.1 \cos \frac{5\pi}{3} \right) + 4 \left(35.7 \cos \frac{11\pi}{6} \right) + 30.9 \cos 2\pi \right] \approx -23.88
\end{aligned}$$

$$\begin{aligned}
b_1 &\approx \frac{12-0}{6(3)(12)} \left[30.9 \sin 0 + 4 \left(32.2 \sin \frac{\pi}{6} \right) + 2 \left(41.1 \sin \frac{\pi}{3} \right) + 4 \left(53.7 \sin \frac{\pi}{2} \right) + 2 \left(64.6 \sin \frac{2\pi}{3} \right) + \right. \\
&\quad \left. 4 \left(74.0 \sin \frac{5\pi}{6} \right) + 2(78.2 \sin \pi) + 4 \left(77.0 \sin \frac{7\pi}{6} \right) + 2 \left(71.0 \sin \frac{4\pi}{3} \right) + \right. \\
&\quad \left. 4 \left(60.1 \sin \frac{3\pi}{2} \right) + 2 \left(47.1 \sin \frac{5\pi}{3} \right) + 4 \left(35.7 \sin \frac{11\pi}{6} \right) + 30.9 \sin 2\pi \right] \approx -3.34
\end{aligned}$$

$$H(t) \approx 55.46 - 23.88 \cos \frac{\pi t}{6} - 3.34 \sin \frac{\pi t}{6}$$

—CONTINUED—

95. —CONTINUED—

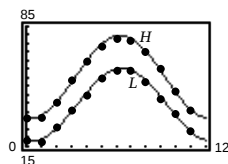
$$(b) a_0 \approx \frac{12-0}{3(12)^2} [18.0 + 4(17.7) + 2(25.8) + 4(36.1) + 2(45.4) + 4(55.2) + 2(59.9) + 4(59.4) + 2(53.1) + 4(43.2) + 2(34.3) + 4(24.2) + 18.0] \approx 39.34$$

$$a_1 \approx \frac{12-0}{6(3)(12)} \left[18.0 \cos 0 + 4 \left(17.7 \cos \frac{\pi}{6} \right) + 2 \left(25.8 \cos \frac{\pi}{3} \right) + 4 \left(36.1 \cos \frac{\pi}{2} \right) + 2 \left(45.4 \cos \frac{2\pi}{3} \right) + 4 \left(55.2 \cos \frac{5\pi}{6} \right) + 2(59.9 \cos \pi) + 4 \left(59.4 \cos \frac{7\pi}{6} \right) + 2 \left(53.1 \cos \frac{4\pi}{3} \right) + 4 \left(43.2 \cos \frac{3\pi}{2} \right) + 2 \left(34.3 \cos \frac{5\pi}{3} \right) + 4 \left(24.2 \cos \frac{11\pi}{6} \right) + 18 \cos 2\pi \right] \approx -20.78$$

$$b_1 \approx \frac{12-0}{6(3)(12)} \left[18.0 \sin 0 + 4 \left(17.7 \sin \frac{\pi}{6} \right) + 2 \left(25.8 \sin \frac{\pi}{3} \right) + 4 \left(36.1 \sin \frac{\pi}{2} \right) + 2 \left(45.4 \sin \frac{2\pi}{3} \right) + 4 \left(55.2 \sin \frac{5\pi}{6} \right) + 2(59.9 \sin \pi) + 4 \left(59.4 \sin \frac{7\pi}{6} \right) + 2 \left(53.1 \sin \frac{4\pi}{3} \right) + 4 \left(43.2 \sin \frac{3\pi}{2} \right) + 2 \left(34.3 \sin \frac{5\pi}{3} \right) + 4 \left(24.2 \sin \frac{11\pi}{6} \right) + 18 \sin 2\pi \right] \approx -4.33$$

$$L(t) \approx 39.34 - 20.78 \cos \frac{\pi t}{6} - 4.33 \sin \frac{\pi t}{6}$$

(c) The difference between the maximum and minimum temperatures is greatest in the summer.



$$97. \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] dx \\ &= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n) \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x] dx \\ &= -\frac{1}{2} \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right]_{-\pi}^{\pi}, \quad (m \neq n) \\ &= -\frac{1}{2} \left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} \right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n} \right) \right] \\ &= 0, \text{ since } \cos(-\theta) = \cos\theta. \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin(mx) \cos(mx) dx = \frac{1}{m} \left[\frac{\sin^2(mx)}{2} \right]_{-\pi}^{\pi} = 0$$

Section 7.4 Trigonometric Substitution

$$\begin{aligned}
 1. \quad \frac{d}{dx} \left[4 \ln \left| \frac{\sqrt{x^2 + 16} - 4}{x} \right| + \sqrt{x^2 + 16} + C \right] &= \frac{d}{dx} \left[4 \ln |\sqrt{x^2 + 16} - 4| - 4 \ln |x| + \sqrt{x^2 + 16} + C \right] \\
 &= 4 \left[\frac{x/\sqrt{x^2 + 16}}{\sqrt{x^2 + 16} - 4} \right] - \frac{4}{x} + \frac{x}{\sqrt{x^2 + 16}} \\
 &= \frac{4x}{\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} - \frac{4}{x} + \frac{x}{\sqrt{x^2 + 16}} \\
 &= \frac{4x^2 - 4\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4) + x^2(\sqrt{x^2 + 16} - 4)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
 &= \frac{4x^2 - 4(x^2 + 16) + 16\sqrt{x^2 + 16} + x^2\sqrt{x^2 + 16} - 4x^2}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
 &= \frac{\sqrt{x^2 + 16}(x^2 + 16) - 4(x^2 + 16)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
 &= \frac{(x^2 + 16)(\sqrt{x^2 + 16} - 4)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} = \frac{\sqrt{x^2 + 16}}{x}
 \end{aligned}$$

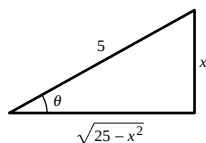
Indefinite integral: $\int \frac{\sqrt{x^2 + 16}}{x} dx$ Matches (b)

$$\begin{aligned}
 3. \quad \frac{d}{dx} \left[8 \arcsin \frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2} + C \right] &= 8 \frac{1/4}{\sqrt{1 - (x/4)^2}} - \frac{x(1/2)(16 - x^2)^{-1/2}(-2x) + \sqrt{16 - x^2}}{2} \\
 &= \frac{8}{\sqrt{16 - x^2}} + \frac{x^2}{2\sqrt{16 - x^2}} - \frac{\sqrt{16 - x^2}}{2} \\
 &= \frac{16}{2\sqrt{16 - x^2}} + \frac{x^2}{2\sqrt{16 - x^2}} - \frac{(16 - x^2)}{2\sqrt{16 - x^2}} = \frac{x^2}{\sqrt{16 - x^2}}
 \end{aligned}$$

Matches (a)

5. Let $x = 5 \sin \theta$, $dx = 5 \cos \theta d\theta$, $\sqrt{25 - x^2} = 5 \cos \theta$.

$$\begin{aligned}
 \int \frac{1}{(25 - x^2)^{3/2}} dx &= \int \frac{5 \cos \theta}{(5 \cos \theta)^3} d\theta \\
 &= \frac{1}{25} \int \sec^2 \theta d\theta \\
 &= \frac{1}{25} \tan \theta + C \\
 &= \frac{x}{25\sqrt{25 - x^2}} + C
 \end{aligned}$$

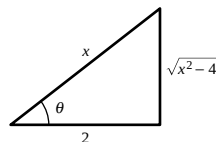


7. Same substitution as in Exercise 5

$$\begin{aligned}
 \int \frac{\sqrt{25 - x^2}}{x} dx &= \int \frac{25 \cos^2 \theta d\theta}{5 \sin \theta} = 5 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta = 5 \int (\csc \theta - \sin \theta) d\theta \\
 &= 5[\ln |\csc \theta - \cot \theta| + \cos \theta] + C = 5 \ln \left| \frac{5 - \sqrt{25 - x^2}}{x} \right| + \sqrt{25 - x^2} + C
 \end{aligned}$$

9. Let $x = 2 \sec \theta$, $dx = 2 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 4} = 2 \tan \theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - 4}} dx &= \int \frac{2 \sec \theta \tan \theta d\theta}{2 \tan \theta} = \int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C_1 \\ &= \ln \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + C_1 \\ &= \ln|x + \sqrt{x^2 - 4}| - \ln 2 + C_1 = \ln|x + \sqrt{x^2 - 4}| + C\end{aligned}$$



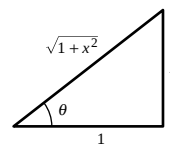
11. Same substitution as in Exercise 9

$$\begin{aligned}\int x^3 \sqrt{x^2 - 4} dx &= \int (8 \sec^3 \theta)(2 \tan \theta)(2 \sec \theta \tan \theta) d\theta = 32 \int \tan^2 \theta \sec^4 \theta d\theta \\ &= 32 \int \tan^2 \theta (1 + \tan^2 \theta) \sec^2 \theta d\theta = 32 \left(\frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} \right) + C \\ &= \frac{32}{15} \tan^3 \theta [5 + 3 \tan^2 \theta] + C = \frac{32}{15} \frac{(x^2 - 4)^{3/2}}{8} \left[5 + 3 \frac{(x^2 - 4)}{4} \right] + C \\ &= \frac{1}{15} (x^2 - 4)^{3/2} [20 + 3(x^2 - 4)] + C = \frac{1}{15} (x^2 - 4)^{3/2} (3x^2 + 8) + C\end{aligned}$$

13. Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $\sqrt{1 + x^2} = \sec \theta$.

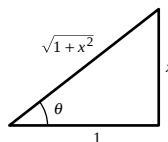
$$\int x \sqrt{1 + x^2} dx = \int \tan \theta (\sec \theta) \sec^2 \theta d\theta = \frac{\sec^3 \theta}{3} + C = \frac{1}{3} (1 + x^2)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.



15. Same substitution as in Exercise 13

$$\begin{aligned}\int \frac{1}{(1 + x^2)^2} dx &= \int \frac{1}{(\sec^2 \theta)^2} dx \\ &= \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\ &= \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] \\ &= \frac{1}{2} [\theta + \sin \theta \cos \theta] + C \\ &= \frac{1}{2} \left[\arctan x + \left(\frac{x}{\sqrt{1 + x^2}} \right) \left(\frac{1}{\sqrt{1 + x^2}} \right) \right] + C \\ &= \frac{1}{2} \left[\arctan x + \frac{x}{1 + x^2} \right] + C\end{aligned}$$



17. Let $u = 3x$, $a = 2$, and $du = 3 dx$.

$$\begin{aligned}\int \sqrt{4 + 9x^2} dx &= \frac{1}{3} \int \sqrt{(2)^2 + (3x)^2} 3 dx \\ &= \frac{1}{3} \left(\frac{1}{2} \right) (3x \sqrt{4 + 9x^2} + 4 \ln|3x + \sqrt{4 + 9x^2}|) + C \\ &= \frac{1}{2} x \sqrt{4 + 9x^2} + \frac{2}{3} \ln|3x + \sqrt{4 + 9x^2}| + C\end{aligned}$$

$$19. \int \frac{x}{\sqrt{x^2 + 9}} dx = \frac{1}{2} \int (x^2 + 9)^{-1/2} (2x) dx$$

$$= \sqrt{x^2 + 9} + C$$

(Power Rule)

$$21. \int \frac{1}{\sqrt{16 - x^2}} dx = \arcsin\left(\frac{x}{4}\right) + C$$

23. Let $x = 2 \sin \theta$, $dx = 2 \cos \theta d\theta$, $\sqrt{4 - x^2} = 2 \cos \theta$.

$$\int \sqrt{16 - 4x^2} dx = 2 \int \sqrt{4 - x^2} dx$$

$$= 2 \int 2 \cos \theta (2 \cos \theta d\theta)$$

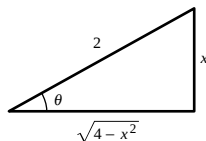
$$= 8 \int \cos^2 \theta d\theta$$

$$= 4 \int (1 + \cos 2\theta) d\theta$$

$$= 4 \left[\theta + \frac{1}{2} \sin 2\theta \right] + C$$

$$= 4\theta + 4 \sin \theta \cos \theta + C$$

$$= 4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4 - x^2} + C$$

25. Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$,

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

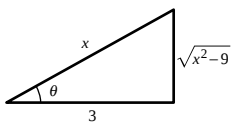
$$\int \frac{1}{\sqrt{x^2 - 9}} dx = \int \frac{3 \sec \theta \tan \theta d\theta}{3 \tan \theta}$$

$$= \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + C_1$$

$$= \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + C_1$$

$$= \ln |x + \sqrt{x^2 - 9}| + C$$

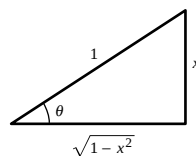
27. Let $x = \sin \theta$, $dx = \cos \theta d\theta$, $\sqrt{1 - x^2} = \cos \theta$.

$$\int \frac{\sqrt{1 - x^2}}{x^4} dx = \int \frac{\cos \theta (\cos \theta d\theta)}{\sin^4 \theta}$$

$$= \int \cot^2 \theta \csc^2 \theta d\theta$$

$$= -\frac{1}{3} \cot^3 \theta + C$$

$$= \frac{-(1 - x^2)^{3/2}}{3x^3} + C$$



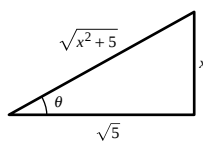
29. Same substitutions as in Exercise 28

$$\int \frac{1}{x \sqrt{4x^2 + 9}} dx = \int \frac{(3/2) \sec^2 \theta d\theta}{(3/2) \tan \theta 3 \sec \theta}$$

$$= \frac{1}{3} \int \csc \theta d\theta = -\frac{1}{3} \ln |\csc \theta + \cot \theta| + C = -\frac{1}{3} \ln \left| \frac{\sqrt{4x^2 + 9} + 3}{2x} \right| + C$$

31. Let $x = \sqrt{5} \tan \theta$, $dx = \sqrt{5} \sec^2 \theta d\theta$, $x^2 + 5 = 5 \sec^2 \theta$.

$$\begin{aligned} \int \frac{-5x}{(x^2 + 5)^{3/2}} dx &= \int \frac{-5\sqrt{5} \tan \theta}{(5 \sec^2 \theta)^{3/2}} \sqrt{5} \sec^2 \theta d\theta \\ &= -\sqrt{5} \int \frac{\tan \theta}{\sec \theta} d\theta \\ &= -\sqrt{5} \int \sin \theta d\theta \\ &= \sqrt{5} \cos \theta + C \\ &= \sqrt{5} \frac{\sqrt{5}}{\sqrt{x^2 + 5}} + C \\ &= \frac{5}{\sqrt{x^2 + 5}} + C \end{aligned}$$

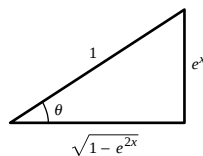


33. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int e^{2x} \sqrt{1 + e^{2x}} dx = \frac{1}{2} \int (1 + e^{2x})^{1/2} (2e^{2x}) dx = \frac{1}{3} (1 + e^{2x})^{3/2} + C$$

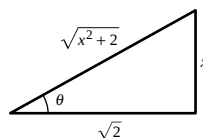
35. Let $e^x = \sin \theta$, $e^x dx = \cos \theta d\theta$, $\sqrt{1 - e^{2x}} = \cos \theta$.

$$\begin{aligned} \int e^x \sqrt{1 - e^{2x}} dx &= \int \cos^2 \theta d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] \\ &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C = \frac{1}{2} (\arcsin e^x + e^x \sqrt{1 - e^{2x}}) + C \end{aligned}$$



37. Let $x = \sqrt{2} \tan \theta$, $dx = \sqrt{2} \sec^2 \theta d\theta$, $x^2 + 2 = 2 \sec^2 \theta$.

$$\begin{aligned} \int \frac{1}{4 + 4x^2 + x^4} dx &= \int \frac{1}{(x^2 + 2)^2} dx \\ &= \int \frac{\sqrt{2} \sec^2 \theta d\theta}{4 \sec^4 \theta} \\ &= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\ &= \frac{\sqrt{2}}{4} \left(\frac{1}{2} \right) \int (1 + \cos 2\theta) d\theta \\ &= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{\sqrt{2}}{8} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{4} \left[\frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} \right] + C \end{aligned}$$



39. Since $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x, \Rightarrow du = \frac{1}{x\sqrt{4x^2 - 1}} dx, dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2 - 1}} dx$$

$$2x = \sec \theta, dx = \frac{1}{2} \sec \theta \tan \theta d\theta, \sqrt{4x^2 - 1} = \tan \theta$$

$$\begin{aligned} \int \operatorname{arcsec} 2x dx &= x \operatorname{arcsec} 2x - \int \frac{(1/2) \sec \theta \tan \theta d\theta}{\tan \theta} = x \operatorname{arcsec} 2x - \frac{1}{2} \int \sec \theta d\theta \\ &= x \operatorname{arcsec} 2x - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = x \operatorname{arcsec} 2x - \frac{1}{2} \ln |2x + \sqrt{4x^2 - 1}| + C. \end{aligned}$$

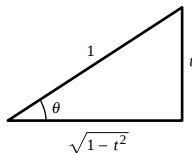
41. $\int \frac{1}{\sqrt{4x - x^2}} dx = \int \frac{1}{\sqrt{4 - (x - 2)^2}} dx = \arcsin\left(\frac{x - 2}{2}\right) + C$

43. Let $x + 2 = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{(x + 2)^2 + 4} = 2 \sec \theta$.

$$\begin{aligned} \int \frac{x}{\sqrt{x^2 + 4x + 8}} dx &= \int \frac{x}{\sqrt{(x + 2)^2 + 4}} dx = \int \frac{(2 \tan \theta - 2)(2 \sec^2 \theta) d\theta}{2 \sec \theta} \\ &= 2 \int (\tan \theta - 1)(\sec \theta) d\theta \\ &= 2 [\sec \theta - \ln |\sec \theta + \tan \theta|] + C_1 \\ &= 2 \left[\frac{\sqrt{(x + 2)^2 + 4}}{2} - \ln \left| \frac{\sqrt{(x + 2)^2 + 4}}{2} + \frac{x + 2}{2} \right| \right] + C_1 \\ &= \sqrt{x^2 + 4x + 8} - 2 \left[\ln \left| \sqrt{x^2 + 4x + 8} + (x + 2) \right| - \ln 2 \right] + C_1 \\ &= \sqrt{x^2 + 4x + 8} - 2 \ln \left| \sqrt{x^2 + 4x + 8} + (x + 2) \right| + C \end{aligned}$$

45. Let $t = \sin \theta$, $dt = \cos \theta d\theta$, $1 - t^2 = \cos^2 \theta$.

$$\begin{aligned} \text{(a)} \quad \int \frac{t^2}{(1 - t^2)^{3/2}} dt &= \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos^3 \theta} \\ &= \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta \\ &= \tan \theta - \theta + C \\ &= \frac{t}{\sqrt{1 - t^2}} - \arcsin t + C \end{aligned}$$



$$\text{Thus, } \int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}} dt = \left[\frac{t}{\sqrt{1 - t^2}} - \arcsin t \right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}} dt = \left[\tan \theta - \theta \right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

47. (a) Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 9}} dx &= \int \frac{(27 \tan^3 \theta)(3 \sec^2 \theta d\theta)}{3 \sec \theta} \\ &= 27 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta \\ &= 27 \left[\frac{1}{3} \sec^3 \theta - \sec \theta \right] + C = 9[\sec^3 \theta - 3 \sec \theta] + C \\ &= 9 \left[\left(\frac{\sqrt{x^2 + 9}}{3} \right)^3 - 3 \left(\frac{\sqrt{x^2 + 9}}{3} \right) \right] + C = \frac{1}{3}(x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} + C \end{aligned}$$

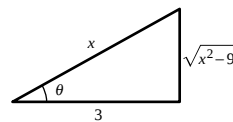
$$\begin{aligned} \text{Thus, } \int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx &= \left[\frac{1}{3}(x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} \right]_0^3 \\ &= \left(\frac{1}{3}(54\sqrt{2}) - 27\sqrt{2} \right) - (9 - 27) \\ &= 18 - 9\sqrt{2} = 9(2 - \sqrt{2}) \approx 5.272. \end{aligned}$$

- (b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. Thus,

$$\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx = 9 \left[\sec^3 \theta - 3 \sec \theta \right]_0^{\pi/4} = 9(2\sqrt{2} - 3\sqrt{2}) - 9(1 - 3) = 9(2 - \sqrt{2}) \approx 5.272.$$

49. (a) Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 9} = 3 \tan \theta$.

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 - 9}} dx &= \int \frac{9 \sec^2 \theta}{3 \tan \theta} 3 \sec \theta \tan \theta d\theta \\ &= 9 \int \sec^3 \theta d\theta \\ &= 9 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] \quad (7.3 \text{ Exercise 90}) \\ &= \frac{9}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|] \\ &= \frac{9}{2} \left[\frac{x}{3} \cdot \frac{\sqrt{x^2 - 9}}{3} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right] \end{aligned}$$



Hence,

$$\begin{aligned} \int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} dx &= \frac{9}{2} \left[\frac{x\sqrt{x^2 - 9}}{9} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right]_4^6 \\ &= \frac{9}{2} \left[\left(\frac{6\sqrt{27}}{9} + \ln \left| 2 + \frac{\sqrt{27}}{3} \right| \right) - \left(\frac{4\sqrt{7}}{9} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right) \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \left(\ln \left(\frac{6 + \sqrt{27}}{3} \right) - \ln \left(\frac{4 + \sqrt{7}}{3} \right) \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{(4 - \sqrt{7})(2 + \sqrt{3})}{3} \right) \approx 12.644. \end{aligned}$$

—CONTINUED—

49. —CONTINUED—

(b) When $x = 4$, $\theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$.

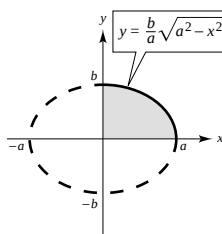
When $x = 6$, $\theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

$$\begin{aligned}\int_4^6 \frac{x^2}{\sqrt{x^2-9}} dx &= \frac{9}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{\operatorname{arcsec}(4/3)}^{\pi/3} \\ &= \frac{9}{2} \left[2 \cdot \sqrt{3} + \ln |2 + \sqrt{3}| \right] - \frac{9}{2} \left[\frac{4}{3} \cdot \frac{\sqrt{7}}{3} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \approx 12.644\end{aligned}$$

51. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}} dx = \frac{1}{2} \sqrt{x^2 + 10x + 9} (x - 15) + 33 \ln |(x + 5) + \sqrt{x^2 + 10x + 9}| + C$

53. $\int \frac{x^2}{\sqrt{x^2 - 1}} dx = \frac{1}{2} (x\sqrt{x^2 - 1} + \ln |x + \sqrt{x^2 - 1}|) + C$ 55. (a) $u = a \sin \theta$ (b) $u = a \tan \theta$ (c) $u = a \sec \theta$

57.
$$\begin{aligned}A &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\ &= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &= \left[\frac{4b}{a} \left(\frac{1}{2} \right) \left(a^2 \arcsin \frac{x}{a} + x\sqrt{a^2 - x^2} \right) \right]_0^a \\ &= \frac{2b}{a} \left(a^2 \left(\frac{\pi}{2} \right) \right) \\ &= \pi ab\end{aligned}$$



Note: See Theorem 7.2 for $\int \sqrt{a^2 - x^2} dx$.

59. $x^2 + y^2 = a^2$

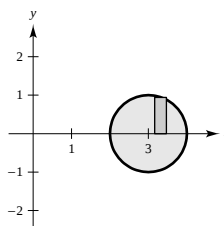
$$x = \pm \sqrt{a^2 - y^2}$$

$$\begin{aligned}A &= 2 \int_h^a \sqrt{a^2 - y^2} dy = \left[a^2 \arcsin \left(\frac{y}{a} \right) + y\sqrt{a^2 - y^2} \right]_h^a && \text{(Theorem 7.2)} \\ &= \left(a^2 \frac{\pi}{2} \right) - \left(a^2 \arcsin \left(\frac{h}{a} \right) + h\sqrt{a^2 - h^2} \right) \\ &= \frac{a^2 \pi}{2} - a^2 \arcsin \left(\frac{h}{a} \right) - h\sqrt{a^2 - h^2}\end{aligned}$$

61. Let $x - 3 = \sin \theta$, $dx = \cos \theta d\theta$, $\sqrt{1 - (x - 3)^2} = \cos \theta$.

Shell Method:

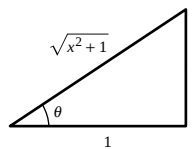
$$\begin{aligned}V &= 4\pi \int_2^4 x \sqrt{1 - (x - 3)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (3 + \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi \left[\frac{3}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + \int_{-\pi/2}^{\pi/2} \cos^2 \theta \sin \theta d\theta \right] \\ &= 4\pi \left[\frac{3}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) - \frac{1}{3} \cos^3 \theta \right]_{-\pi/2}^{\pi/2} = 6\pi^2\end{aligned}$$



63. $y = \ln x, y' = \frac{1}{x}, 1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

Let $x = \tan \theta, dx = \sec^2 \theta d\theta, \sqrt{x^2 + 1} = \sec \theta$.

$$\begin{aligned} s &= \int_1^5 \sqrt{\frac{x^2 + 1}{x^2}} dx = \int_1^5 \frac{\sqrt{x^2 + 1}}{x} dx \\ &= \int_a^b \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta = \int_a^b \frac{\sec \theta}{\tan \theta} (1 + \tan^2 \theta) d\theta \\ &= \int_a^b (\csc \theta + \sec \theta \tan \theta) d\theta \\ &= \left[-\ln|\csc \theta + \cot \theta| + \sec \theta \right]_a^b \\ &= \left[-\ln \left| \frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x} \right| + \sqrt{x^2 + 1} \right]_1^5 \\ &= \left[-\ln \left(\frac{\sqrt{26} + 1}{5} \right) + \sqrt{26} \right] - \left[-\ln(\sqrt{2} + 1) + \sqrt{2} \right] \\ &= \ln \left[\frac{5(\sqrt{2} + 1)}{\sqrt{26} + 1} \right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln \left[\frac{\sqrt{26} - 1}{5(\sqrt{2} - 1)} \right] + \sqrt{26} - \sqrt{2} \end{aligned}$$



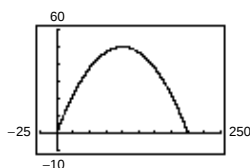
65. Length of one arch of sine curve: $y = \sin x, y' = \cos x$

$$L_1 = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

Length of one arch of cosine curve: $y = \cos x, y' = -\sin x$

$$\begin{aligned} L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} dx \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2 \left(x - \frac{\pi}{2} \right)} dx \quad u = x - \frac{\pi}{2}, du = dx \\ &= \int_{-\pi}^0 \sqrt{1 + \cos^2 u} du \\ &= \int_0^\pi \sqrt{1 + \cos^2 u} du = L_1 \end{aligned}$$

67. (a)



(b) $y = 0$ for $x = 200$ (range)

(c) $y = x - 0.005x^2, y' = 1 - 0.01x, 1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let $u = 1 - 0.01x, du = -0.01 dx, a = 1$. (See Theorem 7.2.)

$$\begin{aligned} s &= \int_0^{200} \sqrt{1 + (1 - 0.01x)^2} dx = -100 \int_0^{200} \sqrt{(1 - 0.01x)^2 + 1} (-0.01) dx \\ &= -50 \left[(1 - 0.01x) \sqrt{(1 - 0.01x)^2 + 1} + \ln \left| (1 - 0.01x) + \sqrt{(1 - 0.01x)^2 + 1} \right| \right]_0^{200} \\ &= -50 \left[(-\sqrt{2} + \ln|-1 + \sqrt{2}|) - (\sqrt{2} + \ln|1 + \sqrt{2}|) \right] \\ &= 100\sqrt{2} + 50 \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 229.559 \end{aligned}$$

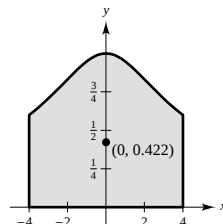
69. Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned} A &= 2 \int_0^4 \frac{3}{\sqrt{x^2 + 9}} dx = 6 \int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = 6 \int_a^b \frac{3 \sec^2 \theta d\theta}{3 \sec \theta} \\ &= 6 \int_a^b \sec \theta d\theta = \left[6 \ln |\sec \theta + \tan \theta| \right]_a^b = \left[6 \ln \left| \frac{\sqrt{x^2 + 9} + x}{3} \right| \right]_0^4 = 6 \ln 3 \end{aligned}$$

$\bar{x} = 0$ (by symmetry)

$$\begin{aligned} \bar{y} &= \frac{1}{2} \left(\frac{1}{A} \right) \int_{-4}^4 \left(\frac{3}{\sqrt{x^2 + 9}} \right)^2 dx \\ &= \frac{9}{12 \ln 3} \int_{-4}^4 \frac{1}{x^2 + 9} dx \\ &= \frac{3}{4 \ln 3} \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{-4}^4 \\ &= \frac{2}{4 \ln 3} \arctan \frac{4}{3} \approx 0.422 \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2 \ln 3} \arctan \frac{4}{3} \right) \approx (0, 0.422)$$



71. $y = x^2$, $y' = 2x$, $1 + (y')^2 = 1 + 4x^2$

$$2x = \tan \theta, dx = \frac{1}{2} \sec^2 \theta d\theta, \sqrt{1 + 4x^2} = \sec \theta$$

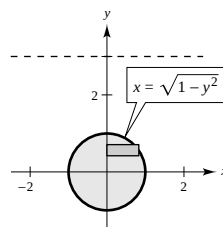
(For $\int \sec^5 \theta d\theta$ and $\int \sec^3 \theta d\theta$, see Exercise 80 in Section 7.3)

$$\begin{aligned} S &= 2\pi \int_0^{\sqrt{2}} x^2 \sqrt{1 + 4x^2} dx = 2\pi \int_a^b \left(\frac{\tan \theta}{2} \right)^2 (\sec \theta) \left(\frac{1}{2} \sec^2 \theta \right) d\theta \\ &= \frac{\pi}{4} \int_a^b \sec^3 \theta \tan^2 \theta d\theta = \frac{\pi}{4} \left[\int_a^b \sec^5 \theta d\theta - \int_a^b \sec^3 \theta d\theta \right] \\ &= \frac{\pi}{4} \left[\frac{1}{4} \left[\sec^3 \theta \tan \theta + \frac{3}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] \Big|_a^b \\ &= \frac{\pi}{4} \left[\frac{1}{4} [(1 + 4x^2)^{3/2} (2x)] - \frac{1}{8} [(1 + 4x^2)^{1/2} (2x) + \ln |\sqrt{1 + 4x^2} + 2x|] \right] \Big|_0^{\sqrt{2}} \\ &= \frac{\pi}{4} \left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{6} = \frac{1}{8} \ln(3 + 2\sqrt{2}) \right] \\ &= \frac{\pi}{4} \left(\frac{51\sqrt{2}}{4} - \frac{\ln(3 + 2\sqrt{2})}{8} \right) = \frac{\pi}{32} [102\sqrt{2} - \ln(3 + 2\sqrt{2})] \approx 13.989 \end{aligned}$$

73. (a) Area of representative rectangle: $2\sqrt{1 - y^2} \Delta y$

Pressure: $2(62.4)(3 - y)\sqrt{1 - y^2} \Delta y$

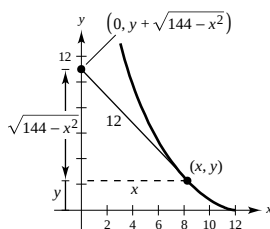
$$\begin{aligned} F &= 124.8 \int_{-1}^1 (3 - y)\sqrt{1 - y^2} dy \\ &= 124.8 \left[3 \int_{-1}^1 \sqrt{1 - y^2} dy - \int_{-1}^1 y\sqrt{1 - y^2} dy \right] \\ &= 124.8 \left[\frac{3}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{2} \left(\frac{2}{3} \right) (1 - y^2)^{3/2} \right]_{-1}^1 \\ &= (62.4)3[\arcsin 1 - \arcsin(-1)] = 187.2\pi \text{ lb} \end{aligned}$$



$$\begin{aligned} \text{(b) } F &= 124.8 \int_{-1}^1 (d - y)\sqrt{1 - y^2} dy = 124.8 \int_{-1}^1 \sqrt{1 - y^2} dy - 124.8 \int_{-1}^1 y\sqrt{1 - y^2} dy \\ &= 124.8 \left(\frac{d}{2} \right) \left[\arcsin y + y\sqrt{1 - y^2} \right]_{-1}^1 - 124.8(0) = 62.4\pi d \text{ lb} \end{aligned}$$

$$75. (a) m = \frac{dy}{dx} = \frac{y - (y + \sqrt{144 - x^2})}{x - 0}$$

$$= -\frac{\sqrt{144 - x^2}}{x}$$



$$(b) y = -\int \frac{\sqrt{144 - x^2}}{x} dx$$

Let $x = 12 \sin \theta$, $dx = 12 \cos \theta d\theta$, $\sqrt{144 - x^2} = 12 \cos \theta$.

$$y = -\int \frac{12 \cos \theta}{12 \sin \theta} 12 \cos \theta d\theta = -12 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta$$

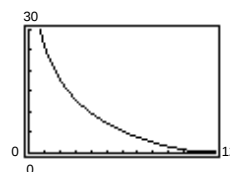
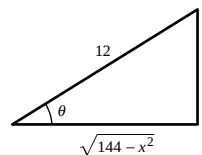
$$= -12 \int (\csc \theta - \sin \theta) d\theta = -12 \ln |\csc \theta - \cot \theta| - 12 \cos \theta + C$$

$$= -12 \ln \left| \frac{12}{x} - \frac{\sqrt{144 - x^2}}{x} \right| - 12 \left(\frac{\sqrt{144 - x^2}}{12} \right) + C$$

$$= -12 \ln \left| \frac{12 - \sqrt{144 - x^2}}{x} \right| - \sqrt{144 - x^2} + C$$

When $x = 12$, $y = 0 \Rightarrow C = 0$. Thus, $y = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$.

Note: $\frac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \leq 12$



(c) Vertical asymptote: $x = 0$

$$(d) y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$$

Thus,

$$12 - \sqrt{144 - x^2} = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$$

$$-1 = \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right)$$

$$xe^{-1} = 12 - \sqrt{144 - x^2}$$

$$(xe^{-1} - 12)^2 = (-\sqrt{144 - x^2})^2$$

$$x^2 e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$

$$x^2(e^{-2} + 1) - 24xe^{-1} = 0$$

$$x[x(e^{-2} + 1) - 24e^{-1}] = 0$$

$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore,

$$s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x} \right)^2} dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}} dx$$

$$= \int_{7.77665}^{12} \frac{12}{x} dx = \left[12 \ln |x| \right]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$

77. True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta$$

79. False

$$\int_0^{\sqrt{3}} \frac{dx}{(\sqrt{1+x^2})^3} = \int_0^{\pi/3} \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int_0^{\pi/3} \cos \theta d\theta$$

81. Let $u = a \sin \theta$, $du = a \cos \theta d\theta$, $\sqrt{a^2 - u^2} = a \cos \theta$.

$$\begin{aligned} \int \sqrt{a^2 - u^2} du &= \int a^2 \cos^2 \theta d\theta = a^2 \int \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{a^2}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{a^2}{2} \left[\arcsin \frac{u}{a} + \left(\frac{u}{a} \right) \left(\frac{\sqrt{a^2 - u^2}}{a} \right) \right] + C = \frac{1}{2} \left[a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right] + C \end{aligned}$$

Let $u = a \sec \theta$, $du = a \sec \theta \tan \theta d\theta$, $\sqrt{u^2 - a^2} = a \tan \theta$.

$$\begin{aligned} \int \sqrt{u^2 - a^2} du &= \int a \tan \theta (a \sec \theta \tan \theta) d\theta = a^2 \int \tan^2 \theta \sec \theta d\theta \\ &= a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta = a^2 \int (\sec^3 \theta - \sec \theta) d\theta \\ &= a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] - a^2 \int \sec \theta d\theta = a^2 \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] \\ &= \frac{a^2}{2} \left[\frac{u}{a} \cdot \frac{\sqrt{u^2 - a^2}}{a} - \ln \left| \frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a} \right| \right] + C_1 \\ &= \frac{1}{2} [u \sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 - a^2}|] + C \end{aligned}$$

Let $u = a \tan \theta$, $du = a \sec^2 \theta d\theta$, $\sqrt{u^2 + a^2} = a \sec \theta d\theta$.

$$\begin{aligned} \int \sqrt{u^2 + a^2} du &= \int (a \sec \theta)(a \sec^2 \theta) d\theta \\ &= a^2 \int \sec^3 \theta d\theta = a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C_1 \\ &= \frac{a^2}{2} \left[\frac{\sqrt{u^2 + a^2}}{a} \cdot \frac{u}{a} + \ln \left| \frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a} \right| \right] + C_1 = \frac{1}{2} [u \sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}|] + C \end{aligned}$$

Section 7.5 Partial Fractions

$$1. \frac{5}{x^2 - 10x} = \frac{5}{x(x - 10)} = \frac{A}{x} + \frac{B}{x - 10}$$

$$3. \frac{2x - 3}{x^3 + 10x} = \frac{2x - 3}{x(x^2 + 10)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 10}$$

$$5. \frac{16x}{x^3 - 10x^2} = \frac{16x}{x^2(x - 10)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 10}$$

$$7. \frac{1}{x^2 - 1} = \frac{1}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

$$1 = A(x - 1) + B(x + 1)$$

$$\text{When } x = -1, 1 = -2A, A = -\frac{1}{2}.$$

$$\text{When } x = 1, 1 = 2B, B = \frac{1}{2}.$$

$$\begin{aligned} \int \frac{1}{x^2 - 1} dx &= -\frac{1}{2} \int \frac{1}{x + 1} dx + \frac{1}{2} \int \frac{1}{x - 1} dx \\ &= -\frac{1}{2} \ln |x + 1| + \frac{1}{2} \ln |x - 1| + C \\ &= \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C \end{aligned}$$

$$9. \frac{3}{x^2 + x - 2} = \frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$3 = (x+2) + B(x-1)$$

When $x = 1$, $3 = 3A$, $A = 1$.

When $x = -2$, $3 = -3B$, $B = -1$.

$$\begin{aligned} \int \frac{3}{x^2 + x - 2} dx &= \int \frac{1}{x-1} dx - \int \frac{1}{x+2} dx \\ &= \ln|x-1| - \ln|x+2| + C \\ &= \ln\left|\frac{x-1}{x+2}\right| + C \end{aligned}$$

$$11. \frac{5-x}{2x^2+x-1} = \frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$

$$5-x = A(x+1) + B(2x-1)$$

When $x = \frac{1}{2}$, $\frac{9}{2} = \frac{3}{2}A$, $A = 3$.

When $x = -1$, $6 = -3B$, $B = -2$.

$$\begin{aligned} \int \frac{5-x}{2x^2+x-1} dx &= 3 \int \frac{1}{2x-1} dx - 2 \int \frac{1}{x+1} dx \\ &= \frac{3}{2} \ln|2x-1| - 2 \ln|x+1| + C \end{aligned}$$

$$13. \frac{x^2 + 12x + 12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

When $x = 0$, $12 = -4A$, $A = -3$. When $x = -2$, $-8 = 8B$, $B = -1$. When $x = 2$, $40 = 8C$, $C = 5$.

$$\begin{aligned} \int \frac{x^2 + 12x + 12}{x^3 - 4x} dx &= 5 \int \frac{1}{x-2} dx - \int \frac{1}{x+2} dx - 3 \int \frac{1}{x} dx \\ &= 5 \ln|x-2| - \ln|x+2| - 3 \ln|x| + C \end{aligned}$$

$$15. \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \frac{x+5}{(x-4)(x+2)} = 2x + \frac{A}{x-4} + \frac{B}{x+2}$$

$$x+5 = A(x+2) + B(x-4)$$

When $x = 4$, $9 = 6A$, $A = \frac{3}{2}$. When $x = -2$, $3 = -6B$, $B = -\frac{1}{2}$.

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} dx &= \int \left[2x + \frac{3/2}{x-4} - \frac{1/2}{x+2} \right] dx \\ &= x^2 + \frac{3}{2} \ln|x-4| - \frac{1}{2} \ln|x+2| + C \end{aligned}$$

$$17. \frac{4x^2 + 2x - 1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

When $x = 0$, $B = -1$. When $x = -1$, $C = 1$. When $x = 1$, $A = 3$.

$$\begin{aligned} \int \frac{4x^2 + 2x - 1}{x^3 + x^2} dx &= \int \left[\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1} \right] dx = 3 \ln|x| + \frac{1}{x} + \ln|x+1| + C \\ &= \frac{1}{x} + \ln|x^4 + x^3| + C \end{aligned}$$

$$19. \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} = \frac{x^2 + 3x - 4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$$

When $x = 0$, $-4 = -4A \Rightarrow A = -1$. When $x = 2$, $6 = 2C \Rightarrow C = 3$. When $x = 1$, $0 = -1 - B + 3 \Rightarrow B = 2$.

$$\begin{aligned} \int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} dx &= \int \frac{-1}{x} dx + \int \frac{2}{(x-2)} dx + \int \frac{3}{(x-2)^2} dx \\ &= -\ln|x| + 2 \ln|x-2| - \frac{3}{(x-2)} + C \end{aligned}$$

$$21. \frac{x^2 - 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$, $A = -1$. When $x = 1$, $0 = -2 + B + C$. When $x = -1$, $0 = -2 + B + C$. Solving these equations we have $A = -1$, $B = 2$, $C = 0$.

$$\begin{aligned} \int \frac{x^2 - 1}{x^3 + x} dx &= -\int \frac{1}{x} dx + \int \frac{2x}{x^2 + 1} dx \\ &= \ln|x^2 + 1| - \ln|x| + C \\ &= \ln\left|\frac{x^2 + 1}{x}\right| + C \end{aligned}$$

$$23. \frac{x^2}{x^4 - 2x^2 - 8} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 2}$$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2$, $4 = 24A$. When $x = -2$, $4 = -24B$. When $x = 0$, $0 = 4A - 4B - 4D$, and when $x = 1$, $1 = 9A - 3B - 3C - 3D$. Solving these equations we have $A = \frac{1}{6}$, $B = -\frac{1}{6}$, $C = 0$, $D = \frac{1}{3}$.

$$\begin{aligned} \int \frac{x^2}{x^4 - 2x^2 - 8} dx &= \frac{1}{6} \left[\int \frac{1}{x - 2} dx - \int \frac{1}{x + 2} dx + 2 \int \frac{1}{x^2 + 2} dx \right] \\ &= \frac{1}{6} \left[\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2} \arctan \frac{x}{\sqrt{2}} \right] + C \end{aligned}$$

$$25. \frac{x}{(2x - 1)(2x + 1)(4x^2 + 1)} = \frac{A}{2x - 1} + \frac{B}{2x + 1} + \frac{Cx + D}{4x^2 + 1}$$

$$x = A(2x + 1)(4x^2 + 1) + B(2x - 1)(4x^2 + 1) + (Cx + D)(2x - 1)(2x + 1)$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 4A$. When $x = -\frac{1}{2}$, $-\frac{1}{2} = -4B$. When $x = 0$, $0 = A - B - D$, and when $x = 1$, $1 = 15A + 5B + 3C + 3D$. Solving these equations we have $A = \frac{1}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = 0$.

$$\begin{aligned} \int \frac{x}{16x^4 - 1} dx &= \frac{1}{8} \left[\int \frac{1}{2x - 1} dx + \int \frac{1}{2x + 1} dx - 4 \int \frac{x}{4x^2 + 1} dx \right] \\ &= \frac{1}{16} \ln\left|\frac{4x^2 - 1}{4x^2 + 1}\right| + C \end{aligned}$$

$$27. \frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$\begin{aligned} x^2 + 5 &= A(x^2 - 2x + 3) + (Bx + C)(x + 1) \\ &= (A + B)x^2 + (-2A + B + C)x + (3A + C) \end{aligned}$$

When $x = -1$, $A = 1$. By equating coefficients of like terms, we have $A + B = 1$, $-2A + B + C = 0$, $3A + C = 5$. Solving these equations we have $A = 1$, $B = 0$, $C = 2$.

$$\begin{aligned} \int \frac{x^2 + 5}{x^3 - x^2 + x + 3} dx &= \int \frac{1}{x + 1} dx + 2 \int \frac{1}{(x - 1)^2 + 2} dx \\ &= \ln|x + 1| + \sqrt{2} \arctan\left(\frac{x - 1}{\sqrt{2}}\right) + C \end{aligned}$$

$$29. \frac{3}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

$$3 = A(x+2) + B(2x+1)$$

When $x = -\frac{1}{2}$, $A = 2$. When $x = -2$, $B = -1$.

$$\begin{aligned} \int_0^1 \frac{3}{2x^2 + 5x + 2} dx &= \int_0^1 \frac{2}{2x+1} dx - \int_0^1 \frac{1}{x+2} dx \\ &= \left[\ln|2x-1| - \ln|x+2| \right]_0^1 \\ &= \ln 2 \end{aligned}$$

$$31. \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

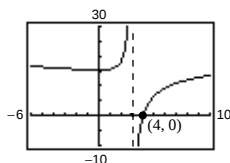
$$x+1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$. When $x = 1$, $2 = 2A + B + C$. When $x = -1$, $0 = 2A + B - C$. Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

$$\begin{aligned} \int_1^2 \frac{x+1}{x(x^2+1)} dx &= \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{x^2+1} dx + \int_1^2 \frac{1}{x^2+1} dx \\ &= \left[\ln|x| - \frac{1}{2} \ln(x^2+1) + \arctan x \right]_1^2 \\ &= \frac{1}{2} \ln \frac{8}{5} - \frac{\pi}{4} + \arctan 2 \\ &\approx 0.557 \end{aligned}$$

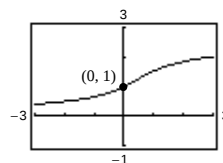
$$33. \int \frac{3x dx}{x^2 - 6x + 9} = 3 \ln|x-3| - \frac{9}{x-3} + C$$

$$(4, 0): 3 \ln|4-3| - \frac{9}{4-3} + C = 0 \Rightarrow C = 9$$



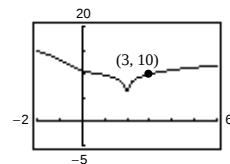
$$35. \int \frac{x^2 + x + 2}{(x^2 + 2)^2} dx = \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} - \frac{1}{2(x^2 + 2)} + C$$

$$(0, 1): 0 - \frac{1}{4} + C = 1 \Rightarrow C = \frac{5}{4}$$



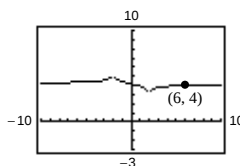
$$37. \int \frac{2x^2 - 2x + 3}{x^3 - x^2 - x - 2} dx = \ln|x-2| + \frac{1}{2} \ln|x^2 + x + 1| - \sqrt{3} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$(3, 10): 0 + \frac{1}{2} \ln 13 - \sqrt{3} \arctan \frac{7}{\sqrt{3}} + C = 10 \Rightarrow C = 10 - \frac{1}{2} \ln 13 + \sqrt{3} \arctan \frac{7}{\sqrt{3}}$$



$$39. \int \frac{1}{x^2 - 4} dx = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

$$(6, 4): \frac{1}{4} \ln \left| \frac{4}{8} \right| + C = 4 \Rightarrow C = 4 - \frac{1}{4} \ln \frac{1}{2} = 4 + \frac{1}{4} \ln 2$$



41. Let
- $u = \cos x$
- ,
- $du = -\sin x \, dx$
- .

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $A = -1$. When $u = 1$, $B = 1$, $u = \cos x$,
 $du = -\sin x \, dx$.

$$\begin{aligned} \int \frac{\sin x}{\cos x(\cos x - 1)} \, dx &= - \int \frac{1}{u(u-1)} \, du \\ &= \int \frac{1}{u} \, du - \int \frac{1}{u-1} \, du \\ &= \ln|u| - \ln|u-1| + C \\ &= \ln \left| \frac{u}{u-1} \right| + C \\ &= \ln \left| \frac{\cos x}{\cos x - 1} \right| + C \end{aligned}$$

45. Let
- $u = e^x$
- ,
- $du = e^x \, dx$
- .

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$. When $u = -4$, $B = -\frac{1}{5}$, $u = e^x$,
 $du = e^x \, dx$.

$$\begin{aligned} \int \frac{e^x}{(e^x-1)(e^x+4)} \, dx &= \int \frac{1}{(u-1)(u+4)} \, du \\ &= \frac{1}{5} \left(\int \frac{1}{u-1} \, du - \int \frac{1}{u+4} \, du \right) \\ &= \frac{1}{5} \ln \left| \frac{u-1}{u+4} \right| + C \\ &= \frac{1}{5} \ln \left| \frac{e^x-1}{e^x+4} \right| + C \end{aligned}$$

- 49.
- $\frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2}$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.

When $x = 0$, $0 = aA + B \Rightarrow A = 1/b$.

$$\begin{aligned} \int \frac{x}{(a+bx)^2} \, dx &= \int \left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2} \right) \, dx \\ &= \frac{1}{b} \int \frac{1}{a+bx} \, dx - \frac{a}{b} \int \frac{1}{(a+bx)^2} \, dx \\ &= \frac{1}{b^2} \ln|a+bx| + \frac{a}{b^2} \left(\frac{1}{a+bx} \right) + C \\ &= \frac{1}{b^2} \left(\frac{a}{a+bx} + \ln|a+bx| \right) + C \end{aligned}$$

- 43.
- $\int \frac{3 \cos x}{\sin^2 x + \sin x - 2} \, dx = 3 \int \frac{1}{u^2 + u - 2} \, du$

$$= \ln \left| \frac{u-1}{u+2} \right| + C$$

$$= \ln \left| \frac{-1 + \sin x}{2 + \sin x} \right| + C$$

(From Exercise 9 with $u = \sin x$, $du = \cos x \, dx$)

- 47.
- $\frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$

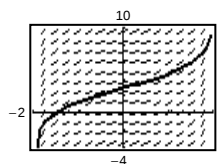
$$1 = A(a+bx) + Bx$$

When $x = 0$, $1 = aA \Rightarrow A = 1/a$.

When $x = -a/b$, $1 = -(a/b)B \Rightarrow B = -b/a$.

$$\begin{aligned} \int \frac{1}{x(a+bx)} \, dx &= \frac{1}{a} \int \left(\frac{1}{x} - \frac{b}{a+bx} \right) \, dx \\ &= \frac{1}{a} (\ln|x| - \ln|a+bx|) + C \\ &= \frac{1}{a} \ln \left| \frac{x}{a+bx} \right| + C \end{aligned}$$

- 51.
- $\frac{dy}{dx} = \frac{6}{4-x^2}$
- ,
- $y(0) = 3$



53. Dividing x^3 by $x - 5$.

55. (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

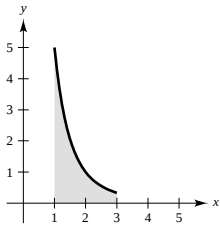
(c) Trigonometric substitution (tan) or inverse tangent rule

$$\begin{aligned}
 57. \text{ Average Cost} &= \frac{1}{80 - 75} \int_{75}^{80} \frac{124p}{(10 + p)(100 - p)} dp \\
 &= \frac{1}{5} \int_{75}^{80} \left(\frac{-124}{(10 + p)11} + \frac{1240}{(100 - p)11} \right) dp \\
 &= \frac{1}{5} \left[\frac{-124}{11} \ln(10 + p) - \frac{1240}{11} \ln(100 - p) \right]_{75}^{80} \\
 &\approx \frac{1}{5} (24.51) = 4.9
 \end{aligned}$$

Approximately \$490,000.

$$59. A = \int_1^3 \frac{10}{x(x^2 + 1)} dx \approx 3$$

Matches (c)



$$61. \quad \frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}, A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1} \int \left(\frac{1}{x+1} + \frac{1}{n-x} \right) dx = kt + C$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + C$$

$$\text{When } t = 0, x = 0, C = \frac{1}{n+1} \ln \frac{1}{n}.$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + \frac{1}{n+1} \ln \frac{1}{n}$$

$$\frac{1}{n+1} \left[\ln \left| \frac{x+1}{n-x} \right| - \ln \frac{1}{n} \right] = kt$$

$$\ln \frac{nx + n}{n - x} = (n+1)kt$$

$$\frac{nx + n}{n - x} = e^{(n+1)kt}$$

$$x = \frac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}}$$

$$\text{Note: } \lim_{t \rightarrow \infty} x = n$$

$$\begin{aligned}
 63. \quad \frac{x}{1+x^4} &= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \\
 x &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) \\
 &= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D)
 \end{aligned}$$

$$0 = A + C \Rightarrow C = -A$$

$$0 = B + D - \sqrt{2}A + \sqrt{2}C \quad \left\{ \begin{array}{l} -2\sqrt{2}A = 0 \Rightarrow A = 0 \text{ and } C = 0 \end{array} \right.$$

$$1 = A + C - \sqrt{2}B + \sqrt{2}D \quad \left\{ \begin{array}{l} -2\sqrt{2}B = 1 \Rightarrow B = -\frac{\sqrt{2}}{4} \text{ and } D = \frac{\sqrt{2}}{4} \end{array} \right.$$

$$0 = B + D \Rightarrow D = -B$$

Thus,

$$\begin{aligned}
 \int_0^1 \frac{x}{1+x^4} dx &= \int_0^1 \left[\frac{-\sqrt{2}/4}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}x+1} \right] dx \\
 &= \frac{\sqrt{2}}{4} \int_0^1 \left[\frac{-1}{\left[x + (\sqrt{2}/2)\right]^2 + (1/2)} + \frac{1}{\left[x - (\sqrt{2}/2)\right]^2 + (1/2)} \right] dx \\
 &= \frac{\sqrt{2}}{4} \cdot \frac{1}{1/\sqrt{2}} \left[-\arctan\left(\frac{x + (\sqrt{2}/2)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x - (\sqrt{2}/2)}{1/\sqrt{2}}\right) \right]_0^1 \\
 &= \frac{1}{2} \left[-\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1) \right]_0^1 \\
 &= \frac{1}{2} \left[(-\arctan(\sqrt{2}+1) + \arctan(\sqrt{2}-1)) - (-\arctan 1 + \arctan(-1)) \right] \\
 &= \frac{1}{2} \left[\arctan(\sqrt{2}-1) - \arctan(\sqrt{2}+1) + \frac{\pi}{4} + \frac{\pi}{4} \right].
 \end{aligned}$$

Since $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$, we have:

$$\int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \left[\arctan\left(\frac{(\sqrt{2}-1) - (\sqrt{2}+1)}{1 + (\sqrt{2}-1)(\sqrt{2}+1)}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[-\frac{\pi}{4} + \frac{\pi}{2} \right] = \frac{\pi}{8}$$

Section 7.6 Integration by Tables and Other Integration Techniques

$$1. \text{ By Formula 6: } \int \frac{x^2}{1+x} dx = -\frac{x}{2}(2-x) + \ln|1+x| + C$$

$$\begin{aligned}
 3. \text{ By Formula 26: } \int e^x \sqrt{1+e^{2x}} dx &= \frac{1}{2} [e^x \sqrt{e^{2x}+1} + \ln|e^x + \sqrt{e^{2x}+1}|] + C \\
 u &= e^x, du = e^x dx
 \end{aligned}$$

$$5. \text{ By Formula 44: } \int \frac{1}{x^2 \sqrt{1-x^2}} dx = -\frac{\sqrt{1-x^2}}{x} + C$$

$$\begin{aligned}
7. \text{ By Formulas 50 and 48: } \int \sin^4(2x) dx &= \frac{1}{2} \int \sin^4(2x)(2) dx \\
&= \frac{1}{2} \left[\frac{-\sin^3(2x) \cos(2x)}{4} + \frac{3}{4} \int \sin^2(2x)(2) dx \right] \\
&= \frac{1}{2} \left[\frac{-\sin^3(2x) \cos(2x)}{4} + \frac{3}{8} (2x - \sin 2x \cos 2x) \right] + C \\
&= \frac{1}{16} (6x - 3 \sin 2x \cos 2x - 2 \sin^3 2x \cos 2x) + C
\end{aligned}$$

$$\begin{aligned}
9. \text{ By Formula 57: } \int \frac{1}{\sqrt{x}(1 - \cos \sqrt{x})} dx &= 2 \int \frac{1}{1 - \cos \sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx \\
&= -2(\cot \sqrt{x} + \csc \sqrt{x}) + C \\
u &= \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx
\end{aligned}$$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}} dx = x - \frac{1}{2} \ln(1 + e^{2x}) + C$$

13. By Formula 89:

$$\int x^3 \ln x dx = \frac{x^4}{16} (4 \ln|x| - 1) + C$$

$$\begin{aligned}
15. (a) \text{ By Formulas 83 and 82: } \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\
&= x^2 e^x - 2[(x - 1)e^x + C_1] \\
&= x^2 e^x - 2x e^x + 2e^x + C
\end{aligned}$$

(b) Integration by parts: $u = x^2$, $du = 2x dx$, $dv = e^x dx$, $v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

Parts again: $u = 2x$, $du = 2 dx$, $dv = e^x dx$, $v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \left[2x e^x - \int 2e^x dx \right] = x^2 e^x - 2x e^x + 2e^x + C$$

17. (a) By Formula: 12, $a = b = 1$, $u = x$, and

$$\begin{aligned}
\int \frac{1}{x^2(x+1)} dx &= \frac{-1}{1} \left(\frac{1}{x} + \frac{1}{1} \ln \left| \frac{x}{1+x} \right| \right) + C \\
&= \frac{-1}{x} - \ln \left| \frac{x}{1+x} \right| + C \\
&= \frac{-1}{x} + \ln \left| \frac{x+1}{x} \right| + C
\end{aligned}$$

(b) Partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$$x = 0: 1 = B$$

$$x = -1: 1 = C$$

$$x = 1: 1 = 2A + 2 + 1 \Rightarrow A = -1$$

$$\begin{aligned}
\int \frac{1}{x^2(x+1)} dx &= \int \left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right] dx \\
&= -\ln|x| - \frac{1}{x} + \ln|x+1| + C \\
&= -\frac{1}{x} - \ln \left| \frac{x}{x+1} \right| + C
\end{aligned}$$

19. By Formula 81: $\int x e^{x^2} = \frac{1}{2} e^{x^2} + C$

21. By Formula 79: $\int x \operatorname{arcsec}(x^2 + 1) dx = \frac{1}{2} \int \operatorname{arcsec}(x^2 + 1)(2x) dx$
 $= \frac{1}{2} [(x^2 + 1) \operatorname{arcsec}(x^2 + 1) - \ln((x^2 + 1) + \sqrt{x^4 + 2x^2})] + C$
 $u = x^2 + 1, du = 2x dx$

23. By Formula 89: $\int x^2 \ln x dx = \frac{x^3}{9}(-1 + 3 \ln|x|) + C$

25. By Formula 35: $\int \frac{1}{x^2 \sqrt{x^2 - 4}} dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

27. By Formula 4: $\int \frac{2x}{(1 - 3x)^2} dx = 2 \int \frac{x}{(1 - 3x)^2} dx = \frac{2}{9} \left(\ln|1 - 3x| + \frac{1}{1 - 3x} \right) + C$

29. By Formula 76:

$$\int e^x \arccos e^x dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

$u = e^x, du = e^x dx$

31. By Formula 73:

$$\int \frac{x}{1 - \sec x^2} dx = \frac{1}{2} \int \frac{2x}{1 - \sec x^2} dx$$

$$= \frac{1}{2} (x^2 + \cot x^2 + \csc x^2) + C$$

33. By Formula 23: $\int \frac{\cos x}{1 + \sin^2 x} dx = \arctan(\sin x) + C$
 $u = \sin x, du = \cos x dx$

35. By Formula 14: $\int \frac{\cos \theta}{3 + 2 \sin \theta + \sin^2 \theta} d\theta = \frac{\sqrt{2}}{2} \arctan\left(\frac{1 + \sin \theta}{\sqrt{2}}\right) + C$
 $u = \sin \theta, du = \cos \theta d\theta$

37. By Formula 35: $\int \frac{1}{x^2 \sqrt{2 + 9x^2}} dx = 3 \int \frac{3}{(3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2}} dx$
 $= -\frac{3\sqrt{2 + 9x^2}}{6x} + C$
 $= -\frac{\sqrt{2 + 9x^2}}{2x} + C$

39. By Formulas 54 and 55:

$$\int t^3 \cos t dt = t^3 \sin t - 3 \int t^2 \sin t dt$$

$$= t^3 \sin t - 3 \left[-t^2 \cos t + 2 \int t \cos t dt \right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6 \left[t \sin t - \int \sin t dt \right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t + C$$

41. By Formula 3: $\int \frac{\ln x}{x(3 + 2 \ln x)} dx = \frac{1}{4}(2 \ln|x| - 3 \ln|3 + 2 \ln|x||) + C$

$$u = \ln x, du = \frac{1}{x} dx$$

43. By Formulas 1, 25, and 33: $\int \frac{x}{(x^2 - 6x + 10)^2} dx = \frac{1}{2} \int \frac{2x - 6 + 6}{(x^2 - 6x + 10)^2} dx$

$$= \frac{1}{2} \int (x^2 - 6x + 10)^{-2} (2x - 6) dx + 3 \int \frac{1}{[(x - 3)^2 + 1]^2} dx$$

$$= -\frac{1}{2(x^2 - 6x + 10)} + \frac{3}{2} \left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3) \right] + C$$

$$= \frac{3x - 10}{2(x^2 - 6x + 10)} + \frac{3}{2} \arctan(x - 3) + C$$

45. By Formula 31: $\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{(x^2 - 3)^2 - 4}} dx$

$$= \frac{1}{2} \ln|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}| + C$$

$$u = x^2 - 3, du = 2x dx$$

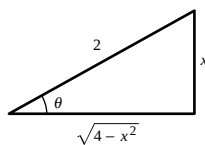
47. $\int \frac{x^3}{\sqrt{4 - x^2}} dx = \int \frac{8 \sin^3 \theta (2 \cos \theta d\theta)}{2 \cos \theta}$

$$= 8 \int (1 - \cos^2 \theta) \sin \theta d\theta$$

$$= 8 \int [\sin \theta - \cos^2 \theta (\sin \theta)] d\theta$$

$$= -8 \cos \theta + \frac{8 \cos^3 \theta}{3} + C$$

$$= \frac{-\sqrt{4 - x^2}}{3} (x^2 + 8) + C$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4 - x^2} = 2 \cos \theta$$

49. By Formula 8: $\int \frac{e^{3x}}{(1 + e^x)^3} dx = \int \frac{(e^x)^2}{(1 + e^x)^3} (e^x) dx$

$$= \frac{2}{1 + e^x} - \frac{1}{2(1 + e^x)^2} + \ln|1 + e^x| + C$$

$$u = e^x, du = e^x dx$$

51. $\frac{u^2}{(a + bu)^2} = \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a + bu)^2} = \frac{1}{b^2} + \frac{A}{a + bu} + \frac{B}{(a + bu)^2}$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a + bu) + B = (aA + B) + bAu$$

Equating the coefficients of like terms we have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations we have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\int \frac{u^2}{(a + bu)^2} du = \frac{1}{b^2} \int du - \frac{2a}{b^2} \left(\frac{1}{b} \right) \int \frac{1}{a + bu} b du + \frac{a^2}{b^2} \left(\frac{1}{b} \right) \int \frac{1}{(a + bu)^2} b du = \frac{1}{b^2} u - \frac{2a}{b^3} \ln|a + bu| - \frac{a^2}{b^3} \left(\frac{1}{a + bu} \right) + C$$

$$= \frac{1}{b^3} \left(bu - \frac{a^2}{a + bu} - 2a \ln|a + bu| \right) + C$$

53. When we have $u^2 + a^2$:

$$u = a \tan \theta$$

$$du = a \sec^2 \theta d\theta$$

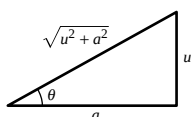
$$u^2 + a^2 = a^2 \sec^2 \theta$$

$$\int \frac{1}{(u^2 + a^2)^{3/2}} du = \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta}$$

$$= \frac{1}{a^2} \int \cos \theta d\theta$$

$$= \frac{1}{a^2} \sin \theta + C$$

$$= \frac{u}{a^2 \sqrt{u^2 + a^2}} + C$$


 When we have $u^2 - a^2$:

$$u = a \sec \theta$$

$$du = a \sec \theta \tan \theta d\theta$$

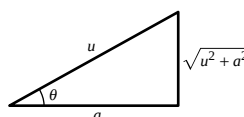
$$u^2 - a^2 = a^2 \tan^2 \theta$$

$$\int \frac{1}{(u^2 - a^2)^{3/2}} du = \int \frac{a \sec \theta \tan \theta d\theta}{a^3 \tan^3 \theta}$$

$$= \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$$= -\frac{1}{a^2} \csc \theta + C$$

$$= \frac{-u}{a^2 \sqrt{u^2 - a^2}} + C$$

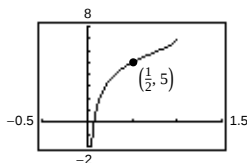


$$\begin{aligned} 55. \int (\arctan u) du &= u \arctan u - \frac{1}{2} \int \frac{2u}{1+u^2} du \\ &= u \arctan u - \frac{1}{2} \ln(1+u^2) + C \end{aligned}$$

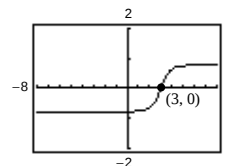
$$= u \arctan u - \ln \sqrt{1+u^2} + C$$

$$w = \arctan u, dv = du, dw = \frac{du}{1+u^2}, v = u$$

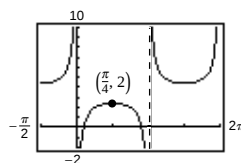
$$\begin{aligned} 57. \int \frac{1}{x^{3/2} \sqrt{1-x}} dx &= \frac{-2\sqrt{1-x}}{\sqrt{x}} + C \\ \left(\frac{1}{2}, 5\right): \frac{-2\sqrt{1/2}}{\sqrt{1/2}} + C &= 5 \Rightarrow C = 7 \\ y &= \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7 \end{aligned}$$



$$\begin{aligned} 59. \int \frac{1}{(x^2 - 6x + 10)^2} dx &= \frac{1}{2} \left[\tan^{-1}(x-3) + \frac{x-3}{x^2 - 6x + 10} \right] + C \\ (3, 0): \frac{1}{2} \left[0 + \frac{0}{10} \right] + C &= 0 \Rightarrow C = 0 \\ y &= \frac{1}{2} \left[\tan^{-1}(x-3) + \frac{x-3}{x^2 - 6x + 10} \right] \end{aligned}$$



$$\begin{aligned} 61. \int \frac{1}{\sin \theta \tan \theta} d\theta &= -\csc \theta + C \\ \left(\frac{\pi}{4}, 2\right): -\frac{2}{\sqrt{2}} + C &= 2 \Rightarrow C = 2 + \sqrt{2} \\ y &= -\csc \theta + 2 + \sqrt{2} \end{aligned}$$



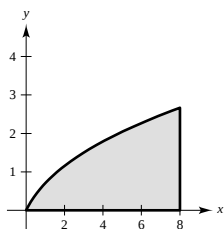
$$\begin{aligned}
 63. \int \frac{1}{2 - 3 \sin \theta} d\theta &= \int \left[\frac{\frac{2 du}{1 + u^2}}{2 - 3 \left(\frac{2u}{1 + u^2} \right)} \right] \\
 &= \int \frac{2}{(1 + u^2) - 6u} du \\
 &= \int \frac{1}{u^2 - 3u + 1} du \\
 &= \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}} du \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}} \right| + C \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}} \right| + C \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{2 \tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2 \tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}} \right| + C
 \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned}
 67. \int \frac{\sin \theta}{3 - 2 \cos \theta} d\theta &= \frac{1}{2} \int \frac{2 \sin \theta}{3 - 2 \cos \theta} d\theta \\
 &= \frac{1}{2} \ln |u| + C \\
 &= \frac{1}{2} \ln(3 - 2 \cos \theta) + C
 \end{aligned}$$

$$u = 3 - 2 \cos \theta, du = 2 \sin \theta d\theta$$

$$\begin{aligned}
 71. A &= \int_0^8 \frac{x}{\sqrt{x+1}} dx \\
 &= \left[\frac{-2(2-x)}{3} \sqrt{x+1} \right]_0^8 \\
 &= 12 - \left(-\frac{4}{3}\right) \\
 &= \frac{40}{3} \approx 13.333 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 65. \int_0^{\pi/2} \frac{1}{1 + \sin \theta + \cos \theta} d\theta &= \int_0^1 \left[\frac{\frac{2 du}{1 + u^2}}{1 + \frac{2u}{1 + u^2} + \frac{1 - u^2}{1 + u^2}} \right] \\
 &= \int_0^1 \frac{1}{1 + u} du \\
 &= \left[\ln |1 + u| \right]_0^1 \\
 &= \ln 2
 \end{aligned}$$

$$u = \tan \frac{\theta}{2}$$

$$\begin{aligned}
 69. \int \frac{\cos \sqrt{\theta}}{\sqrt{\theta}} d\theta &= 2 \int \cos \sqrt{\theta} \left(\frac{1}{2\sqrt{\theta}} \right) d\theta \\
 &= 2 \sin \sqrt{\theta} + C \\
 u &= \sqrt{\theta}, du = \frac{1}{2\sqrt{\theta}} d\theta
 \end{aligned}$$

73. Arctangent Formula, Formula 23,

$$\int \frac{1}{u^2 + 1} du, u = e^x$$

75. Substitution: $u = x^2, du = 2x dx$
Then Formula 81.

77. Cannot be integrated.

79. Answers will vary. For example,

$$\int (2x)e^{2x} dx$$

can be integrated by first letting
 $u = 2x$ and then using Formula 82.

$$\begin{aligned}
 81. \quad W &= \int_0^5 2000xe^{-x} dx \\
 &= -2000 \int_0^5 -xe^{-x} dx \\
 &= 2000 \int_0^5 (-x)e^{-x}(-1) dx \\
 &= 2000 \left[(-x)e^{-x} - e^{-x} \right]_0^5 \\
 &= 2000 \left(-\frac{6}{e^5} + 1 \right) \\
 &\approx 1919.145 \text{ ft} \cdot \text{lbs}
 \end{aligned}$$

$$\begin{aligned}
 83. \quad (a) \quad V &= 20(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy & W &= 148(80 \ln(3 + \sqrt{10})) \\
 &= \left[80 \ln|y + \sqrt{1+y^2}| \right]_0^3 & &= 11,840 \ln(3 + \sqrt{10}) \\
 &= 80 \ln(3 + \sqrt{10}) & &\approx 21,530.4 \text{ lb} \\
 &\approx 145.5 \text{ cubic feet}
 \end{aligned}$$

(b) By symmetry, $\bar{x} = 0$.

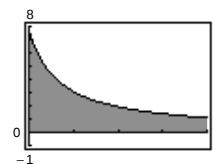
$$\begin{aligned}
 M &= \rho(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy = \left[4\rho \ln|y + \sqrt{1+y^2}| \right]_0^3 = 4\rho \ln(3 + \sqrt{10}) \\
 M_x &= 2\rho \int_0^3 \frac{2y}{\sqrt{1+y^2}} dy = \left[4\rho \sqrt{1+y^2} \right]_0^3 = 4\rho(\sqrt{10} - 1) \\
 \bar{y} &= \frac{M_x}{M} = \frac{4\rho(\sqrt{10} - 1)}{4\rho \ln(3 + \sqrt{10})} \approx 1.19
 \end{aligned}$$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

$$85. \quad (a) \quad \int_0^4 \frac{k}{2+3x} dx = 10$$

$$\begin{aligned}
 k &= \frac{10}{\int_0^4 \frac{1}{2+3x} dx} \approx \frac{10}{0.6486} \\
 &= 15.417 \quad \left(= \frac{30}{\ln 7} \right)
 \end{aligned}$$

$$(b) \quad \int_0^4 \frac{15.417}{2+3x} dx$$

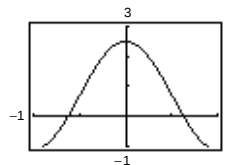


87. False. You might need to convert your integral using substitution or algebra.

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

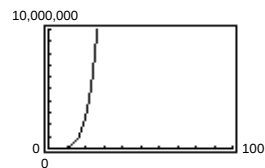
$$1. \quad \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x} \approx 2.5 \left(\text{exact: } \frac{5}{2} \right)$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.4132	2.4991	2.500	2.500	2.4991	2.4132



3. $\lim_{x \rightarrow \infty} x^5 e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9901	90,484	3.7×10^9	4.5×10^{10}	0	0



5. (a) $\lim_{x \rightarrow 3} \frac{2(x-3)}{x^2-9} = \lim_{x \rightarrow 3} \frac{2(x-3)}{(x+3)(x-3)} = \lim_{x \rightarrow 3} \frac{2}{x+3} = \frac{1}{3}$

(b) $\lim_{x \rightarrow 3} \frac{2(x-3)}{x^2-9} = \lim_{x \rightarrow 3} \frac{(d/dx)[2(x-3)]}{(d/dx)[x^2-9]} = \lim_{x \rightarrow 3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$

7. (a) $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x \rightarrow 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$

(b) $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{(d/dx)[\sqrt{x+1}-2]}{(d/dx)[x-3]} = \lim_{x \rightarrow 3} \frac{1/(2\sqrt{x+1})}{1} = \frac{1}{4}$

9. (a) $\lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \rightarrow \infty} \frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$

(b) $\lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \rightarrow \infty} \frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x \rightarrow \infty} \frac{10x-3}{6x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x \rightarrow \infty} \frac{10}{6} = \frac{5}{3}$

11. $\lim_{x \rightarrow 2} \frac{x^2-x-2}{x-2} = \lim_{x \rightarrow 2} \frac{2x-1}{1} = 3$

13. $\lim_{x \rightarrow 0} \frac{\sqrt{4-x^2}-2}{x} = \lim_{x \rightarrow 0} \frac{-x/\sqrt{4-x^2}}{1} = 0$

15. $\lim_{x \rightarrow 0} \frac{e^x - (1-x)}{x} = \lim_{x \rightarrow 0} \frac{e^x + 1}{1} = 2$

17. Case 1: $n = 1$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{1} = 0$$

Case 2: $n = 2$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x^2} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0^+} \frac{e^x}{2} = \frac{1}{2}$$

Case 3: $n \geq 3$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x^n} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{nx^{n-1}} = \lim_{x \rightarrow 0^+} \frac{e^x}{n(n-1)x^{n-2}} = \infty$$

19. $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{2 \cos 2x}{3 \cos 3x} = \frac{2}{3}$

21. $\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{1/\sqrt{1-x^2}}{1} = 1$

23. $\lim_{x \rightarrow \infty} \frac{3x^2-2x+1}{2x^2+3} = \lim_{x \rightarrow \infty} \frac{6x-2}{4x}$

$$= \lim_{x \rightarrow \infty} \frac{6}{4} = \frac{3}{2}$$

25. $\lim_{x \rightarrow \infty} \frac{x^2+2x+3}{x-1} = \lim_{x \rightarrow \infty} \frac{2x+2}{1} = \infty$

27. $\lim_{x \rightarrow \infty} \frac{x^3}{e^{x/2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{(1/2)e^{x/2}}$

$$= \lim_{x \rightarrow \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x \rightarrow \infty} \frac{6}{(1/8)e^{x/2}} = 0$$

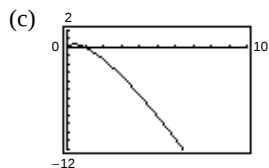
$$29. \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$$

Note: L'Hôpital's Rule does not work on this limit.
See Exercise 79.

$$33. \lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0$$

$$37. (a) \lim_{x \rightarrow 0^+} (-x \ln x) = (-0)(-\infty) = (0)(\infty)$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} (-x \ln x) &= \lim_{x \rightarrow 0^+} \frac{\ln x}{-1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{1/x}{1/x^2} \\ &= \lim_{x \rightarrow 0^+} x = 0 \end{aligned}$$



$$41. (a) \lim_{x \rightarrow 0^+} x^{1/x} = 0^\infty = 0, \text{ not indeterminate}$$

(See Exercise 95)

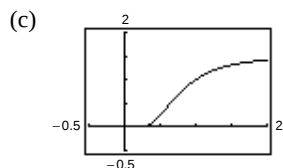
$$(b) \text{ Let } y = x^{1/x}$$

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x.$$

Since $x \rightarrow 0^+$, $\frac{1}{x} \ln x \rightarrow (\infty)(-\infty) = -\infty$. Hence,

$$\ln y \rightarrow -\infty \Rightarrow y \rightarrow 0^+.$$

Therefore, $\lim_{x \rightarrow 0^+} x^{1/x} = 0$.



$$45. (a) \lim_{x \rightarrow 0^+} (1+x)^{1/x} = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow 0^+} (1+x)^{1/x}.$$

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow 0^+} \left(\frac{1/(1+x)}{1} \right) = 1 \end{aligned}$$

Thus, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$.

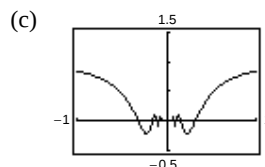
$$31. \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0 \text{ by Squeeze Theorem}$$

$$\left(\frac{\cos x}{x} \leq \frac{1}{x} \right)$$

$$35. \lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

$$39. (a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) = (\infty)(0)$$

$$\begin{aligned} (b) \lim_{x \rightarrow \infty} x \sin \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = 1 \end{aligned}$$



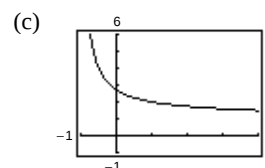
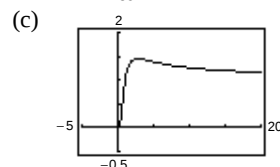
$$43. (a) \lim_{x \rightarrow \infty} x^{1/x} = \infty^0$$

$$(b) \text{ Let } y = \lim_{x \rightarrow \infty} x^{1/x}.$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \left(\frac{1/x}{1} \right) = 0$$

Thus, $\ln y = 0 \Rightarrow y = e^0 = 1$. Therefore,

$$\lim_{x \rightarrow \infty} x^{1/x} = 1.$$



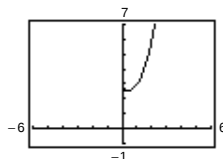
47. (a) $\lim_{x \rightarrow 0^+} [3(x)^{x/2}] = 0^0$

(b) Let $y = \lim_{x \rightarrow 0^+} 3(x)^{x/2}$.

$$\begin{aligned}\ln y &= \lim_{x \rightarrow 0^+} \left[\ln 3 + \frac{x}{2} \ln x \right] \\ &= \lim_{x \rightarrow 0^+} \left[\ln 3 + \frac{\ln x}{2/x} \right] \\ &= \lim_{x \rightarrow 0^+} \ln 3 + \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^2} \\ &= \lim_{x \rightarrow 0^+} \ln 3 - \lim_{x \rightarrow 0^+} \frac{x}{2} \\ &= \ln 3\end{aligned}$$

Hence, $\lim_{x \rightarrow 0^+} 3(x)^{x/2} = 3$.

(c)

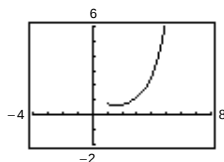


49. (a) $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = \lim_{x \rightarrow 1^+} (\ln x)^{x-1}$
 $= \lim_{x \rightarrow 1^+} (x-1) \ln x = 0$

Hence, $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 1$

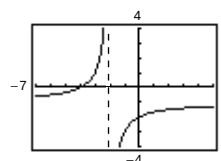
(c)



51. (a) $\lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x-2} \right) = \infty - \infty$

(b) $\lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x-2} \right) = \lim_{x \rightarrow 2^+} \frac{8 - x(x+2)}{x^2 - 4}$
 $= \lim_{x \rightarrow 2^+} \frac{(2-x)(4+x)}{(x+2)(x-2)}$
 $= \lim_{x \rightarrow 2^+} \frac{-(x+4)}{x+2} = \frac{-3}{2}$

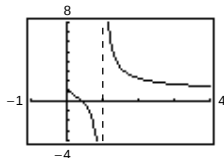
(c)



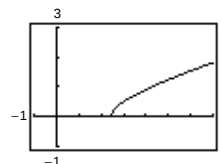
53. (a) $\lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right) = \infty - \infty$

(b) $\lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right) = \lim_{x \rightarrow 1^+} \frac{3x - 3 - 2 \ln x}{(x-1) \ln x}$
 $= \lim_{x \rightarrow 1^+} \frac{3 - (2/x)}{[(x-1)/x] + \ln x} = \infty$

(c)

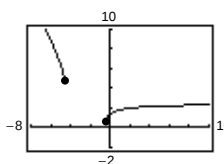


55. (a)



(b) $\lim_{x \rightarrow 3} \frac{x-3}{\ln(2x-5)} = \lim_{x \rightarrow 3} \frac{1}{2/(2x-5)}$
 $= \lim_{x \rightarrow 3} \frac{2x-5}{2} = \frac{1}{2}$

57. (a)



(b) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) = \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) \frac{(\sqrt{x^2 + 5x + 2} + x)}{(\sqrt{x^2 + 5x + 2} + x)}$
 $= \lim_{x \rightarrow \infty} \frac{(x^2 + 5x + 2) - x^2}{\sqrt{x^2 + 5x + 2} + x}$
 $= \lim_{x \rightarrow \infty} \frac{5x + 2}{\sqrt{x^2 + 5x + 2} + x}$
 $= \lim_{x \rightarrow \infty} \frac{5 + (2/x)}{\sqrt{1 + (5/x) + (2/x^2)} + 1} = \frac{5}{2}$

59. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty$

63. $\lim_{x \rightarrow \infty} \frac{x^2}{e^{5x}} = \lim_{x \rightarrow \infty} \frac{2x}{5e^{5x}} = \lim_{x \rightarrow \infty} \frac{2}{25e^{5x}} = 0$

65. $\lim_{x \rightarrow \infty} \frac{(\ln x)^3}{x} = \lim_{x \rightarrow \infty} \frac{3(\ln x)^2(1/x)}{1}$
 $= \lim_{x \rightarrow \infty} \frac{3(\ln x)^2}{x}$
 $= \lim_{x \rightarrow \infty} \frac{6(\ln x)(1/x)}{1}$
 $= \lim_{x \rightarrow \infty} \frac{6(\ln x)}{x} = \lim_{x \rightarrow \infty} \frac{6}{x} = 0$

69.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

71. $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 37)

$$\ln y = \frac{1}{x} \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \left(\frac{1}{x} \right) + (\ln x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = x^{1/x} \left(\frac{1}{x^2} \right) (1 - \ln x) = x^{(1/x)-2} (1 - \ln x) = 0$$

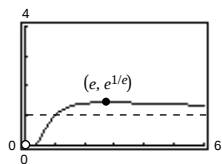
Critical number: $x = e$

Intervals: $(0, e)$ (e, ∞)

Sign of dy/dx : $+$ $-$

$y = f(x)$: Increasing Decreasing

Relative maximum: $(e, e^{1/e})$



75. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞ .
 L'Hôpital's Rule does not apply.

61. (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x - 5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x - 5)^3$.

67. $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^m} = \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$
 $= \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}}{mx^m}$
 $= \lim_{x \rightarrow \infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2 x^m}$
 $= \dots = \lim_{x \rightarrow \infty} \frac{n!}{m^n x^m} = 0$

73. $y = 2xe^{-x}$

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$$

$$= 2e^{-x}(1 - x) = 0$$

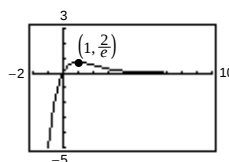
Critical number: $x = 1$

Intervals: $(-\infty, 1)$ $(1, \infty)$

Sign of dy/dx : $+$ $-$

$y = f(x)$: Increasing Decreasing

Relative maximum: $\left(1, \frac{2}{e}\right)$



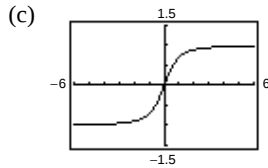
77. $\lim_{x \rightarrow \infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞ .
 L'Hôpital's Rule does not apply.

$$\begin{aligned}
 79. (a) \quad \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2 + 1/x}} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1}/\sqrt{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} \\
 &= \frac{1}{\sqrt{1 + 0}} = 1
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{1}{x/\sqrt{x^2 + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \rightarrow \infty} \frac{x/\sqrt{x^2 + 1}}{1} \\
 &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}
 \end{aligned}$$

Applying L'Hôpital's rule twice results in the original limit, so L'Hôpital's rule fails.



$$\begin{aligned}
 81. \quad \lim_{k \rightarrow 0} \frac{32 \left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32} \right)}{k} &= \lim_{k \rightarrow 0} \frac{32(1 - e^{-kt})}{k} + \lim_{k \rightarrow 0} (v_0 e^{-kt}) \\
 &= \lim_{k \rightarrow 0} \frac{32(0 + t e^{-kt})}{1} + \lim_{k \rightarrow 0} \left(\frac{v_0}{e^{kt}} \right) = 32t + v_0
 \end{aligned}$$

83. Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x \cos x$

Shaded area: Area of rectangle - Area under curve

$$\begin{aligned}
 2x(1 - \cos x) - 2 \int_0^x (1 - \cos t) dt &= 2x(1 - \cos x) - 2 \left[t - \sin t \right]_0^x \\
 &= 2x(1 - \cos x) - 2(x - \sin x) = 2 \sin x - 2x \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{Ratio: } \lim_{x \rightarrow 0} \frac{x - x \cos x}{2 \sin x - 2x \cos x} &= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2 \cos x + 2x \sin x - 2 \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos x + \sin x + \sin x}{2x \cos x + 2 \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{2x \cos x + 2 \sin x} \cdot \frac{1/\cos x}{1/\cos x} \\
 &= \lim_{x \rightarrow 0} \frac{x + 2 \tan x}{2x + 2 \tan x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + 2 \sec^2 x}{2 + 2 \sec^2 x} = \frac{3}{4}
 \end{aligned}$$

85. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\begin{aligned}
 \frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\
 \frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{3c^2}{2c} \\
 \frac{1}{1} &= \frac{3c}{2} \\
 c &= \frac{2}{3}
 \end{aligned}$$

87. $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2} \right]$

$$\begin{aligned}
 \frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} &= \frac{f'(c)}{g'(c)} \\
 \frac{1}{-1} &= \frac{\cos c}{-\sin c} \\
 -1 &= -\cot c \\
 c &= \frac{\pi}{4}
 \end{aligned}$$

89. False. L'Hôpital's Rule does not apply since

$$\lim_{x \rightarrow 0} (x^2 + x + 1) \neq 0.$$

$$\lim_{x \rightarrow 0} \frac{x^2 + x + 1}{x} = \lim_{x \rightarrow 0} \left(x + 1 + \frac{1}{x} \right) = 1 + \infty = \infty$$

91. True

93. (a) $\sin \theta = BD$

$$\cos \theta = DO \Rightarrow AD = 1 - \cos \theta$$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos \theta) \sin \theta = \frac{1}{2} \sin \theta - \frac{1}{2} \sin \theta \cos \theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos \theta)(\sin \theta) = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta$$

$$(c) R = \frac{(1/2) \sin \theta - (1/2) \sin \theta \cos \theta}{(1/2)\theta - (1/2) \sin \theta \cos \theta} = \frac{\sin \theta - \sin \theta \cos \theta}{\theta - \sin \theta \cos \theta}$$

$$(d) \lim_{\theta \rightarrow 0} R = \lim_{\theta \rightarrow 0} \frac{\sin \theta - (1/2) \sin 2\theta}{\theta - (1/2) \sin 2\theta}$$

$$= \lim_{\theta \rightarrow 0} \frac{\cos \theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\sin \theta + 2 \sin 2\theta}{2 \sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\cos \theta + 4 \cos 2\theta}{4 \cos 2\theta} = \frac{3}{4}$$

95. $\lim_{x \rightarrow a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$$

As $x \rightarrow a$, $\ln y \Rightarrow -\infty$, and hence $y = 0$. Thus,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = 0.$$

$$97. f'(a)(b-a) - \int_a^b f''(t)(t-b) dt = f'(a)(b-a) - \left\{ \left[f'(t)(t-b) \right]_a^b - \int_a^b f'(t) dt \right\}$$

$$= f'(a)(b-a) + f'(a)(a-b) + \left[f(t) \right]_a^b = f(b) - f(a)$$

$$dv = f''(t)dt \Rightarrow v = f'(t)$$

$$u = t - b \Rightarrow du = dt$$

Section 7.8 Improper Integrals

1. Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow 0^+} \int_b^4 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{b \rightarrow 0^+} \left[2\sqrt{x} \right]_b^4$$

$$= \lim_{b \rightarrow 0^+} (4 - 2\sqrt{b}) = 4$$

Converges

3. Infinite discontinuity at $x = 1$.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^2} dx &= \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left[-\frac{1}{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[-\frac{1}{x-1} \right]_c^2 = (\infty - 1) + (-1 + \infty)\end{aligned}$$

Diverges

5. Infinite limit of integration.

$$\begin{aligned}\int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 0 + 1 = 1\end{aligned}$$

Converges

$$7. \int_{-1}^1 \frac{1}{x^2} dx \neq -2$$

because the integrand is not defined at $x = 0$.

Diverges

$$\begin{aligned}9. \int_1^\infty \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1\end{aligned}$$

$$\begin{aligned}11. \int_1^\infty \frac{3}{\sqrt[3]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b 3x^{-1/3} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{9}{2} x^{2/3} \right]_1^b = \infty\end{aligned}$$

Diverges

$$13. \int_{-\infty}^0 x e^{-2x} dx = \lim_{b \rightarrow -\infty} \int_b^0 x e^{-2x} dx = \lim_{b \rightarrow -\infty} \frac{1}{4} \left[(-2x - 1) e^{-2x} \right]_b^0 = \lim_{b \rightarrow -\infty} \frac{1}{4} [-1 + (2b + 1) e^{-2b}] = -\infty \quad (\text{Integration by parts})$$

Diverges

$$15. \int_0^\infty x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x}(x^2 + 2x + 2) \right]_0^b = \lim_{b \rightarrow \infty} \left(-\frac{b^2 + 2b + 2}{e^b} + 2 \right) = 2$$

Since $\lim_{b \rightarrow \infty} \left(-\frac{b^2 + 2b + 2}{e^b} \right) = 0$ by L'Hôpital's Rule.

$$\begin{aligned}17. \int_0^\infty e^{-x} \cos x dx &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[e^{-x} (-\cos x + \sin x) \right]_0^b \\ &= \frac{1}{2} [0 - (-1)] = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}19. \int_4^\infty \frac{1}{x(\ln x)^3} dx &= \lim_{b \rightarrow \infty} \int_4^b (\ln x)^{-3} \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} (\ln x)^{-2} \right]_4^b \\ &= -\frac{1}{2} (\ln b)^{-2} + \frac{1}{2} (\ln 4)^{-2} \\ &= \frac{1}{2} \frac{1}{(\ln 2)^2} = \frac{1}{8(\ln 2)^2}\end{aligned}$$

$$\begin{aligned}21. \int_{-\infty}^\infty \frac{2}{4+x^2} dx &= \int_{-\infty}^0 \frac{2}{4+x^2} dx + \int_0^\infty \frac{2}{4+x^2} dx \\ &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{2}{4+x^2} dx + \lim_{c \rightarrow \infty} \int_0^c \frac{2}{4+x^2} dx \\ &= \lim_{b \rightarrow -\infty} \left[\arctan\left(\frac{x}{2}\right) \right]_b^0 + \lim_{c \rightarrow \infty} \left[\arctan\left(\frac{x}{2}\right) \right]_0^c \\ &= \left(0 - \left(-\frac{\pi}{2} \right) \right) + \left(\frac{\pi}{2} - 0 \right) = \pi\end{aligned}$$

$$\begin{aligned}
 23. \int_0^\infty \frac{1}{e^x + e^{-x}} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{1 + e^{2x}} dx \\
 &= \lim_{b \rightarrow \infty} \left[\arctan(e^x) \right]_0^b \\
 &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
 \end{aligned}$$

$$25. \int_0^\infty \cos \pi x dx = \lim_{b \rightarrow \infty} \left[\frac{1}{\pi} \sin \pi x \right]_0^b$$

Diverges since $\sin \pi x$ does not approach a limit as $x \rightarrow \infty$.

$$27. \int_0^1 \frac{1}{x^2} dx = \lim_{b \rightarrow 0^+} \left[-\frac{1}{x} \right]_b^1 = -1 + \infty$$

Diverges

$$29. \int_0^8 \frac{1}{\sqrt[3]{8-x}} dx = \lim_{b \rightarrow 8^-} \int_0^b \frac{1}{\sqrt[3]{8-x}} dx = \lim_{b \rightarrow 8^-} \left[\frac{-3}{2} (8-x)^{2/3} \right]_0^b = 6$$

$$31. \int_0^1 x \ln x dx = \lim_{b \rightarrow 0^+} \left[\frac{x^2}{2} \ln|x| - \frac{x^2}{4} \right]_b^1 = \lim_{b \rightarrow 0^+} \left[\frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4} \right] = \frac{-1}{4} \text{ since } \lim_{b \rightarrow 0^+} (b^2 \ln b) = 0 \text{ by L'Hôpital's Rule.}$$

$$33. \int_0^{\pi/2} \tan \theta d\theta = \lim_{b \rightarrow (\pi/2)^-} \left[\ln|\sec \theta| \right]_0^b = \infty$$

Diverges

$$\begin{aligned}
 35. \int_2^4 \frac{2}{x\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \int_b^4 \frac{2}{x\sqrt{x^2-4}} dx \\
 &= \lim_{b \rightarrow 2^+} \left[\operatorname{arcsec} \left| \frac{x}{2} \right| \right]_b^4 \\
 &= \lim_{b \rightarrow 2^+} \left(\operatorname{arcsec} 2 - \operatorname{arcsec} \left(\frac{b}{2} \right) \right) \\
 &= \frac{\pi}{3} - 0 = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 37. \int_2^4 \frac{1}{\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \left[\ln|x + \sqrt{x^2-4}| \right]_b^4 \\
 &= \ln(4 + 2\sqrt{3}) - \ln 2 \\
 &= \ln(2 + \sqrt{3}) \approx 1.317
 \end{aligned}$$

$$\begin{aligned}
 39. \int_0^2 \frac{1}{\sqrt[3]{x-1}} dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}} dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}} dx \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{3}{2} (x-1)^{2/3} \right]_0^b + \lim_{c \rightarrow 1^+} \left[\frac{3}{2} (x-1)^{2/3} \right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0
 \end{aligned}$$

$$41. \int_0^\infty \frac{4}{\sqrt{x}(x+6)} dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)} dx + \int_1^\infty \frac{4}{\sqrt{x}(x+6)} dx$$

Let $u = \sqrt{x}$, $u^2 = x$, $2u du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)} dx = \int \frac{4(2u du)}{u(u^2+6)} = 8 \int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}} \arctan\left(\frac{4}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

$$\begin{aligned}
 \text{Thus, } \int_0^\infty \frac{4}{\sqrt{x}(x+6)} dx &= \lim_{b \rightarrow 0^+} \left[\frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_b^1 + \lim_{c \rightarrow \infty} \left[\frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_1^c \\
 &= \left(\frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}} 0 \right) + \left(\frac{8}{\sqrt{6}} \frac{\pi}{2} - \frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) \right) \\
 &= \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.
 \end{aligned}$$

43. If $p = 1$, $\int_1^\infty \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln x \Big|_1^b$.

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right].$$

This converges to $\frac{1}{p-1}$ if $1-p < 0$ or $p > 1$.

45. For $n = 1$ we have

$$\begin{aligned} \int_0^\infty x e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} x - e^{-x} \right]_0^b \quad (\text{Parts: } u = x, dv = e^{-x} dx) \\ &= \lim_{b \rightarrow \infty} [-e^{-b} b - e^{-b} + 1] \\ &= \lim_{b \rightarrow \infty} \left[\frac{-b}{e^b} - \frac{1}{e^b} + 1 \right] = 1 \quad (\text{L'Hôpital's Rule}) \end{aligned}$$

Assume that $\int_0^\infty x^n e^{-x} dx$ converges. Then for $n+1$ we have

$$\int x^{n+1} e^{-x} dx = -x^{n+1} e^{-x} + (n+1) \int x^n e^{-x} dx$$

by parts ($u = x^{n+1}$, $du = (n+1)x^n dx$, $dv = e^{-x} dx$, $v = -e^{-x}$).

Thus,

$$\int_0^\infty x^{n+1} e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^{n+1} e^{-x} \right]_0^b + (n+1) \int_0^\infty x^n e^{-x} dx = 0 + (n+1) \int_0^\infty x^n e^{-x} dx, \text{ which converges.}$$

47. $\int_0^1 \frac{1}{x^3} dx$ diverges.

(See Exercise 44, $p = 3 \nless 1$.)

49. $\int_1^\infty \frac{1}{x^3} dx = \frac{1}{3-1} = \frac{1}{2}$ converges.

(See Exercise 43, $p = 3$.)

51. Since $\frac{1}{x^2+5} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx$ converges by Exercise 43, $\int_1^\infty \frac{1}{x^2+5} dx$ converges.

53. Since $\frac{1}{\sqrt[3]{x(x-1)}} \geq \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and $\int_2^\infty \frac{1}{\sqrt[3]{x^2}} dx$ diverges by Exercise 43, $\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}} dx$ diverges.

55. Since $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$ and $\int_0^\infty e^{-x} dx$ converges (see Exercise 5), $\int_0^\infty e^{-x^2} dx$ converges.

57. Answers will vary. See pages 540, 543.

59. $\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$

These two integrals diverge by Exercise 44.

61. $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st} dt = \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \right]_0^b = \frac{1}{s}, s > 0$$

63. $f(t) = t^2$

$$\begin{aligned} F(s) &= \int_0^\infty t^2 e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{1}{s^3} (-s^2 t^2 - 2st - 2) e^{-st} \right]_0^b \\ &= \frac{2}{s^3}, s > 0 \end{aligned}$$

65. $f(t) = \cos at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \cos at \, dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^b \\
 &= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2}, s > 0
 \end{aligned}$$

67. $f(t) = \cosh at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \cosh at \, dt = \int_0^{\infty} e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} \left[e^{t(-s+a)} + e^{t(-s-a)} \right] dt \\
 &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\frac{1}{(-s+a)} e^{t(-s+a)} + \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] \\
 &= \frac{-1}{2} \left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] = \frac{s}{s^2 - a^2}, s > |a|
 \end{aligned}$$

69. (a) $A = \int_0^{\infty} e^{-x} \, dx$

$$= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 0 - (-1) = 1$$

(b) **Disk:**

$$\begin{aligned}
 V &= \pi \int_0^{\infty} (e^{-x})^2 \, dx \\
 &= \lim_{b \rightarrow \infty} \pi \left[-\frac{1}{2} e^{-2x} \right]_0^b = \frac{\pi}{2}
 \end{aligned}$$

(c) **Shell:**

$$\begin{aligned}
 V &= 2\pi \int_0^{\infty} x e^{-x} \, dx \\
 &= \lim_{b \rightarrow \infty} \left\{ 2\pi \left[-e^{-x}(x+1) \right]_0^b \right\} = 2\pi
 \end{aligned}$$

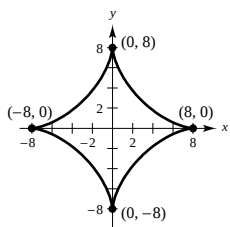
71. $x^{2/3} + y^{2/3} = 4$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}$$

$$s = 4 \int_0^8 \frac{2}{x^{1/3}} \, dx = \lim_{b \rightarrow 0^+} \left[8 \cdot \frac{3}{2} x^{2/3} \right]_b^8 = 48$$



$$73. \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$(a) \Gamma(1) = \int_0^\infty e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 1$$

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x}(x+1) \right]_0^b = 1$$

$$\Gamma(3) = \int_0^\infty x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^b = 2$$

$$(b) \Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^n e^{-x} \right]_0^b + \lim_{b \rightarrow \infty} n \int_0^b x^{n-1} e^{-x} dx = 0 + n\Gamma(n) \quad (u = x^n, dv = e^{-x} dx)$$

$$(c) \Gamma(n) = (n-1)!$$

$$75. (a) \int_{-\infty}^\infty \frac{1}{7} e^{-t/7} dt = \int_0^\infty \frac{1}{7} e^{-t/7} dt = \lim_{b \rightarrow \infty} \left[-e^{-t/7} \right]_0^b = 1$$

$$(b) \int_0^4 \frac{1}{7} e^{-t/7} dt = \left[-e^{-t/7} \right]_0^4 = -e^{-4/7} + 1$$

$$\approx 0.4353 = 43.53\%$$

$$(c) \int_0^\infty t \left[\frac{1}{7} e^{-t/7} \right] dt = \lim_{b \rightarrow \infty} \left[-te^{-t/7} - 7e^{-t/7} \right]_0^b$$

$$= 0 + 7 = 7$$

$$77. (a) C = 650,000 + \int_0^5 25,000 e^{-0.06t} dt = 650,000 - \left[\frac{25,000}{0.06} e^{-0.06t} \right]_0^5 \approx \$757,992.41$$

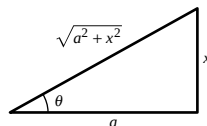
$$(b) C = 650,000 + \int_0^{10} 25,000 e^{-0.06t} dt \approx \$837,995.15$$

$$(c) C = 650,000 + \int_0^\infty 25,000 e^{-0.06t} dt = 650,000 - \lim_{b \rightarrow \infty} \left[\frac{25,000}{0.06} e^{-0.06t} \right]_0^b \approx \$1,066,666.67$$

$$79. \text{ Let } x = a \tan \theta, dx = a \sec^2 \theta d\theta, \sqrt{a^2 + x^2} = a \sec \theta.$$

$$\int \frac{1}{(a^2 + x^2)^{3/2}} dx = \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2} \int \cos \theta d\theta$$

$$= \frac{1}{a^2} \sin \theta = \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}}$$



Hence,

$$P = k \int_1^\infty \frac{1}{(a^2 + x^2)^{3/2}} dx = \frac{k}{a^2} \lim_{b \rightarrow \infty} \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_1^b$$

$$= \frac{k}{a^2} \left[1 - \frac{1}{\sqrt{a^2 + 1}} \right] = \frac{k(\sqrt{a^2 + 1} - 1)}{a^2 \sqrt{a^2 + 1}}.$$

$$81. \frac{10}{x^2 - 2x} = \frac{10}{x(x-2)} \Rightarrow x = 0, 2.$$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx$$

83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b (x^2 + 1)^{-4} (2x dx) = \lim_{b \rightarrow \infty} \left[-\frac{1}{6} \frac{1}{(x^2 + 1)^3} \right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx = \lim_{b \rightarrow \infty} \left[\frac{-x^{2n-2}}{2(n+2)(x^2 + 1)^n} + 2 \right]_0^b + \frac{n-1}{n+2} \int_0^\infty \frac{x^{2n-3}}{(x^2 + 1)^{n+2}} dx = 0 + \frac{n-1}{n+2} (I_{n-1})$$

$$u = x^{2n-2}, du = (2n-2)x^{2n-3} dx, dv = \frac{x}{(x^2 + 1)^{n+3}} dx, v = \frac{-1}{2(n+2)(x^2 + 1)^{n+2}}$$

$$(a) \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{6(x^2 + 1)^3} \right]_0^b = \frac{1}{6}$$

$$(b) \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{1}{4} \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \frac{1}{4} \left(\frac{1}{6} \right) = \frac{1}{24}$$

$$(c) \int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx = \frac{2}{5} \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{2}{5} \left(\frac{1}{24} \right) = \frac{1}{60}$$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x \rightarrow \infty} 1/(x+1) = 0$, but $\int_0^\infty \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_0^b = \infty$.

Diverges

87. True

Review Exercises for Chapter 7

$$\begin{aligned} 1. \int x\sqrt{x^2-1} dx &= \frac{1}{2} \int (x^2-1)^{1/2} (2x) dx \\ &= \frac{1}{2} \frac{(x^2-1)^{3/2}}{3/2} + C \\ &= \frac{1}{3} (x^2-1)^{3/2} + C \end{aligned}$$

$$\begin{aligned} 3. \int \frac{x}{x^2-1} dx &= \frac{1}{2} \int \frac{2x}{x^2-1} dx \\ &= \frac{1}{2} \ln|x^2-1| + C \end{aligned}$$

$$5. \int \frac{\ln(2x)}{x} dx = \frac{(\ln 2x)^2}{2} + C$$

$$7. \int \frac{16}{\sqrt{16-x^2}} dx = 16 \arcsin\left(\frac{x}{4}\right) + C$$

$$\begin{aligned} 9. \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\ &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(\frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \end{aligned}$$

$$\frac{13}{9} \int e^{2x} \sin 3x dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

$$(1) dv = \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$(2) dv = \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$11. u = x, du = dx, dv = (x - 5)^{1/2} dx, v = \frac{2}{3}(x - 5)^{3/2}$$

$$\begin{aligned}\int x\sqrt{x-5} dx &= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2} dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C \\ &= (x-5)^{3/2} \left[\frac{2}{3}x - \frac{4}{15}(x-5) \right] + C \\ &= (x-5)^{3/2} \left[\frac{6}{15}x + \frac{4}{3} \right] + C \\ &= \frac{2}{15}(x-5)^{3/2}[3x+10] + C\end{aligned}$$

$$\begin{aligned}13. \int x^2 \sin 2x dx &= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C\end{aligned}$$

$$(1) dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) dv = \cos 2x dx \Rightarrow v = \frac{1}{2} \sin 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}15. \int x \arcsin 2x dx &= \frac{x^2}{2} \arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \int \frac{2(2x)^2}{\sqrt{1-(2x)^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \left(\frac{1}{2} \right) [-(2x)\sqrt{1-4x^2} + \arcsin 2x] + C \quad (\text{by Formula 43 of Integration Tables}) \\ &= \frac{1}{16} [(8x^2 - 1)\arcsin 2x + 2x\sqrt{1-4x^2}] + C\end{aligned}$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

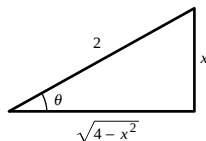
$$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned}17. \int \cos^3(\pi x - 1) dx &= \int [1 - \sin^2(\pi x - 1)] \cos(\pi x - 1) dx \\ &= \frac{1}{\pi} \left[\sin(\pi x - 1) - \frac{1}{3} \sin^3(\pi x - 1) \right] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - \sin^2(\pi x - 1)] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - (1 - \cos^2(\pi x - 1))] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [2 + \cos^2(\pi x - 1)] + C\end{aligned}$$

$$\begin{aligned}19. \int \sec^4\left(\frac{x}{2}\right) dx &= \int \left[\tan^2\left(\frac{x}{2}\right) + 1 \right] \sec^2\left(\frac{x}{2}\right) dx \\ &= \int \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx + \int \sec^2\left(\frac{x}{2}\right) dx \\ &= \frac{2}{3} \tan^3\left(\frac{x}{2}\right) + 2 \tan\left(\frac{x}{2}\right) + C = \frac{2}{3} \left[\tan^3\left(\frac{x}{2}\right) + 3 \tan\left(\frac{x}{2}\right) \right] + C\end{aligned}$$

$$21. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$\begin{aligned}
 23. \int \frac{-12}{x^2 \sqrt{4-x^2}} dx &= \int \frac{-24 \cos \theta d\theta}{(4 \sin^2 \theta)(2 \cos \theta)} \\
 &= -3 \int \csc^2 \theta d\theta \\
 &= 3 \cot \theta + C \\
 &= \frac{3\sqrt{4-x^2}}{x} + C
 \end{aligned}$$



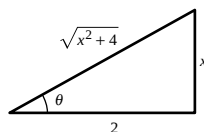
$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$25. \quad x = 2 \tan \theta$$

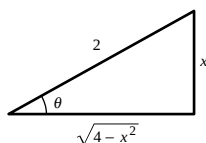
$$dx = 2 \sec^2 \theta d\theta$$

$$4 + x^2 = 4 \sec^2 \theta$$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\
 &= 8 \int \tan^3 \theta \sec \theta d\theta \\
 &= 8 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta \\
 &= 8 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 8 \left[\frac{(x^2 + 4)^{3/2}}{24} - \frac{\sqrt{x^2 + 4}}{2} \right] + C \\
 &= \sqrt{x^2 + 4} \left[\frac{1}{3}(x^2 + 4) - 4 \right] + C \\
 &= \frac{1}{3} x^2 \sqrt{x^2 + 4} - \frac{8}{3} \sqrt{x^2 + 4} + C \\
 &= \frac{1}{3} (x^2 + 4)^{1/2} (x^2 - 8) + C
 \end{aligned}$$



$$\begin{aligned}
 27. \int \sqrt{4-x^2} dx &= \int (2 \cos \theta)(2 \cos \theta) d\theta \\
 &= 2 \int (1 + \cos 2\theta) d\theta \\
 &= 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= 2(\theta + \sin \theta \cos \theta) + C \\
 &= 2 \left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \left(\frac{\sqrt{4-x^2}}{2} \right) \right] + C \\
 &= \frac{1}{2} \left[4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4-x^2} \right] + C
 \end{aligned}$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 29. \text{ (a) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= 8 \int \frac{\sin^3 \theta}{\cos^4 \theta} d\theta \\
 &= 8 \int (\cos^{-4} \theta - \cos^{-2} \theta) \sin \theta d\theta \\
 &= \frac{8}{3} \sec \theta (\sec^2 \theta - 3) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$x = 2 \tan \theta, dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned}
 \text{(b) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int (u^2 - 4) du \\
 &= \frac{1}{3} u^3 - 4u + C \\
 &= \frac{u}{3} (u^2 - 12) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$u^2 = 4 + x^2, 2u du = 2x dx$$

$$\begin{aligned}
 \text{(c) } \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - \int 2x \sqrt{4+x^2} dx \\
 &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$31. \frac{x-28}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$x-28 = A(x+2) + B(x-3)$$

$$x = -2 \Rightarrow -30 = B(-5) \Rightarrow B = 6$$

$$x = 3 \Rightarrow -25 = A(5) \Rightarrow A = -5$$

$$\int \frac{x-28}{x^2-x-6} dx = \int \left(\frac{-5}{x-3} + \frac{6}{x+2} \right) dx = -5 \ln|x-3| + 6 \ln|x+2| + C$$

$$33. \frac{x^2+2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+2x = A(x^2+1) + (Bx+C)(x-1)$$

$$\text{Let } x = 1: 3 = 2A \Rightarrow A = \frac{3}{2}$$

$$\text{Let } x = 0: 0 = A - C \Rightarrow C = \frac{3}{2}$$

$$\text{Let } x = 2: 8 = 5A + 2B + C \Rightarrow B = -\frac{1}{2}$$

$$\begin{aligned}
 \int \frac{x^2+2x}{x^3-x^2+x-1} dx &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x-3}{x^2+1} dx \\
 &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{3}{2} \int \frac{1}{x^2+1} dx \\
 &= \frac{3}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{3}{2} \arctan x + C \\
 &= \frac{1}{4} [6 \ln|x-1| - \ln(x^2+1) + 6 \arctan x] + C
 \end{aligned}$$

$$35. \frac{x^2}{x^2 + 2x - 15} = 1 + \frac{15 - 2x}{x^2 + 2x - 15}$$

$$\frac{15 - 2x}{(x - 3)(x + 5)} = \frac{A}{x - 3} + \frac{B}{x + 5}$$

$$15 - 2x = A(x + 5) + B(x - 3)$$

$$\text{Let } x = 3: 9 = 8A \Rightarrow A = \frac{9}{8}$$

$$\text{Let } x = -5: 25 = -8B \Rightarrow B = -\frac{25}{8}$$

$$\begin{aligned} \int \frac{x^2}{x^2 + 2x - 15} dx &= \int dx + \frac{9}{8} \int \frac{1}{x - 3} dx - \frac{25}{8} \int \frac{1}{x + 5} dx \\ &= x + \frac{9}{8} \ln|x - 3| - \frac{25}{8} \ln|x + 5| + C \end{aligned}$$

$$37. \int \frac{x}{(2 + 3x)^2} dx = \frac{1}{9} \left[\frac{2}{2 + 3x} + \ln|2 + 3x| \right] + C$$

(Formula 4)

$$\begin{aligned} 39. \int \frac{x}{1 + \sin x^2} dx &= \frac{1}{2} \int \frac{1}{1 + \sin u} du \quad (u = x^2) \\ &= \frac{1}{2} [\tan u - \sec u] + C \quad (\text{Formula 56}) \\ &= \frac{1}{2} [\tan x^2 - \sec x^2] + C \end{aligned}$$

$$\begin{aligned} 41. \int \frac{x}{x^2 + 4x + 8} dx &= \frac{1}{2} \left[\ln|x^2 + 4x + 8| - 4 \int \frac{1}{x^2 + 4x + 8} dx \right] \quad (\text{Formula 15}) \\ &= \frac{1}{2} [\ln|x^2 + 4x + 8|] - 2 \left[\frac{2}{\sqrt{32 - 16}} \arctan \left(\frac{2x + 4}{\sqrt{32 - 16}} \right) \right] + C \quad (\text{Formula 14}) \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan \left(1 + \frac{x}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 43. \int \frac{1}{\sin \pi x \cos \pi x} dx &= \frac{1}{\pi} \int \frac{1}{\sin \pi x \cos \pi x} (\pi) dx \quad (u = \pi x) \\ &= \frac{1}{\pi} \ln|\tan \pi x| + C \quad (\text{Formula 58}) \end{aligned}$$

$$\begin{aligned} 45. dv &= dx \Rightarrow v = x \\ u &= (\ln x)^n \Rightarrow du = n(\ln x)^{n-1} \frac{1}{x} dx \\ \int (\ln x)^n dx &= x(\ln x)^n - n \int (\ln x)^{n-1} dx \end{aligned}$$

$$\begin{aligned} 47. \int \theta \sin \theta \cos \theta d\theta &= \frac{1}{2} \int \theta \sin 2\theta d\theta \\ &= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C \end{aligned}$$

$$dv = \sin 2\theta d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned}
 49. \int \frac{x^{1/4}}{1+x^{1/2}} dx &= 4 \int \frac{u(u^3)}{1+u^2} du \\
 &= 4 \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du \\
 &= 4 \left(\frac{1}{3} u^3 - u + \arctan u \right) + C \\
 &= \frac{4}{3} [x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C
 \end{aligned}$$

$$y = \sqrt[4]{x}, x = u^4, dx = 4u^3 du$$

$$\begin{aligned}
 53. \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\
 &= \sin x \ln(\sin x) - \sin x + C
 \end{aligned}$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$\begin{aligned}
 57. y &= \int \ln(x^2 + x) dx = x \ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x} dx \\
 &= x \ln|x^2 + x| - \int \frac{2x + 1}{x + 1} dx \\
 &= x \ln|x^2 + x| - \int 2 dx + \int \frac{1}{x + 1} dx \\
 &= x \ln|x^2 + x| - 2x + \ln|x + 1| + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 + x) \Rightarrow du = \frac{2x + 1}{x^2 + x} dx$$

$$61. \int_1^4 \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} (\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$$

$$\begin{aligned}
 65. A &= \int_0^4 x \sqrt{4-x} dx = \int_2^0 (4-u^2)u(-2u) du \\
 &= \int_2^0 2(u^4 - 4u^2) du \\
 &= \left[2 \left(\frac{u^5}{5} - \frac{4u^3}{3} \right) \right]_2^0 = \frac{128}{15} \\
 u &= \sqrt{4-x}, x = 4-u^2, dx = -2u du
 \end{aligned}$$

$$69. s = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$\begin{aligned}
 51. \int \sqrt{1 + \cos x} dx &= \int \frac{\sin x}{\sqrt{1 - \cos x}} dx \\
 &= \int (1 - \cos x)^{-1/2} (\sin x) dx \\
 &= 2\sqrt{1 - \cos x} + C \\
 u &= 1 - \cos x, du = \sin x dx
 \end{aligned}$$

$$\begin{aligned}
 55. y &= \int \frac{9}{x^2 - 9} dx = \frac{3}{2} \ln \left| \frac{x-3}{x+3} \right| + C \\
 &\text{(by Formula 24 of Integration Tables)}
 \end{aligned}$$

$$59. \int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} dx = \left[\frac{1}{5} (x^2 - 4)^{5/2} \right]_2^{\sqrt{5}} = \frac{1}{5}$$

$$63. \int_0^\pi x \sin x dx = \left[-x \cos x + \sin x \right]_0^\pi = \pi$$

$$67. \text{By symmetry, } \bar{x} = 0, A = \frac{1}{2} \pi.$$

$$\begin{aligned}
 \bar{y} &= \frac{2}{\pi} \left(\frac{1}{2} \right) \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \frac{1}{\pi} \left[x - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{4}{3\pi} \\
 (\bar{x}, \bar{y}) &= \left(0, \frac{4}{3\pi} \right)
 \end{aligned}$$

$$71. \lim_{x \rightarrow 1} \left[\frac{(\ln x)^2}{x-1} \right] = \lim_{x \rightarrow 1} \left[\frac{2(1/x) \ln x}{1} \right] = 0$$

$$\begin{aligned}
 75. y &= \lim_{x \rightarrow \infty} (\ln x)^{2/x} \\
 \ln y &= \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[\frac{2/(x \ln x)}{1} \right] = 0
 \end{aligned}$$

Since $\ln y = 0, y = 1$.

83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b (x^2 + 1)^{-4} (2x dx) = \lim_{b \rightarrow \infty} \left[-\frac{1}{6} \frac{1}{(x^2 + 1)^3} \right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx = \lim_{b \rightarrow \infty} \left[\frac{-x^{2n-2}}{2(n+2)(x^2 + 1)^n} + 2 \right]_0^b + \frac{n-1}{n+2} \int_0^\infty \frac{x^{2n-3}}{(x^2 + 1)^{n+2}} dx = 0 + \frac{n-1}{n+2} (I_{n-1})$$

$$u = x^{2n-2}, du = (2n-2)x^{2n-3} dx, dv = \frac{x}{(x^2 + 1)^{n+3}} dx, v = \frac{-1}{2(n+2)(x^2 + 1)^{n+2}}$$

$$(a) \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{6(x^2 + 1)^3} \right]_0^b = \frac{1}{6}$$

$$(b) \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{1}{4} \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \frac{1}{4} \left(\frac{1}{6} \right) = \frac{1}{24}$$

$$(c) \int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx = \frac{2}{5} \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{2}{5} \left(\frac{1}{24} \right) = \frac{1}{60}$$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x \rightarrow \infty} 1/(x+1) = 0$, but $\int_0^\infty \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_0^b = \infty$.

Diverges

87. True

Review Exercises for Chapter 7

$$\begin{aligned} 1. \int x\sqrt{x^2-1} dx &= \frac{1}{2} \int (x^2-1)^{1/2} (2x) dx \\ &= \frac{1}{2} \frac{(x^2-1)^{3/2}}{3/2} + C \\ &= \frac{1}{3} (x^2-1)^{3/2} + C \end{aligned}$$

$$\begin{aligned} 3. \int \frac{x}{x^2-1} dx &= \frac{1}{2} \int \frac{2x}{x^2-1} dx \\ &= \frac{1}{2} \ln|x^2-1| + C \end{aligned}$$

$$5. \int \frac{\ln(2x)}{x} dx = \frac{(\ln 2x)^2}{2} + C$$

$$7. \int \frac{16}{\sqrt{16-x^2}} dx = 16 \arcsin\left(\frac{x}{4}\right) + C$$

$$\begin{aligned} 9. \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\ &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(\frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \end{aligned}$$

$$\frac{13}{9} \int e^{2x} \sin 3x dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

$$(1) dv = \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$(2) dv = \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$11. u = x, du = dx, dv = (x - 5)^{1/2} dx, v = \frac{2}{3}(x - 5)^{3/2}$$

$$\begin{aligned}\int x\sqrt{x-5} dx &= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2} dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C \\ &= (x-5)^{3/2} \left[\frac{2}{3}x - \frac{4}{15}(x-5) \right] + C \\ &= (x-5)^{3/2} \left[\frac{6}{15}x + \frac{4}{3} \right] + C \\ &= \frac{2}{15}(x-5)^{3/2}[3x+10] + C\end{aligned}$$

$$\begin{aligned}13. \int x^2 \sin 2x dx &= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C\end{aligned}$$

$$(1) dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) dv = \cos 2x dx \Rightarrow v = \frac{1}{2} \sin 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}15. \int x \arcsin 2x dx &= \frac{x^2}{2} \arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \int \frac{2(2x)^2}{\sqrt{1-(2x)^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \left(\frac{1}{2} \right) [-(2x)\sqrt{1-4x^2} + \arcsin 2x] + C \quad (\text{by Formula 43 of Integration Tables}) \\ &= \frac{1}{16} [(8x^2 - 1)\arcsin 2x + 2x\sqrt{1-4x^2}] + C\end{aligned}$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

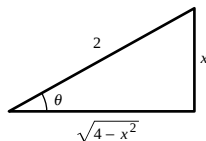
$$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned}17. \int \cos^3(\pi x - 1) dx &= \int [1 - \sin^2(\pi x - 1)] \cos(\pi x - 1) dx \\ &= \frac{1}{\pi} \left[\sin(\pi x - 1) - \frac{1}{3} \sin^3(\pi x - 1) \right] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - \sin^2(\pi x - 1)] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - (1 - \cos^2(\pi x - 1))] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [2 + \cos^2(\pi x - 1)] + C\end{aligned}$$

$$\begin{aligned}19. \int \sec^4\left(\frac{x}{2}\right) dx &= \int \left[\tan^2\left(\frac{x}{2}\right) + 1 \right] \sec^2\left(\frac{x}{2}\right) dx \\ &= \int \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx + \int \sec^2\left(\frac{x}{2}\right) dx \\ &= \frac{2}{3} \tan^3\left(\frac{x}{2}\right) + 2 \tan\left(\frac{x}{2}\right) + C = \frac{2}{3} \left[\tan^3\left(\frac{x}{2}\right) + 3 \tan\left(\frac{x}{2}\right) \right] + C\end{aligned}$$

$$21. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$\begin{aligned}
 23. \int \frac{-12}{x^2 \sqrt{4-x^2}} dx &= \int \frac{-24 \cos \theta d\theta}{(4 \sin^2 \theta)(2 \cos \theta)} \\
 &= -3 \int \csc^2 \theta d\theta \\
 &= 3 \cot \theta + C \\
 &= \frac{3\sqrt{4-x^2}}{x} + C
 \end{aligned}$$



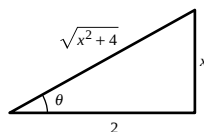
$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$25. \quad x = 2 \tan \theta$$

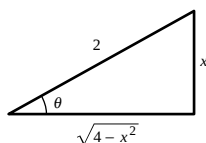
$$dx = 2 \sec^2 \theta d\theta$$

$$4 + x^2 = 4 \sec^2 \theta$$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\
 &= 8 \int \tan^3 \theta \sec \theta d\theta \\
 &= 8 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta \\
 &= 8 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 8 \left[\frac{(x^2 + 4)^{3/2}}{24} - \frac{\sqrt{x^2 + 4}}{2} \right] + C \\
 &= \sqrt{x^2 + 4} \left[\frac{1}{3}(x^2 + 4) - 4 \right] + C \\
 &= \frac{1}{3} x^2 \sqrt{x^2 + 4} - \frac{8}{3} \sqrt{x^2 + 4} + C \\
 &= \frac{1}{3} (x^2 + 4)^{1/2} (x^2 - 8) + C
 \end{aligned}$$



$$\begin{aligned}
 27. \int \sqrt{4-x^2} dx &= \int (2 \cos \theta)(2 \cos \theta) d\theta \\
 &= 2 \int (1 + \cos 2\theta) d\theta \\
 &= 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= 2(\theta + \sin \theta \cos \theta) + C \\
 &= 2 \left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \left(\frac{\sqrt{4-x^2}}{2} \right) \right] + C \\
 &= \frac{1}{2} \left[4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4-x^2} \right] + C
 \end{aligned}$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$\begin{aligned}
 29. \quad (a) \quad \int \frac{x^3}{\sqrt{4+x^2}} dx &= 8 \int \frac{\sin^3 \theta}{\cos^4 \theta} d\theta \\
 &= 8 \int (\cos^{-4} \theta - \cos^{-2} \theta) \sin \theta d\theta \\
 &= \frac{8}{3} \sec \theta (\sec^2 \theta - 3) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$x = 2 \tan \theta, dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned}
 (b) \quad \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int (u^2 - 4) du \\
 &= \frac{1}{3} u^3 - 4u + C \\
 &= \frac{u}{3} (u^2 - 12) + C \\
 &= \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$u^2 = 4 + x^2, 2u du = 2x dx$$

$$\begin{aligned}
 (c) \quad \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2 \sqrt{4+x^2} - \int 2x \sqrt{4+x^2} dx \\
 &= x^2 \sqrt{4+x^2} - \frac{2}{3} (4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3} (x^2 - 8) + C
 \end{aligned}$$

$$dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$31. \quad \frac{x-28}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$x-28 = A(x+2) + B(x-3)$$

$$x = -2 \Rightarrow -30 = B(-5) \Rightarrow B = 6$$

$$x = 3 \Rightarrow -25 = A(5) \Rightarrow A = -5$$

$$\int \frac{x-28}{x^2-x-6} dx = \int \left(\frac{-5}{x-3} + \frac{6}{x+2} \right) dx = -5 \ln|x-3| + 6 \ln|x+2| + C$$

$$33. \quad \frac{x^2+2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+2x = A(x^2+1) + (Bx+C)(x-1)$$

$$\text{Let } x = 1: 3 = 2A \Rightarrow A = \frac{3}{2}$$

$$\text{Let } x = 0: 0 = A - C \Rightarrow C = \frac{3}{2}$$

$$\text{Let } x = 2: 8 = 5A + 2B + C \Rightarrow B = -\frac{1}{2}$$

$$\begin{aligned}
 \int \frac{x^2+2x}{x^3-x^2+x-1} dx &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x-3}{x^2+1} dx \\
 &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{3}{2} \int \frac{1}{x^2+1} dx \\
 &= \frac{3}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{3}{2} \arctan x + C \\
 &= \frac{1}{4} [6 \ln|x-1| - \ln(x^2+1) + 6 \arctan x] + C
 \end{aligned}$$

$$35. \frac{x^2}{x^2 + 2x - 15} = 1 + \frac{15 - 2x}{x^2 + 2x - 15}$$

$$\frac{15 - 2x}{(x - 3)(x + 5)} = \frac{A}{x - 3} + \frac{B}{x + 5}$$

$$15 - 2x = A(x + 5) + B(x - 3)$$

$$\text{Let } x = 3: 9 = 8A \Rightarrow A = \frac{9}{8}$$

$$\text{Let } x = -5: 25 = -8B \Rightarrow B = -\frac{25}{8}$$

$$\begin{aligned} \int \frac{x^2}{x^2 + 2x - 15} dx &= \int dx + \frac{9}{8} \int \frac{1}{x - 3} dx - \frac{25}{8} \int \frac{1}{x + 5} dx \\ &= x + \frac{9}{8} \ln|x - 3| - \frac{25}{8} \ln|x + 5| + C \end{aligned}$$

$$37. \int \frac{x}{(2 + 3x)^2} dx = \frac{1}{9} \left[\frac{2}{2 + 3x} + \ln|2 + 3x| \right] + C$$

(Formula 4)

$$\begin{aligned} 39. \int \frac{x}{1 + \sin x^2} dx &= \frac{1}{2} \int \frac{1}{1 + \sin u} du \quad (u = x^2) \\ &= \frac{1}{2} [\tan u - \sec u] + C \quad (\text{Formula 56}) \\ &= \frac{1}{2} [\tan x^2 - \sec x^2] + C \end{aligned}$$

$$\begin{aligned} 41. \int \frac{x}{x^2 + 4x + 8} dx &= \frac{1}{2} \left[\ln|x^2 + 4x + 8| - 4 \int \frac{1}{x^2 + 4x + 8} dx \right] \quad (\text{Formula 15}) \\ &= \frac{1}{2} [\ln|x^2 + 4x + 8|] - 2 \left[\frac{2}{\sqrt{32 - 16}} \arctan \left(\frac{2x + 4}{\sqrt{32 - 16}} \right) \right] + C \quad (\text{Formula 14}) \\ &= \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan \left(1 + \frac{x}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 43. \int \frac{1}{\sin \pi x \cos \pi x} dx &= \frac{1}{\pi} \int \frac{1}{\sin \pi x \cos \pi x} (\pi) dx \quad (u = \pi x) \\ &= \frac{1}{\pi} \ln|\tan \pi x| + C \quad (\text{Formula 58}) \end{aligned}$$

$$\begin{aligned} 45. dv &= dx \Rightarrow v = x \\ u &= (\ln x)^n \Rightarrow du = n(\ln x)^{n-1} \frac{1}{x} dx \\ \int (\ln x)^n dx &= x(\ln x)^n - n \int (\ln x)^{n-1} dx \end{aligned}$$

$$\begin{aligned} 47. \int \theta \sin \theta \cos \theta d\theta &= \frac{1}{2} \int \theta \sin 2\theta d\theta \\ &= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C \end{aligned}$$

$$dv = \sin 2\theta d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned}
 49. \int \frac{x^{1/4}}{1+x^{1/2}} dx &= 4 \int \frac{u(u^3)}{1+u^2} du \\
 &= 4 \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du \\
 &= 4 \left(\frac{1}{3} u^3 - u + \arctan u \right) + C \\
 &= \frac{4}{3} [x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C
 \end{aligned}$$

$$y = \sqrt[4]{x}, x = u^4, dx = 4u^3 du$$

$$\begin{aligned}
 53. \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\
 &= \sin x \ln(\sin x) - \sin x + C
 \end{aligned}$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$\begin{aligned}
 57. y &= \int \ln(x^2 + x) dx = x \ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x} dx \\
 &= x \ln|x^2 + x| - \int \frac{2x + 1}{x + 1} dx \\
 &= x \ln|x^2 + x| - \int 2 dx + \int \frac{1}{x + 1} dx \\
 &= x \ln|x^2 + x| - 2x + \ln|x + 1| + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 + x) \Rightarrow du = \frac{2x + 1}{x^2 + x} dx$$

$$61. \int_1^4 \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} (\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$$

$$\begin{aligned}
 65. A &= \int_0^4 x \sqrt{4-x} dx = \int_2^0 (4-u^2)u(-2u) du \\
 &= \int_2^0 2(u^4 - 4u^2) du \\
 &= \left[2 \left(\frac{u^5}{5} - \frac{4u^3}{3} \right) \right]_2^0 = \frac{128}{15} \\
 u &= \sqrt{4-x}, x = 4-u^2, dx = -2u du
 \end{aligned}$$

$$69. s = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$\begin{aligned}
 51. \int \sqrt{1 + \cos x} dx &= \int \frac{\sin x}{\sqrt{1 - \cos x}} dx \\
 &= \int (1 - \cos x)^{-1/2} (\sin x) dx \\
 &= 2\sqrt{1 - \cos x} + C \\
 u &= 1 - \cos x, du = \sin x dx
 \end{aligned}$$

$$\begin{aligned}
 55. y &= \int \frac{9}{x^2 - 9} dx = \frac{3}{2} \ln \left| \frac{x-3}{x+3} \right| + C \\
 &\text{(by Formula 24 of Integration Tables)}
 \end{aligned}$$

$$59. \int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} dx = \left[\frac{1}{5} (x^2 - 4)^{5/2} \right]_2^{\sqrt{5}} = \frac{1}{5}$$

$$63. \int_0^\pi x \sin x dx = \left[-x \cos x + \sin x \right]_0^\pi = \pi$$

$$67. \text{By symmetry, } \bar{x} = 0, A = \frac{1}{2} \pi.$$

$$\begin{aligned}
 \bar{y} &= \frac{2}{\pi} \left(\frac{1}{2} \right) \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \frac{1}{\pi} \left[x - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{4}{3\pi} \\
 (\bar{x}, \bar{y}) &= \left(0, \frac{4}{3\pi} \right)
 \end{aligned}$$

$$71. \lim_{x \rightarrow 1} \left[\frac{(\ln x)^2}{x-1} \right] = \lim_{x \rightarrow 1} \left[\frac{2(1/x) \ln x}{1} \right] = 0$$

$$\begin{aligned}
 75. y &= \lim_{x \rightarrow \infty} (\ln x)^{2/x} \\
 \ln y &= \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[\frac{2/(x \ln x)}{1} \right] = 0
 \end{aligned}$$

Since $\ln y = 0, y = 1$.

$$77. \lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n}\right)^n = 1000 \lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n}\right)^n$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n}\right)^n.$$

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{0.09}{n}\right) = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{0.09}{1 + \left(\frac{0.09}{n}\right)} = 0.09$$

$$\text{Thus, } \ln y = 0.09 \Rightarrow y = e^{0.09} \text{ and } \lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17.$$

$$79. \int_0^{16} \frac{1}{\sqrt[4]{x}} dx = \lim_{b \rightarrow 0^+} \left[\frac{4}{3} x^{3/4} \right]_b^{16} = \frac{32}{3}$$

Converges

$$81. \int_1^{\infty} x^2 \ln x dx = \lim_{b \rightarrow \infty} \left[\frac{x^3}{9} (-1 + 3 \ln x) \right]_1^b = \infty$$

Diverges

$$\begin{aligned} 83. \int_0^{t_0} 500,000 e^{-0.05t} dt &= \left[\frac{500,000}{-0.05} e^{-0.05t} \right]_0^{t_0} \\ &= \frac{-500,000}{0.05} (e^{-0.05t_0} - 1) \\ &= 10,000,000 (1 - e^{-0.05t_0}) \end{aligned}$$

$$(a) t_0 = 20: \$6,321,205.59$$

$$(b) t_0 \rightarrow \infty: \$10,000,000$$

$$85. (a) P(13 \leq x < \infty) = \frac{1}{0.95\sqrt{2\pi}} \int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.4581$$

$$(b) P(15 \leq x < 20) = \frac{1}{0.95\sqrt{2\pi}} \int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.0135$$

Problem Solving for Chapter 7

$$1. (a) \int_{-1}^1 (1 - x^2) dx = \left[x - \frac{x^3}{3} \right]_{-1}^1 = 2 \left(1 - \frac{1}{3} \right) = \frac{4}{3}$$

$$\int_{-1}^1 (1 - x^2)^2 dx = \int_{-1}^1 (1 - 2x^2 + x^4) dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 = 2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15}$$

$$(b) \text{ Let } x = \sin u, dx = \cos u du, 1 - x^2 = 1 - \sin^2 u = \cos^2 u.$$

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{-\pi/2}^{\pi/2} (\cos^2 u)^n \cos u du \\ &= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} u du \\ &= 2 \left[\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{(2n)}{(2n+1)} \right] \quad (\text{Wallis's Formula}) \\ &= 2 \left[\frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)(2n+1)} \right] \\ &= \frac{2(2^n)(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \end{aligned}$$

$$\begin{aligned}
 3. \quad & \lim_{x \rightarrow \infty} \left(\frac{x+c}{x-c} \right)^x = 9 \\
 & \lim_{x \rightarrow \infty} x \ln \left(\frac{x+c}{x-c} \right) = \ln 9 \\
 & \lim_{x \rightarrow \infty} \frac{\ln(x+c) - \ln(x-c)}{1/x} = \ln 9 \\
 & \lim_{x \rightarrow \infty} \frac{\frac{1}{x+c} - \frac{1}{x-c}}{-\frac{1}{x^2}} = \ln 9 \\
 & \lim_{x \rightarrow \infty} \frac{-2c}{(x+c)(x-c)} (-x^2) = \ln 9 \\
 & \lim_{x \rightarrow \infty} \left(\frac{2cx^2}{x^2 - c^2} \right) = \ln 9 \\
 & 2c = \ln 9 \\
 & 2c = 2 \ln 3 \\
 & c = \ln 3
 \end{aligned}$$

$$5. \sin \theta = \frac{PB}{OP} = PB, \cos \theta = OB$$

$$AQ = \widehat{AP} = \theta$$

$$BR = OR + OB = OR + \cos \theta$$

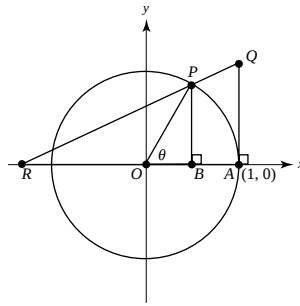
The triangles $\triangle AQR$ and $\triangle BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Rightarrow \frac{OR+1}{\theta} = \frac{OR+\cos \theta}{\sin \theta}$$

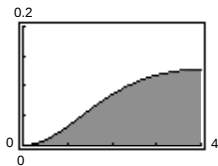
$$\sin \theta (OR) + \sin \theta = (OR)\theta + \theta \cos \theta$$

$$OR = \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta}$$

$$\begin{aligned}
 \lim_{\theta \rightarrow 0^+} OR &= \lim_{\theta \rightarrow 0^+} \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta + \cos \theta - \cos \theta}{\cos \theta - 1} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta}{\cos \theta - 1} \\
 &= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - \theta \cos \theta}{-\sin \theta} \\
 &= \lim_{\theta \rightarrow 0} \frac{\cos \theta + \cos \theta - \theta \sin \theta}{\cos \theta} \\
 &= 2
 \end{aligned}$$

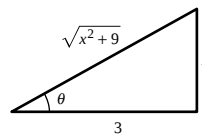


7. (a) Area
- ≈ 0.2986



- (b) Let
- $x = 3 \tan \theta$
- ,
- $dx = 3 \sec^2 \theta d\theta$
- ,
- $x^2 + 9 = 9 \sec^2 \theta$
- .

$$\begin{aligned} \int \frac{x^2}{(x^2 + 9)^{3/2}} dx &= \int \frac{9 \tan^2 \theta}{(9 \sec^2 \theta)^{3/2}} (3 \sec^2 \theta d\theta) \\ &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta \\ &= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \end{aligned}$$



$$\begin{aligned} \text{Area} &= \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \left[\ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\tan^{-1}(4/3)} \\ &= \left[\ln \left(\frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right) - \frac{x}{\sqrt{x^2 + 9}} \right]_0^4 \\ &= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \frac{4}{5} = \ln 3 - \frac{4}{5} \end{aligned}$$

- (c)
- $x = 3 \sinh u$
- ,
- $dx = 3 \cosh u du$
- ,
- $x^2 + 9 = 9 \sinh^2 u + 9 = 9 \cosh^2 u$

$$\begin{aligned} A &= \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \int_0^{\sinh^{-1}(4/3)} \frac{9 \sinh^2 u}{(9 \cosh^2 u)^{3/2}} (3 \cosh u du) \\ &= \int_0^{\sinh^{-1}(4/3)} \tanh^2 u du \\ &= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u) du \\ &= \left[u - \tanh u \right]_0^{\sinh^{-1}(4/3)} \\ &= \sinh^{-1} \left(\frac{4}{3} \right) - \tanh \left(\sinh^{-1} \left(\frac{4}{3} \right) \right) \\ &= \ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) - \tanh \left[\ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) \right] \\ &= \ln \left(\frac{4}{3} + \frac{5}{3} \right) - \tanh \left(\ln \left(\frac{4}{3} + \frac{5}{3} \right) \right) \\ &= \ln 3 - \tanh(\ln 3) \\ &= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)} \\ &= \ln 3 - \frac{4}{5} \end{aligned}$$

$$9. y = \ln(1 - x^2), y' = \frac{-2x}{1 - x^2}$$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

$$\begin{aligned} \text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2} dx \\ &= \int_0^{1/2} \left(\frac{1 + x^2}{1 - x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{1}{x + 1} + \frac{1}{1 - x}\right) dx \\ &= \left[-x + \ln(1 + x) - \ln(1 - x)\right]_0^{1/2} \\ &= \left(-\frac{1}{2} + \ln \frac{3}{2} - \ln \frac{1}{2}\right) \\ &= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\ &= \ln 3 - \frac{1}{2} \approx 0.5986 \end{aligned}$$

$$11. \text{ Consider } \int \frac{1}{\ln x} dx.$$

$$\text{Let } u = \ln x, du = \frac{1}{x} dx, x = e^u. \text{ Then } \int \frac{1}{\ln x} dx = \int \frac{1}{u} e^u du = \int \frac{e^u}{u} du.$$

$$\text{If } \int \frac{1}{\ln x} dx \text{ were elementary, then } \int \frac{e^u}{u} du \text{ would be too, which is false.}$$

$$\text{Hence, } \int \frac{1}{\ln x} dx \text{ is not elementary.}$$

$$13. x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$$a = -c, b = d = 1, a = \sqrt{2}$$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\begin{aligned} \int_0^1 \frac{1}{x^4 + 1} dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1} dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1} dx \\ &= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} dx \\ &= \frac{\sqrt{2}}{4} \left[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1) \right]_0^1 + \frac{\sqrt{2}}{8} \left[\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1) \right]_0^1 \\ &= \frac{\sqrt{2}}{4} [\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1)] + \frac{\sqrt{2}}{8} [\ln(2 + \sqrt{2}) - \ln(2 - \sqrt{2})] - \frac{\sqrt{2}}{4} \left[\frac{\pi}{4} - \frac{\pi}{4} \right] - \frac{\sqrt{2}}{8} [0] \\ &\approx 0.5554 + 0.3116 \\ &\approx 0.8670 \end{aligned}$$

15. Using a graphing utility,

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) = \infty$$

$$(b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right) = 0$$

$$(c) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) \approx -\frac{2}{3}.$$

Analytically,

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) = \infty + \infty = \infty$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \rightarrow 0^+} \frac{x \cot x - 1}{x} = \lim_{x \rightarrow 0^+} \frac{x \cos x - \sin x}{x \sin x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{-x \sin x}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} = 0. \end{aligned}$$

$$\begin{aligned} (c) \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) &= \cot^2 x - \frac{1}{x^2} \\ &= \frac{x^2 \cot^2 x - 1}{x^2} \\ \lim_{x \rightarrow 0^+} \frac{x^2 \cot^2 x - 1}{x^2} &= \lim_{x \rightarrow 0^+} \frac{2x \cot^2 x - 2x^2 \cot x \csc^2 x}{2x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cot^2 x - x \cot x \csc^2 x}{1} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos^2 x \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{(1 - \sin^2 x) \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} - 1 \end{aligned}$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} &= \lim_{x \rightarrow 0^+} \frac{\cos x - \cos x + x \sin x}{3 \sin^2 x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{3 \sin x \cdot \cos x} \\ &= \lim_{x \rightarrow 0^+} \left(\frac{x}{\sin x} \right) \frac{1}{3 \cos x} = \frac{1}{3}. \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) = \frac{1}{3} - 1 = -\frac{2}{3}.$$

The form $0 \cdot \infty$ is indeterminant.

$$17. \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{P_1}{x} + \frac{P_2}{x-1} + \frac{P_3}{x+4} + \frac{P_4}{x-3} \Rightarrow c_1 = 0, c_2 = 1, c_3 = -4, c_4 = 3$$

$$N(x) = x^3 - 3x^2 + 1$$

$$D'(x) = 4x^3 - 26x + 12$$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

$$\text{Thus, } \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{1/12}{x} + \frac{1/10}{x-1} + \frac{111/140}{x+4} + \frac{1/42}{x-3}.$$

19. By parts,

$$\begin{aligned} \int_a^b f(x)g''(x) dx &= \left[f(x)g'(x) \right]_a^b - \int_a^b f'(x)g'(x) dx \\ &= - \int_a^b f'(x)g'(x) dx \\ &= \left[-f'(x)g(x) \right]_a^b + \int_a^b g(x)f''(x) dx \\ &= \int_a^b f''(x)g(x) dx. \end{aligned}$$

CHAPTER 8

Infinite Series

Section 8.1	Sequences	369
Section 8.2	Series and Convergence	373
Section 8.3	The Integral Test and p -Series	378
Section 8.4	Comparisons of Series	381
Section 8.5	Alternating Series	385
Section 8.6	The Ratio and Root Tests	389
Section 8.7	Taylor Polynomials and Approximations	393
Section 8.8	Power Series	398
Section 8.9	Representation of Functions by Power Series	403
Section 8.10	Taylor and Maclaurin Series	408
	Review Exercises	414
	Problem Solving	421

CHAPTER 8

Infinite Series

Section 8.1 Sequences

Solutions to Even-Numbered Exercises

$$2. a_n = \frac{2n}{n+3}$$

$$a_1 = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{4}{5}$$

$$a_3 = \frac{6}{6} = 1$$

$$a_4 = \frac{8}{7}$$

$$a_5 = \frac{10}{8} = \frac{5}{4}$$

$$4. a_n = \left(-\frac{2}{3}\right)^n$$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

$$a_4 = \frac{16}{81}$$

$$a_5 = -\frac{32}{243}$$

$$6. a_n = \cos \frac{n\pi}{2}$$

$$a_1 = \cos \frac{\pi}{2} = 0$$

$$a_2 = \cos \pi = -1$$

$$a_3 = \cos \frac{3\pi}{2} = 0$$

$$a_4 = \cos 2\pi = 1$$

$$a_5 = \cos \frac{5\pi}{2} = 0$$

$$8. a_n = (-1)^{n+1} \left(\frac{2}{n}\right)$$

$$a_1 = \frac{2}{1} = 2$$

$$a_2 = -\frac{2}{2} = -1$$

$$a_3 = \frac{2}{3}$$

$$a_4 = -\frac{2}{4} = -\frac{1}{2}$$

$$a_5 = \frac{2}{5}$$

$$10. a_n = 10 + \frac{2}{n} + \frac{6}{n^2}$$

$$a_1 = 10 + 2 + 6 = 18$$

$$a_2 = 10 + 1 + \frac{3}{2} = \frac{25}{2}$$

$$a_3 = 10 + \frac{2}{3} + \frac{2}{3} = \frac{34}{3}$$

$$a_4 = 10 + \frac{1}{2} + \frac{3}{8} = \frac{87}{8}$$

$$a_5 = 10 + \frac{2}{5} + \frac{6}{25} = \frac{266}{25}$$

$$12. a_n = \frac{3n!}{(n-1)!} = 3n$$

$$a_1 = 3(1) = 3$$

$$a_2 = 3(2) = 6$$

$$a_3 = 3(3) = 9$$

$$a_4 = 3(4) = 12$$

$$a_5 = 3(5) = 15$$

$$14. a_1 = 4, a_{k+1} = \left(\frac{k+1}{2}\right)a_k$$

$$a_2 = \left(\frac{1+1}{2}\right)a_1 = 4$$

$$a_3 = \left(\frac{2+1}{2}\right)a_2 = 6$$

$$a_4 = \left(\frac{3+1}{2}\right)a_3 = 12$$

$$a_5 = \left(\frac{4+1}{2}\right)a_4 = 30$$

$$16. a_1 = 6, a_{k+1} = \frac{1}{3}a_k^2$$

$$a_2 = \frac{1}{3}a_1^2 = \frac{1}{3}(6^2) = 12$$

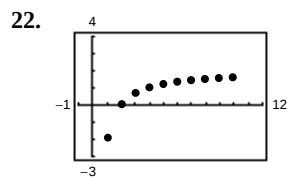
$$a_3 = \frac{1}{3}a_2^2 = \frac{1}{3}(12^2) = 48$$

$$a_4 = \frac{1}{3}a_3^2 = \frac{1}{3}(48^2) = 768$$

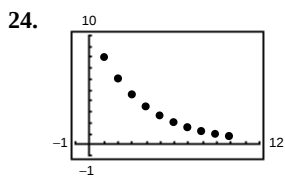
$$a_5 = \frac{1}{3}a_4^2 = \frac{1}{3}(768^2) = 196,608$$

18. Because the sequence tends to 8 as n tends to infinity, it matches (a).

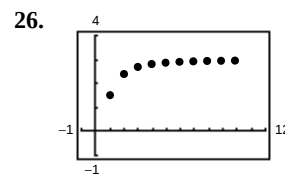
20. This sequence increases for a few terms, then decreases $a_2 = \frac{16}{2} = 8$. Matches (b).



$$a_n = 2 - \frac{4}{n}, n = 1, \dots, 10$$



$$a_n = 8(0.75)^{n-1}, n = 1, 2, \dots, 10$$



$$a_n = \frac{3n^2}{n^2 + 1}, n = 1, \dots, 10$$

28. $a_n = \frac{n+6}{2}$

$$a_5 = \frac{5+6}{2} = \frac{11}{2}$$

$$a_6 = \frac{6+6}{2} = 6$$

30. $a_{n+1} = 2a_n, a_1 = 5$

$$a_5 = 2(40) = 80$$

$$a_6 = 2(80) = 160$$

32. $\frac{25!}{23!} = \frac{23!(24)(25)}{23!}$
 $= (24)(25) = 600$

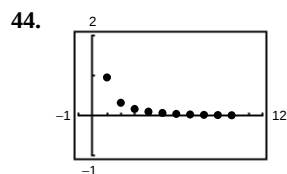
34. $\frac{(n+2)!}{n!} = \frac{n!(n+1)(n+2)}{n!}$
 $= (n+1)(n+2)$

36. $\frac{(2n+2)!}{(2n)!} = \frac{(2n)!(2n+1)(2n+2)}{(2n)!}$
 $= (2n+1)(2n+2)$

38. $\lim_{n \rightarrow \infty} \left(5 - \frac{1}{n^2}\right) = 5 - 0 = 5$

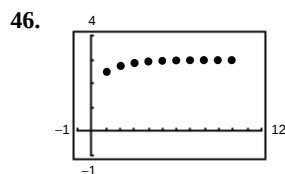
40. $\lim_{n \rightarrow \infty} \frac{5n}{\sqrt{n^2 + 4}} = \lim_{n \rightarrow \infty} \frac{5}{\sqrt{1 + (4/n^2)}}$
 $= \frac{5}{1} = 5$

42. $\lim_{n \rightarrow \infty} \cos\left(\frac{2}{n}\right) = 1$



The graph seems to indicate that the sequence converges to 0. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^{3/2}} = \lim_{x \rightarrow \infty} \frac{1}{x^{3/2}} = 0.$$



The graph seems to indicate that the sequence converges to 3. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(3 - \frac{1}{2^n}\right) = 3 - 0 = 3.$$

48. $\lim_{n \rightarrow \infty} [1 + (-1)^n]$

does not exist, (alternates between 0 and 2), diverges.

50. $\lim_{n \rightarrow \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1} = 1$, converges

52. $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n^2} = 0$, converges

54. $\lim_{n \rightarrow \infty} \frac{\ln \sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{1/2 \ln n}{n}$
 $= \lim_{n \rightarrow \infty} \frac{1}{2n} = 0$, converges

(L'Hôpital's Rule)

56. $\lim_{n \rightarrow \infty} (0.5)^n = 0$, converges

58. $\lim_{n \rightarrow \infty} \frac{(n-2)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n(n-1)} = 0$, converges

$$60. \lim_{n \rightarrow \infty} \left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1} \right) = \lim_{n \rightarrow \infty} \frac{-2n^2}{4n^2-1} = -\frac{1}{2}, \text{ converges}$$

$$62. a_n = n \sin \frac{1}{n}$$

$$\text{Let } f(x) = x \sin \frac{1}{x}.$$

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} = \lim_{x \rightarrow \infty} \cos \frac{1}{x} = \cos 0 = 1 \text{ (L'Hôpital's Rule)}$$

or,

$$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{y \rightarrow 0^+} \frac{\sin(y)}{y} = 1. \text{ Therefore } \lim_{n \rightarrow \infty} n \sin \frac{1}{n} = 1.$$

$$64. \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1, \text{ converges}$$

$$66. \lim_{n \rightarrow \infty} \frac{\cos \pi n}{n^2} = 0, \text{ converges}$$

$$68. a_n = 4n - 1$$

$$70. a_n = \frac{(-1)^{n-1}}{n^2}$$

$$72. a_n = \frac{n+2}{3n-1}$$

$$74. a_n = (-1)^n \frac{3^{n-2}}{2^{n-1}}$$

$$76. a_n = 1 + \frac{2^n - 1}{2^n} \\ = \frac{2^{n+1} - 1}{2^n}$$

$$78. a_n = \frac{1}{n!}$$

$$80. a_n = \frac{x^{n-1}}{(n-1)!}$$

$$82. \text{ Let } f(x) = \frac{3x}{x+2}. \text{ Then } f'(x) = \frac{6}{(x+2)^2}.$$

Thus, f is increasing which implies $\{a_n\}$ is increasing.

$$|a_n| < 3, \text{ bounded}$$

$$84. a_n = ne^{-n/2}$$

$$a_1 = 0.6065$$

$$a_2 = 0.7358$$

$$a_3 = 0.6694$$

Not monotonic; $|a_n| \leq 0.7358$, bounded

$$86. a_n = \left(-\frac{2}{3}\right)^n$$

$$a_1 = -\frac{2}{3}$$

$$a_2 = \frac{4}{9}$$

$$a_3 = -\frac{8}{27}$$

Not monotonic; $|a_n| \leq \frac{2}{3}$, bounded

$$88. a_n = \left(\frac{3}{2}\right)^n < \left(\frac{3}{2}\right)^{n+1} = a_{n+1}$$

Monotonic; $\lim_{n \rightarrow \infty} a_n = \infty$, not bounded

$$90. a_n = \frac{\cos n}{n}$$

$$a_1 = 0.5403$$

$$a_2 = -0.2081$$

$$a_3 = -0.3230$$

$$a_4 = -0.1634$$

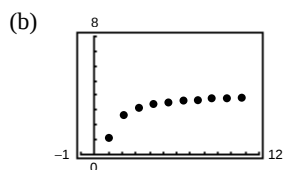
Not monotonic; $|a_n| \leq 1$, bounded

92. (a) $a_n = 4 - \frac{3}{n}$

$$\left| 4 - \frac{3}{n} \right| < 4 \Rightarrow \text{bounded}$$

$$a_n = 4 - \frac{3}{n} < 3 - \frac{4}{n+1} = a_{n+1} \Rightarrow \text{monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(4 - \frac{3}{n} \right) = 4$$

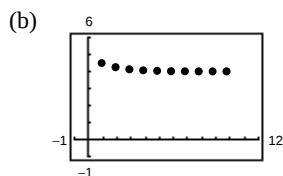
94. (a) $a_n = 4 + \frac{1}{2^n}$

$$\left| 4 + \frac{1}{2^n} \right| \leq 4.5 \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = 4 + \frac{1}{2^n} > 4 + \frac{1}{2^{n+1}}$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(4 + \frac{1}{2^n} \right) = 4$$

96. $A_n = 100(101)[(1.01)^n - 1]$

(a) $A_1 = \$101.00$

(b) $A_{60} = \$8248.64$

$A_2 = \$203.01$

(c) $A_{240} = \$99,914.79$

$A_3 = \$306.04$

$A_4 = \$410.10$

$A_5 = \$515.20$

$A_6 = \$621.35$

98. The first sequence because every other point is below the x -axis.

100. Impossible. The sequence converges by Theorem 8.5.

102. Impossible. An unbounded sequence diverges.

104. $P_n = 16,000(1.045)^n$

$P_1 = \$16,720.00$

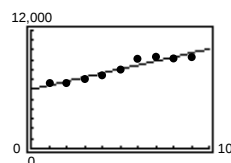
$P_2 = \$17,472.40$

$P_3 \approx \$18,258.66$

$P_4 \approx \$19,080.30$

$P_5 \approx \$19,938.91$

106. (a) $a_n = 410.9212n + 6003.8545$



(b) For 2004, $n = 14$ and $a_n = 11,757$, or \$11,757,000,000.

108. $a_n = \left(1 + \frac{1}{n} \right)^n$

$a_1 = 2.0000$

$a_2 = 2.2500$

$a_3 \approx 2.3704$

$a_4 \approx 2.4414$

$a_5 \approx 2.4883$

$a_6 \approx 2.5216$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$$

110. Since

$$\lim_{n \rightarrow \infty} s_n = L > 0,$$

there exists for each $\epsilon > 0$, an integer N such that $|s_n - L| < \epsilon$ for every $n > N$. Let $\epsilon = L > 0$ and we have,

$$|s_n - L| < L, -L < s_n - L < L, \text{ or } 0 < s_n < 2L$$

for each $n > N$.

112. If $\{a_n\}$ is bounded, monotonic and nonincreasing, then $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. Then

$$-a_1 \leq -a_2 \leq -a_3 \leq \cdots \leq -a_n \leq \cdots$$

is a bounded, monotonic, nondecreasing sequence which converges by the first half of the theorem. Since $\{-a_n\}$ converges, then so does $\{a_n\}$.

114. True

116. True

118. $x_0 = 1, x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, n = 1, 2, \dots$

$$x_1 = 1.5 \quad x_6 = 1.414214$$

$$x_2 = 1.41667 \quad x_7 = 1.414214$$

$$x_3 = 1.414216 \quad x_8 = 1.414114$$

$$x_4 = 1.414214 \quad x_9 = 1.414214$$

$$x_5 = 1.414214 \quad x_{10} = 1.414214$$

The limit of the sequence appears to be $\sqrt{2}$. In fact, this sequence is Newton's Method applied to $f(x) = x^2 - 2$.

Section 8.2 Series and Convergence

2. $S_1 = \frac{1}{6} \approx 0.1667$

$$S_2 = \frac{1}{6} + \frac{1}{6} \approx 0.3333$$

$$S_3 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} \approx 0.4833$$

$$S_4 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} \approx 0.6167$$

$$S_5 = \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{2}{15} + \frac{5}{42} \approx 0.7357$$

4. $S_1 = 1$

$$S_2 = 1 + \frac{1}{3} \approx 1.3333$$

$$S_3 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.5333$$

$$S_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} \approx 1.6444$$

$$S_5 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} \approx 1.7354$$

6. $S_1 = 1$

$$S_2 = 1 - \frac{1}{2} = 0.5$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{6} \approx 0.6667$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \approx 0.6250$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \approx 0.6333$$

8. $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$ Geometric series

$$r = \frac{4}{3} > 1$$

Diverges by Theorem 8.6

10. $\sum_{n=0}^{\infty} 2(-1.03)^n$ Geometric series

$$|r| = 1.03 > 1$$

Diverges by Theorem 8.6

12. $\sum_{n=1}^{\infty} \frac{n}{2n+3}$

$$\lim_{n \rightarrow \infty} \frac{n}{2n+3} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

14. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 8.9

16. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \infty$$

Diverges by Theorem 8.9

$$18. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

$$S_0 = 1, S_1 = \frac{5}{3}, S_2 \approx 2.11, \dots$$

Matches graph (b).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - 2/3} = \frac{1}{1/3} = 3$$

$$22. \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)} \right) = \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{8} \right) + \left(\frac{1}{6} - \frac{1}{10} \right) + \left(\frac{1}{8} - \frac{1}{12} \right) + \left(\frac{1}{10} - \frac{1}{14} \right) + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \right] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$24. \sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n$$

Geometric series with $|r| = \left| -\frac{1}{2} \right| < 1$.

Converges by Theorem 8.6

$$20. \sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17}{3} \left[1 - \frac{8}{9} + \frac{64}{81} - \cdots \right]$$

$$S_0 = \frac{17}{3}, S_1 \approx 0.63, S_3 \approx 5.1, \dots$$

Matches (d).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{17}{3} \left(-\frac{8}{9}\right)^n = \frac{17/3}{1 - (-8/9)} = \frac{17/3}{17/9} = 3$$

$$26. \sum_{n=0}^{\infty} (-0.6)^n$$

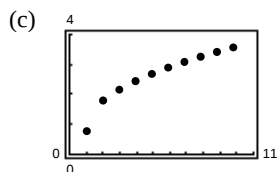
Geometric series with $|r| = |-0.6| < 1$.

Converges by Theorem 8.6

$$\begin{aligned} 28. (a) \sum_{n=1}^{\infty} \frac{4}{n(n+4)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+4} \right) \\ &= \left(1 - \frac{1}{5} \right) + \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{3} - \frac{1}{7} \right) + \left(\frac{1}{4} - \frac{1}{8} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \left(\frac{1}{6} - \frac{1}{10} \right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.0833 \end{aligned}$$

(b)

n	5	10	20	50	100
S_n	1.5377	1.7607	1.9051	2.0071	2.0443

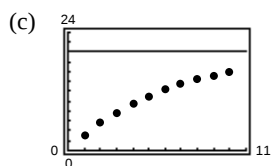


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$30. (a) \sum_{n=1}^{\infty} 3(0.85)^{n-1} = \frac{3}{1 - 0.85} = 20 \quad (\text{Geometric series})$$

(b)

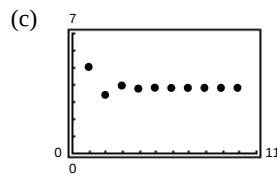
n	5	10	20	50	100
S_n	11.1259	16.0625	19.2248	19.9941	19.99998



32. (a) $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1} = \frac{5}{1 - (-1/3)} = \frac{15}{4} = 3.75$

(b)

n	5	10	20	50	100
S_n	3.7654	3.7499	3.7500	3.7500	3.7500



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

34. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = 2 \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots \right] = 2 \left(1 + \frac{1}{2} \right) = 3$

36. $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right) = \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{9} \right) + \cdots \right] = \frac{1}{2} \left(\frac{1}{3} \right) = \frac{1}{6}$

38. $\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n = \frac{6}{1 - (4/5)} = 30$ (Geometric)

40. $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^n = \frac{2}{1 - (-2/3)} = \frac{6}{5}$

42. $\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1 - (3/4)} = 32$

44. $\sum_{n=0}^{\infty} 4\left(-\frac{1}{2}\right)^n = \frac{4}{1 - (-1/2)} = \frac{8}{3}$

46. $\sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n] = \sum_{n=0}^{\infty} \left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n - 2 = \frac{1}{1 - (7/10)} + \frac{1}{1 - (9/10)} - 2 = \frac{10}{3} + 10 - 2 = \frac{34}{3}$

48. $0.81\overline{81} = \sum_{n=0}^{\infty} \frac{81}{100} \left(\frac{1}{100} \right)^n$

Geometric series with $a = \frac{81}{100}$ and $r = \frac{1}{100}$

$$S = \frac{a}{1 - r} = \frac{81/100}{1 - (1/100)} = \frac{81}{99} = \frac{9}{11}$$

50. $0.215\overline{15} = \frac{1}{5} + \sum_{n=0}^{\infty} \frac{3}{200} \left(\frac{1}{100} \right)^n$

Geometric series with $a = \frac{3}{200}$ and $r = \frac{1}{100}$

$$S = \frac{1}{5} + \frac{a}{1 - r} = \frac{1}{5} + \frac{3/200}{99/100} = \frac{71}{330}$$

52. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$

$$\lim_{n \rightarrow \infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

54. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$

$$= \frac{1}{3} \left[\left(1 - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \left(\frac{1}{6} - \frac{1}{9} \right) + \cdots \right]$$

$$= \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{11}{18}, \text{ converges}$$

56. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n \rightarrow \infty} \frac{3^n}{n^3} = \lim_{n \rightarrow \infty} \frac{(\ln 2)3^n}{3n^2} = \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 3^n}{6n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^3 3^n}{6} = \infty$$

(by L'Hôpital's Rule) Diverges by Theorem 8.9

$$58. \sum_{n=0}^{\infty} \frac{1}{4^n}$$

Geometric series with $r = \frac{1}{4}$

Converges by Theorem 8.6

$$60. \sum_{n=1}^{\infty} \frac{2^n}{100}$$

Geometric series with $r = 2$

Diverges by Theorem 8.6

$$62. \sum_{n=1}^{\infty} \left(1 + \frac{k}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = e^k \neq 0$$

Diverges by Theorem 8.9

64. $\lim_{n \rightarrow \infty} a_n = 5$ means that the limit of the sequence $\{a_n\}$ is 5.

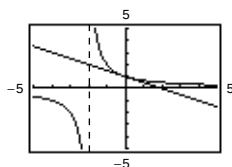
$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = 5$ means that the limit of the partial sums is 5.

66. If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

68. (a) $(-x/2)$ is the common ratio.

$$(c) y_1 = \frac{2}{2+x}$$

$$y_2 = 1 - \frac{x}{2}$$



$$(b) 1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \frac{1}{1 - (-x/2)}$$

$$= \frac{2}{2+x}, |x| < 2$$

Geometric series:

$$a = 1, r = -\frac{x}{2}, \left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$$

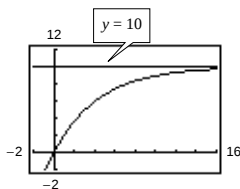
$$70. f(x) = 2 \left[\frac{1 - 0.8^x}{1 - 0.8} \right]$$

Horizontal asymptote: $y = 10$

$$\sum_{n=0}^{\infty} 2 \left(\frac{4}{5}\right)^n$$

$$S = \frac{2}{1 - (4/5)} = 10$$

The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.



$$72. \frac{1}{2^n} < 0.0001$$

$$10,000 < 2^n$$

This inequality is true when $n = 14$.

$$(0.01)^n < 0.0001$$

$$10,000 < 10^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

$$74. V(t) = 225,000(1 - 0.3)^n = (0.7)^n(225,000)$$

$$V(5) = (0.7)^5(225,000) = \$37,815.75$$

$$76. \sum_{i=0}^{n-1} 100(0.60)^i = \frac{100[1 - 0.6^n]}{1 - 0.6}$$

$$= 250(1 - 0.6^n) \quad \text{million dollars}$$

$$\text{Sum} = 250 \text{ million dollars}$$

78. The ball in Exercise 77 takes the following times for each fall.

$$s_1 = -16t^2 + 16$$

$$s_1 = 0 \text{ if } t = 1$$

$$s_2 = -16t^2 + 16(0.81)$$

$$s_2 = 0 \text{ if } t = 0.9$$

$$s_3 = -16t^2 + 16(0.81)^2$$

$$s_3 = 0 \text{ if } t = (0.9)^2$$

$\vdots$

$\vdots$

$$s_n = -16t^2 + 16(0.81)^{n-1}$$

$$s_n = 0 \text{ if } t = (0.9)^{n-1}$$

Beginning with s_2 , the ball takes the same amount of time to bounce up as it takes to fall. The total elapsed time before the ball comes to rest is

$$t = 1 + 2 \sum_{n=1}^{\infty} (0.9)^n = -1 + 2 \sum_{n=0}^{\infty} (0.9)^n$$

$$= -1 + \frac{2}{1 - 0.9} = 19 \text{ seconds.}$$

$$80. P(n) = \frac{1}{3} \left(\frac{2}{3}\right)^n$$

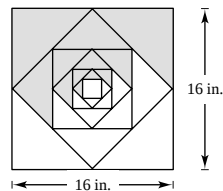
$$P(2) = \frac{1}{3} \left(\frac{2}{3}\right)^2 = \frac{4}{27}$$

$$\sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$$

82. (a) $64 + 32 + 16 + 8 + 4 + 2 = 126 \text{ in.}^2$

(b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1 - (1/2)} = 128 \text{ in.}^2$

Note: This is one-half of the area of the original square!



84. Surface area $= 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2 \cdot 4\pi\left(\frac{1}{9}\right)^2 + \cdots = 4[\pi + \pi + \cdots] = \infty$

86.
$$\sum_{n=0}^{12t-1} P\left(1 + \frac{r}{12}\right)^n = \frac{P\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]}{1 - \left(1 + \frac{r}{12}\right)}$$

$$= P\left(-\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{12t}\right]$$

$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$\sum_{n=0}^{12t-1} P(e^{r/12})^n = \frac{P(1 - (e^{r/12})^{12t})}{1 - e^{r/12}} = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

90. $P = 20, r = 0.06, t = 50$

(a) $A = 20\left(\frac{12}{0.06}\right)\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right] \approx \$75,743.82$

(b) $A = \frac{20(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76,151.45$

88. $P = 75, r = 0.05, t = 25$

(a) $A = 75\left(\frac{12}{0.05}\right)\left[\left(1 + \frac{0.05}{12}\right)^{12(25)} - 1\right] \approx \$44,663.23$

(b) $A = \frac{75(e^{0.05(25)} - 1)}{e^{0.05/12} - 1} \approx \$44,732.85$

92. $T = 40,000 + 40,000(1.04) + \cdots + 40,000(1.04)^{39}$

$$= \sum_{n=0}^{39} 40,000(1.04)^n$$

$$= 40,000\left(\frac{1 - 1.04^{40}}{1 - 1.04}\right)$$

$$\approx \$3,801,020$$

94. $x = 0.a_1a_2a_3 \cdots \overline{a_ka_1a_2a_3 \cdots a_k}$

$$= 0.a_1a_2a_3 \cdots a_k \left[1 + \frac{1}{10^k} + \left(\frac{1}{10^k}\right)^2 + \left(\frac{1}{10^k}\right)^3 + \cdots\right]$$

$$= 0.a_1a_2a_3 \cdots a_k \sum_{n=0}^{\infty} \left(\frac{1}{10^k}\right)^n$$

$$= 0.a_1a_2a_3 \cdots a_k \left[\frac{1}{1 - (1/10^k)}\right] = \text{a rational number}$$

96. Let $\{S_n\}$ be the sequence of partial sums for the convergent series $\sum_{n=1}^{\infty} a_n = L$. Then

$$\lim_{n \rightarrow \infty} S_n = L \text{ and since } R_n = \sum_{k=n+1}^{\infty} a_k = L - S_n,$$

we have

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} (L - S_n) = \lim_{n \rightarrow \infty} L - \lim_{n \rightarrow \infty} S_n = L - L = 0.$$

98. If $\sum (a_n + b_n)$ converged, then $\sum (a_n + b_n) - \sum a_n = \sum b_n$ would converge, which is a contradiction.

Thus, $\sum (a_n + b_n)$ diverges.

100. True

102. True; $\lim_{n \rightarrow \infty} \frac{n}{1000(n+1)} = \frac{1}{1000} \neq 0$

$$104. \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \sum_{n=0}^{\infty} \frac{1}{r} \left(\frac{1}{r}\right)^n = \frac{1/r}{1 - (1/r)} = \frac{1}{r-1} \quad \left(\text{since } \left|\frac{1}{r}\right| < 1\right)$$

This is a geometric series which converges if $\left|\frac{1}{r}\right| < 1 \Leftrightarrow |r| > 1$.

Section 8.3 The Integral Test and p -Series

$$2. \sum_{n=1}^{\infty} \frac{2}{3n+5}$$

$$\text{Let } f(x) = \frac{2}{3x+5}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{2}{3x+5} dx = \left[\frac{2}{3} \ln(3x+5) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$4. \sum_{n=1}^{\infty} ne^{-n/2}$$

$$\text{Let } f(x) = xe^{-x/2}.$$

f is positive, continuous, and decreasing for $x \geq 3$.

$$\text{Since } f'(x) = \frac{2-x}{2e^{x/2}} < 0 \text{ for } x \geq 3.$$

$$\int_3^{\infty} xe^{-x/2} dx = \left[-2(x+2)e^{-x/2} \right]_3^{\infty} = 10e^{-3/2}$$

Converges by Theorem 8.10

$$6. \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$$\text{Let } f(x) = \frac{1}{2x+1}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x+1} dx = \left[\ln \sqrt{2x+1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$8. \sum_{n=1}^{\infty} \frac{n}{n^2+3}$$

$$\text{Let } f(x) = \frac{x}{x^2+3}.$$

$f(x)$ is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{3-x^2}{(x^2+3)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{x}{x^2+3} dx = \left[\ln \sqrt{x^2+3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$10. \sum_{n=1}^{\infty} n^k e^{-n}$$

$$\text{Let } f(x) = \frac{x^k}{e^x}.$$

f is positive, continuous, and decreasing for $x > k$ since

$$f'(x) = \frac{x^{k-1}(k-x)}{e^x} < 0$$

for $x > k$. We use integration by parts.

$$\begin{aligned} \int_1^{\infty} x^k e^{-x} dx &= \left[-x^k e^{-x} \right]_1^{\infty} + k \int_1^{\infty} x^{k-1} e^{-x} dx \\ &= \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e} \end{aligned}$$

Converges by Theorem 8.10

$$12. \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

$$\text{Let } f(x) = \frac{1}{x^{1/3}}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^{1/3}} dx = \left[\frac{3}{2} x^{2/3} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

14. $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

 Convergent p -series with $p = \frac{5}{3} > 1$

16. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

 Convergent p -series with $p = 2 > 1$

18. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

 Divergent p -series with $p = \frac{2}{3} < 1$

20. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

 Convergent p -series with $p = \pi > 1$

22. $\sum_{n=1}^{\infty} \frac{2}{n} = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \cdots$

$S_1 = 2$

$S_2 = 3$

$S_3 \approx 3.67$

Matches (d)

Diverges—harmonic series

24. $\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 + \frac{2}{2^2} + \frac{2}{3^2} + \cdots$

$S_1 = 2$

$S_2 = 2.5$

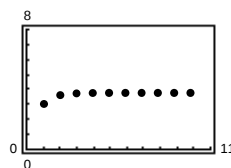
$S_3 \approx 2.722$

Matches (c)

 Converges— p -series with $p = 2 > 1$.

26. (a)

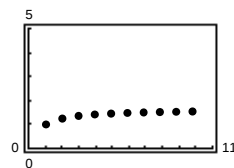
n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75



The partial sums approach the sum 3.75 very rapidly.

(b)

n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635


 The partial sums approach the sum $\frac{\pi^2}{6} \approx 1.6449$ slower than the series in part (a).

28. $\xi(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \frac{1}{n^x}$

 Converges for $x > 1$ by Theorem 8.11.

30. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

 If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{\ln x}{x^p} dx = \int_2^{\infty} x^{-p} \ln x dx = \left[\frac{x^{-p+1}}{(-p+1)^2} [-1 + (-p+1) \ln x] \right]_2^{\infty}. \text{ (Use Integration by Parts.)}$$

 Converges for $-p + 1 < 0$ or $p > 1$.

 32. A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p -series, $p > 0$.

 The p -series converges if $p > 1$ and diverges if $0 < p \leq 1$.

 34. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

36. From Exercise 35, we have:

$$0 \leq S - S_N \leq \int_N^\infty f(x) dx$$

$$S_N \leq S \leq S_N + \int_N^\infty f(x) dx$$

$$\sum_{n=1}^N a_n \leq S \leq \sum_{n=1}^N a_n + \int_N^\infty f(x) dx$$

40. $S_{10} = \frac{1}{2(\ln 2)^3} + \frac{1}{3(\ln 3)^3} + \frac{1}{4(\ln 4)^3} + \cdots + \frac{1}{11(\ln 11)^3} \approx 1.9821$

$$R_{10} \leq \int_{10}^\infty \frac{1}{(x+1)[\ln(x+1)]^3} dx = \left[-\frac{1}{2[\ln(x+1)]^2} \right]_{10}^\infty = \frac{1}{2(\ln 11)^3} \approx 0.0870$$

$$1.9821 \leq \sum_{n=1}^\infty \frac{1}{(n+1)[\ln(n+1)]^3} \leq 1.9821 + 0.0870 = 2.0691$$

42. $S_4 = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} \approx 0.5713$

$$R_4 \leq \int_4^\infty e^{-x} dx = \left[-e^{-x} \right]_4^\infty \approx 0.0183$$

$$0.5713 \leq \sum_{n=0}^\infty e^{-n} \leq 0.5713 + 0.0183 = 0.5896$$

46. $R_N \leq \int_N^\infty e^{-x/2} dx = \left[-2e^{-x/2} \right]_N^\infty = \frac{2}{e^{N/2}} < 0.001$

$$\frac{2}{e^{N/2}} < 0.001$$

$$e^{N/2} > 2000$$

$$\frac{N}{2} > \ln 2000$$

$$N > 2 \ln 2000 \approx 15.2$$

$$N \geq 16$$

38. $S_4 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363$

$$R_4 \leq \int_4^\infty \frac{1}{x^5} dx = \left[-\frac{1}{4x^4} \right]_4^\infty \approx 0.0010$$

$$1.0363 \leq \sum_{n=1}^\infty \frac{1}{n^5} \leq 1.0363 + 0.0010 = 1.0373$$

44. $0 \leq R_N \leq \int_N^\infty \frac{1}{x^{3/2}} dx = \left[-\frac{2}{x^{1/2}} \right]_N^\infty = \frac{2}{\sqrt{N}} < 0.001$

$$N^{-1/2} < 0.0005$$

$$\sqrt{N} > 2000$$

$$N \geq 4,000,000$$

48. $R_n \leq \int_N^\infty \frac{2}{x^2 + 5} dx = 2 \left[\frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) \right]_N^\infty$

$$= \frac{2}{\sqrt{5}} \left(\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) \right) < 0.001$$

$$\frac{\pi}{2} - \arctan\left(\frac{N}{\sqrt{5}}\right) < 0.001118$$

$$1.56968 < \arctan\left(\frac{N}{\sqrt{5}}\right)$$

$$\frac{N}{\sqrt{5}} > \tan 1.56968$$

$$N \geq 2004$$

50. (a) $\int_{10}^\infty \frac{1}{x^p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_{10}^\infty = \frac{1}{(p-1)10^{p-1}}, p > 1$

(b) $f(x) = \frac{1}{x^p}$

$$R_{10}(p) = \sum_{n=11}^\infty \frac{1}{n^p} \leq \text{Area under the graph of } f \text{ over the interval } [10, \infty)$$

(c) The horizontal asymptote is $y = 0$. As n increases, the error decreases.

52. $\sum_{n=2}^\infty \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^\infty \ln\left(\frac{n^2-1}{n^2}\right) = \sum_{n=2}^\infty \ln \frac{(n+1)(n-1)}{n^2} = \sum_{n=2}^\infty [\ln(n+1) + \ln(n-1) - 2 \ln n]$

$$= \ln 3 + \ln 1 - 2 \ln 2 + (\ln 4 + \ln 2 - 2 \ln 3) + (\ln 5 + \ln 3 - 2 \ln 4) + (\ln 6 + \ln 4 - 2 \ln 5)$$

$$+ (\ln 7 + \ln 5 - 2 \ln 6) + (\ln 8 + \ln 6 - 2 \ln 7) + (\ln 9 + \ln 7 - 2 \ln 8) + \cdots = -\ln 2$$

$$54. \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

$$\text{Let } f(x) = \frac{1}{x\sqrt{x^2-1}}.$$

f is positive, continuous, and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx = \left[\operatorname{arcsec} x \right]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3}$$

Converges by Theorem 8.10

$$56. 3 \sum_{n=1}^{\infty} \frac{1}{n^{0.95}}$$

p -series with $p = 0.95$

Diverges by Theorem 8.11

$$58. \sum_{n=0}^{\infty} (1.075)^n$$

Geometric series with $r = 1.075$

Diverges by Theorem 8.6

$$60. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Since these are both convergent p -series, the difference is convergent.

$$62. \sum_{n=2}^{\infty} \ln(n)$$

$$\lim_{n \rightarrow \infty} \ln(n) = \infty$$

Diverges by Theorem 8.9

$$64. \sum_{n=2}^{\infty} \frac{\ln n}{n^3}$$

$$\text{Let } f(x) = \frac{\ln x}{x^3}.$$

f is positive, continuous, and decreasing for $x \geq 2$ since $f'(x) = \frac{1 - 3 \ln x}{x^4} < 0$ for $x \geq 2$.

$$\int_2^{\infty} \frac{\ln x}{x^3} dx = \left[-\frac{\ln x}{2x^2} \right]_2^{\infty} + \frac{1}{2} \int_2^{\infty} \frac{1}{x^3} dx = \frac{\ln 2}{8} + \left[-\frac{1}{4x^2} \right]_2^{\infty} = \frac{\ln 2}{8} + \frac{1}{16} \quad (\text{Use Integration by Parts.})$$

Converges by Theorem 8.10. See Exercise 14.

Section 8.4 Comparisons of Series

$$2. (a) \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} = 2 + \frac{2}{\sqrt{2}} + \cdots \quad S_1 = 2$$

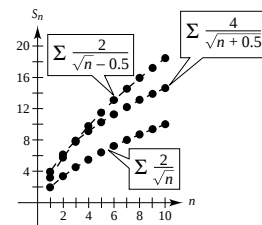
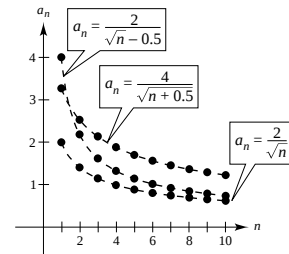
$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n} - 0.5} = \frac{2}{0.5} + \frac{2}{\sqrt{2} - 0.5} + \cdots \quad S_1 = 4$$

$$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n} + 0.5} = \frac{4}{\sqrt{1.5}} + \frac{4}{\sqrt{2.5}} + \cdots \quad S_1 \approx 3.3$$

(b) The first series is a p -series. It diverges ($p = \frac{1}{2} < 1$).

(c) The magnitude of the terms of the other two series are greater than the corresponding terms of the divergent p -series. Hence, the other two series diverge.

(d) The larger the magnitude of the terms, the larger the magnitude of the terms of the sequence of partial sums.



$$4. \frac{1}{3n^2 + 2} < \frac{1}{3n^2}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$$

converges by comparison with the convergent p -series

$$\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$8. \frac{3^n}{4^n + 5} < \left(\frac{3}{4}\right)^n$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^n}{4^n + 5}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$12. \frac{1}{4\sqrt[3]{n} - 1} > \frac{1}{4\sqrt[4]{n}}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n} - 1}$$

diverges by comparison with the divergent p -series

$$\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}.$$

$$16. \lim_{n \rightarrow \infty} \frac{2/(3^n - 5)}{1/3^n} = \lim_{n \rightarrow \infty} \frac{2 \cdot 3^n}{3^n - 5} = 2$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

$$20. \lim_{n \rightarrow \infty} \frac{\frac{5n - 3}{n^2 - 2n + 5}}{1/n} = \lim_{n \rightarrow \infty} \frac{5n^2 - 3n}{n^2 - 2n + 5} = 5$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5n - 3}{n^2 - 2n + 5}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$6. \frac{1}{\sqrt{n} - 1} > \frac{1}{\sqrt{n}} \text{ for } n \geq 2.$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

diverges by comparison with the divergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}.$$

$$10. \frac{1}{\sqrt{n^3 + 1}} < \frac{1}{n^{3/2}}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$14. \frac{4^n}{3^n - 1} > \frac{4^n}{3^n}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$$

diverges by comparison with the divergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n.$$

$$18. \lim_{n \rightarrow \infty} \frac{3/\sqrt{n^2 - 4}}{1/n} = \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{n^2 - 4}} = 3$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2 - 4}}$$

diverges by a limit comparison with the divergent harmonic series

$$\sum_{n=3}^{\infty} \frac{1}{n}.$$

$$22. \lim_{n \rightarrow \infty} \frac{\frac{1}{n(n^2 + 1)}}{1/n^3} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3 + n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2 + 1)}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$24. \lim_{n \rightarrow \infty} \frac{n/[(n+1)2^{n-1}]}{1/(2^{n-1})} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$$

converges by a limit comparison with the convergent geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

$$28. \lim_{n \rightarrow \infty} \frac{\tan(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2) \sec^2(1/n)}{-1/n^2} = \lim_{n \rightarrow \infty} \sec^2\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$30. \sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$$

Converges

Geometric series with $r = -\frac{1}{5}$

$$32. \sum_{n=4}^{\infty} \frac{1}{3n^2 - 2n - 15}$$

Converges

Limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$34. \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots = \frac{1}{2}$$

Converges; telescoping series

$$36. \sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

Converges; telescoping series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

38. If $j < k - 1$, then $k - j > 1$. The p -series with $p = k - j$ converges and since

$$\lim_{n \rightarrow \infty} \frac{P(n)/Q(n)}{1/n^{k-j}} = L > 0,$$

the series $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ converges by the limit comparison test. Similarly, if $j \geq k - 1$, then $k - j \leq 1$ which implies that

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

diverges by the limit comparison test.

$$40. \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1},$$

which converges since the degree of the numerator is two less than the degree of the denominator.

$$42. \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

diverges since the degree of the numerator is only one less than the degree of the denominator.

44. $\lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \lim_{n \rightarrow \infty} n = \infty \neq 0$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ diverges.}$$

46. See Theorem 8.13, page 585.

One example is

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}} \text{ diverges because } \lim_{n \rightarrow \infty} \frac{1/\sqrt{n-1}}{1/\sqrt{n}} = 1$$

and

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (} p\text{-series).}$$

50. $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \cdots = \sum_{n=0}^{\infty} \frac{1}{200 + 10n},$
diverges

48. This is not correct. The beginning terms do not affect the convergence or divergence of a series.

In fact,

$$\frac{1}{1000} + \frac{1}{1001} + \cdots = \sum_{n=1000}^{\infty} \frac{1}{n} \text{ diverges (harmonic)}$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (} p\text{-series).}$$

52. $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200 + n^3},$
converges

54. (a) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 4n + 1}$

converges since the degree of the numerator is two less than the degree of the denominator. (See Exercise 38.)

(b)

n	5	10	20	50	100
S_n	1.1839	1.02087	1.2212	1.2287	1.2312

(c) $\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_2 \approx 0.1226$

(d) $\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - S_9 \approx 0.0277$

56. True

58. False. Let $a_n = 1/n$, $b_n = 1/n$, $c_n = 1/n^2$. Then, $a_n \leq b_n + c_n$, but $\sum_{n=1}^{\infty} c_n$ converges.

60. Since $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n a_n = \sum_{n=1}^{\infty} a_n^2$ converges by Exercise 59.

62. $\sum \frac{1}{n^2}$ converge, and hence so does $\sum \left(\frac{1}{n^2}\right)^2 = \sum \frac{1}{n^4}$.

64. (a) $\sum a_n = \sum \frac{1}{n^3}$ and $\sum b_n = \sum \frac{1}{n^2}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/n^3}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \text{and} \quad \sum \frac{1}{n^2}$$

converges, so does $\sum \frac{1}{n^3}$.

(b) $\sum a_n = \sum \frac{1}{\sqrt{n}}$ and $\sum b_n = \sum \frac{1}{n}$. Since

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1/\sqrt{n}}{1/n} = \lim_{n \rightarrow \infty} \sqrt{n} = \infty \quad \text{and} \quad \sum \frac{1}{n}$$

diverges, so does $\sum \frac{1}{\sqrt{n}}$.

Section 8.5 Alternating Series

$$2. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 6}{n^2} = \frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \dots$$

$$S_1 = 6, S_2 = 4.5$$

Matches (d)

$$4. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 10}{n2^n} = \frac{10}{2} - \frac{10}{8} + \dots$$

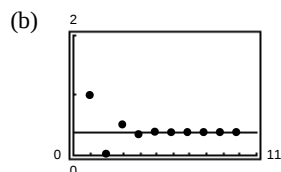
$$S_1 = 5, S_2 = 3.75$$

Matches (a)

$$6. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e} \approx 0.3679$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0	0.5	0.3333	0.375	0.3667	0.3681	0.3679	0.3679	0.3679



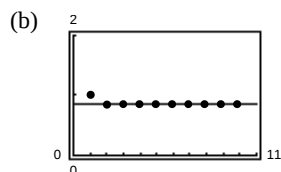
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$8. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin(1) \approx 0.8415$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.8333	0.8417	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415	0.8415



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next series.

$$10. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$$

$$\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2}$$

Diverges by the n th Term Test.

$$12. \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

$$a_{n+1} = \frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$$

Converges by Theorem 8.14

$$14. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 + 1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2 + 1} < \frac{n}{n^2 + 1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = 0$$

Converges by Theorem 8.14

$$16. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 5} = 1$$

Diverges by n th Term Test

$$18. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n+1)}{n+1}$$

$$a_{n+1} = \frac{\ln[(n+1)+1]}{(n+1)+1} < \frac{\ln(n+1)}{n+1} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{n+1} = \lim_{n \rightarrow \infty} \frac{1/(n+1)}{1} = 0$$

Converges by Theorem 8.14

$$22. \sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges; (see Exercise 9)

$$26. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{\sqrt[3]{n}}$$

$$\lim_{n \rightarrow \infty} \frac{n^{1/2}}{n^{1/3}} = \lim_{n \rightarrow \infty} n^{1/6} = \infty$$

Diverges by the n th Term Test

$$30. S_6 = \sum_{n=1}^6 \frac{4(-1)^{n+1}}{\ln(n+1)} \approx 2.7067$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236; 0.7831 \leq S \leq 4.6303$$

$$32. S_6 = \sum_{n=1}^6 \frac{(-1)^{n+1} n}{2^n} = 0.1875$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{7}{2^7} \approx 0.05469; 0.1328 \leq S \leq 0.2422$$

$$34. \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$

(a) By Theorem 8.15,

$$|R_n| \leq a_{N+1} = \frac{1}{2^{N+1}(N+1)!} < 0.001.$$

This inequality is valid when $N = 4$.

(b) We may approximate the series by

$$\sum_{n=0}^4 \frac{(-1)^n}{2^n n!} = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{348} \approx 0.607.$$

(5 terms. Note that the sum begins with $n = 0$.)

$$20. \sum_{n=1}^{\infty} \frac{1}{n} \sin \left[\frac{(2n-1)\pi}{2} \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Converges; (see Exercise 9)

$$24. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$a_{n+1} = \frac{1}{(2n+3)!} < \frac{1}{(2n+1)!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{(2n+1)!} = 0$$

Converges by Theorem 8.14

$$28. \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} + 1}$$

$$\text{Let } f(x) = \frac{2e^x}{e^{2x} + 1}. \text{ Then}$$

$$f'(x) = \frac{2e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^2} < 0 \text{ for } x > 0.$$

Thus, $f(x)$ is decreasing for $x > 0$ which implies

$$a_{n+1} < a_n.$$

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$$

The series converges by Theorem 8.14.

$$36. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(2N+2)!} < 0.001.$$

This inequality is valid when $N = 3$.

(b) We may approximate the series by

$$\sum_{n=0}^3 \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \approx 0.540.$$

(4 terms. Note that the sum begins with $n = 0$.)

$$38. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{4^{N+1}(N+1)} < 0.001.$$

This inequality is valid when $N = 3$.

$$40. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

By Theorem 8.15, $|R_N| \leq a_{N+1} = \frac{1}{(N+1)^4} < 0.001$.

This inequality is valid when $N = 5$.

$$44. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p -series.

Therefore, the given series converges absolutely.

$$48. \sum_{n=0}^{\infty} \frac{(-1)^n}{e^{n^2}}$$

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by a comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

Therefore, the given series converges absolutely.

$$52. \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}}$$

diverges by a limit comparison to the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

Therefore, the given series converges conditionally.

(b) We may approximate the series by

$$\sum_{n=1}^3 \frac{(-1)^{n+1}}{4^n n} = \frac{1}{4} - \frac{1}{32} + \frac{1}{192} \approx 0.224.$$

(3 terms)

$$42. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

diverges by the Integral Test. Therefore, the series converge conditionally.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$$

$\lim_{n \rightarrow \infty} \frac{2n+3}{n+10} = 2$ Therefore, the series diverges by the n th Term Test.

$$50. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p -series.

Therefore, the given series converge absolutely.

$$54. \sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$$

$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0$ Therefore, the series diverges by the n th Term Test.

$$56. \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=1}^{\infty} \left| \frac{\sin[(2n-1)\pi/2]}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent p -series. Therefore, the series converges conditionally.

$$60. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad (\text{Alternating Harmonic Series})$$

$$62. \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

If $p = 0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 1$$

and the series diverges. If $p > 0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \text{ and } \frac{1}{(n+1)^p} < \frac{1}{n^p}.$$

Therefore, the series converge by the Alternating Series Test.

$$66. (a) \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges absolutely (by comparison) for

$$-1 < x < 1,$$

since

$$\left| \frac{x^n}{n} \right| < |x^n| \text{ and } \sum x^n$$

is a convergent geometric series for $-1 < x < 1$.

$$58. |S - S_n| = |R_n| \leq a_{n+1} \quad (\text{Theorem 8.15})$$

$$64. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges, but } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

(b) When $x = -1$, we have the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

When $x = 1$, we have the divergent harmonic series

$$\frac{1}{n}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges conditionally for $x = -1$.

68. True, equivalent to Theorem 8.16

$$70. \sum_{n=1}^{\infty} \frac{3}{n^2 + 5}$$

converges by limit comparison to convergent p -series

$$\sum \frac{1}{n^2}.$$

72. Converges by limit comparison to convergent geometric series $\sum \frac{1}{2^n}$.

74. Diverges by n th Term Test. $\lim_{n \rightarrow \infty} a_n = \frac{3}{2}$

76. Converges (conditionally) by Alternating Series Test.

78. Diverges by comparison to Divergent Harmonic Series:

$$\frac{\ln n}{n} > \frac{1}{n} \text{ for } n \geq 3.$$

Section 8.6 The Ratio and Root Tests

$$2. \frac{(2k-2)!}{(2k)!} = \frac{(2k-2)!}{(2k)(2k-1)(2k-2)!} = \frac{1}{(2k)(2k-1)}$$

4. Use the Principle of Mathematical Induction. When $k = 3$, the formula is valid since $\frac{1}{1} = \frac{2^3 3!(3)(5)}{6!} = 1$. Assume that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} = \frac{2^n n!(2n-3)(2n-1)}{(2n)!}$$

and show that

$$\frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} = \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!}.$$

To do this, note that:

$$\begin{aligned} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)} &= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-5)} \cdot \frac{1}{(2n-3)} \\ &= \frac{2^n n!(2n-3)(2n-1)}{(2n)!} \cdot \frac{1}{(2n-3)} \\ &= \frac{2^n n!(2n-1)}{(2n)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)} \\ &= \frac{2^n (2)(n+1)n!(2n-1)(2n+1)}{(2n)!(2n+1)(2n+2)} \\ &= \frac{2^{n+1}(n+1)!(2n-1)(2n+1)}{(2n+2)!} \end{aligned}$$

The formula is valid for all $n \geq 3$.

$$6. \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \left(\frac{1}{n!}\right) = \frac{3}{4} + \frac{9}{16} \left(\frac{1}{2}\right) + \cdots$$

$$S_1 = \frac{3}{4}, S_2 \approx 1.03$$

Matches (c)

$$8. \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{(2n)!} = \frac{4}{2} - \frac{4}{24} + \cdots$$

$$S_1 = 2$$

Matches (b)

$$10. \sum_{n=0}^{\infty} 4e^{-n} = 4 + \frac{4}{e} + \cdots$$

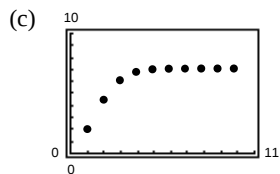
$$S_1 = 4$$

Matches (e)

$$12. (a) \text{ Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2 + 1}{n!}}{\frac{n^2 + 1}{n!}} = \lim_{n \rightarrow \infty} \left(\frac{n^2 + 2n + 2}{n^2 + 1} \right) \left(\frac{1}{n+1} \right) = 0 < 1. \text{ Converges}$$

(b)

n	5	10	15	20	25
S_n	7.0917	7.1548	7.1548	7.1548	7.1548



(d) The sum is approximately 7.15485

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

$$14. \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$18. \sum_{n=1}^{\infty} \frac{n^3}{2^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3/2^{n+1}}{n^3/2^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{2n^3} \right| = \frac{1}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$16. \sum_{n=1}^{\infty} n \left(\frac{3}{2} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)3^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n3^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3(n+1)}{2n} = \frac{3}{2} \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$20. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$$

$$a_{n+1} = \frac{n+3}{(n+1)(n+2)} \leq \frac{n+2}{n(n+1)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{n+2}{n(n+1)} = 0$$

Therefore, by Theorem 8.14, the series converges.

Note: The Ratio Test is inconclusive since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

The series converges conditionally.

$$22. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3/2)^{n+1}}{n^2 + 2n + 1} \cdot \frac{n^2}{(3/2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3n^2}{2(n^2 + 2n + 1)} = \frac{3}{2} > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$24. \sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)n^5}{(n+1)^5} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$26. \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = e > 1 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$28. \sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^2}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$30. \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{4n+4}}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{4n}} \right| = \lim_{n \rightarrow \infty} \frac{2^4}{(2n+3)(2n+2)} = 0$$

Therefore, by the Ratio Test, the series converges.

$$32. \sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 4 \cdot 6 \cdots 2n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots 2n(2n+2)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{2 \cdot 4 \cdots 2n} \right| = \lim_{n \rightarrow \infty} \frac{2n+2}{3n+2} = \frac{2}{3}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-\frac{2}{2} + \frac{2 \cdot 4}{2 \cdot 5} - \frac{2 \cdot 4 \cdot 6}{2 \cdot 5 \cdot 8} + \cdots$

$$34. (a) \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^4} \cdot \frac{n^4}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^4 = 1$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^p = 1$$

$$36. \sum_{n=1}^{\infty} \left(\frac{2n}{n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{2n}{n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \end{aligned}$$

Therefore, by the Root Test, the series diverges.

$$38. \sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1} \right)^{3n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{-3n}{2n+1} \right)^{3n} \right|} \\ &= \lim_{n \rightarrow \infty} \left(\frac{3n}{2n+1} \right)^3 = \left(\frac{3}{2} \right)^3 = \frac{27}{8} \end{aligned}$$

Therefore, by the Root Test, the series diverges.

$$40. \sum_{n=0}^{\infty} e^{-n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{e^n}} = \frac{1}{e}$$

Therefore, by the Root Test, the series converges.

$$42. \sum_{n=0}^{\infty} \frac{n+1}{3^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n+1}{3^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3}$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \sqrt[n]{x+1}$$

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{x+1})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{x} \ln(x+1)$$

$$= \lim_{n \rightarrow \infty} \frac{\ln(x+1)}{x} = \frac{1}{x+1} = 0.$$

Since $\ln y = 0$, $y = e^0 = 1$, so

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n+1}}{3} = \frac{1}{3}.$$

Therefore, by the Root Test, the series converges.

$$44. \sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$$

This is the divergent harmonic series.

$$46. \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$$

Since $\pi/4 < 1$, this is convergent geometric series.

$$48. \sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n/(2n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2} > 0$$

This series diverges by limit comparison to the divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$50. \sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$$

$$\lim_{n \rightarrow \infty} \frac{10/3n^{3/2}}{1/n^{3/2}} = \frac{10}{3}$$

Therefore, the series converges by a limit comparison test with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$52. \sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{4n^2 - 1} = \lim_{n \rightarrow \infty} \frac{(\ln 2)2^n}{8n} = \lim_{n \rightarrow \infty} \frac{(\ln 2)^2 2^n}{8} = \infty$$

Therefore, the series diverges by the n th Term Test for Divergence.

$$54. \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$$

$$a_{n+1} = \frac{1}{(n+1) \ln(n+1)} \leq \frac{1}{n \ln(n)} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln(n)} = 0$$

Therefore, by the Alternating Series Test, the series converges.

$$56. \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

$$\frac{\ln(n)}{n^2} \leq \frac{1}{n^{3/2}}$$

Therefore, the series converges by comparison with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

$$58. \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 2^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3n}{2(n+1)} = \frac{3}{2}$$

Therefore, by the Ratio Test, the series diverges.

$$60. \sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)}{18^{n+1}(2n+1)(2n-1)n!} \cdot \frac{18^n (2n-1)n!}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right| = \lim_{n \rightarrow \infty} \frac{(2n+3)(2n-1)}{18(2n+1)(2n-1)} = \frac{2}{18} = \frac{1}{9}$$

Therefore, by the Ratio Test, the series converge.

62. (b) and (c)

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1) \left(\frac{3}{4}\right)^n &= \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^{n-1} \\ &= 1 + 2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots \end{aligned}$$

64. (a) and (b) are the same.

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}} &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} &= \frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \cdots \end{aligned}$$

66. Replace n with $n + 2$.

$$\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n!}$$

$$\begin{aligned} 68. \sum_{k=0}^{\infty} \frac{(-3)^k}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k+1)} &= \sum_{k=0}^{\infty} \frac{(-3)^k 2^k k!}{(2k)!(2k+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-6)^k k!}{(2k+1)!} \\ &\approx 0.40967 \end{aligned}$$

(See Exercise 3 and use 10 terms, $k = 9$.)

70. See Theorem 8.18.

$$72. \text{ One example is } \sum_{n=1}^{\infty} \left(-100 + \frac{1}{n} \right).$$

74. Assume that

$$\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L > 1 \text{ or that } \lim_{n \rightarrow \infty} |a_{n+1}/a_n| = \infty.$$

Then there exists $N > 0$ such that $|a_{n+1}/a_n| > 1$ for all $n > N$. Therefore,

$$|a_{n+1}| > |a_n|, \quad n > N \Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum a_n \text{ diverges}$$

76. The differentiation test states that if

$$\sum_{n=1}^{\infty} U_n$$

is an infinite series with real terms and $f(x)$ is a real function such that $f(1/n) = U_n$ for all positive integers n and $d^2 f/dx^2$ exists at $x = 0$, then

$$\sum_{n=1}^{\infty} U_n$$

converges absolutely if $f(0) = f'(0) = 0$ and diverges otherwise. Below are some examples.

Convergent Series

$$\sum \frac{1}{n^3}, f(x) = x^3$$

$$\sum \left(1 - \cos \frac{1}{n} \right), f(x) = 1 - \cos x$$

Divergent Series

$$\sum \frac{1}{n}, f(x) = x$$

$$\sum \sin \frac{1}{n}, f(x) = \sin x$$

Section 8.7 Taylor Polynomials and Approximations

$$2. y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$$

y -axis symmetry

Three relative extrema

Matches (c)

$$4. y = e^{-1/2} \left[\frac{1}{3}(x-1)^3 - (x-1) + 1 \right]$$

Cubic

Matches (b)

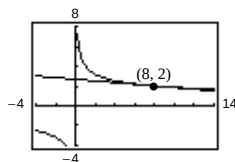
$$6. f(x) = \frac{4}{\sqrt[3]{x}} = 4x^{-1/3} \quad f(8) = 2$$

$$f'(x) = -\frac{4}{3}x^{-4/3} \quad f'(8) = -\frac{1}{12}$$

$$P_1(x) = f(8) + f'(8)(x-8)$$

$$= 2 + \left(-\frac{1}{12} \right)(x-8)$$

$$P_1(x) = -\frac{1}{12}x + \frac{8}{3}$$



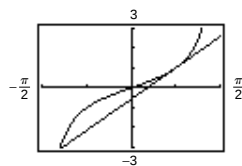
8. $f(x) = \tan x$ $f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $f'\left(\frac{\pi}{4}\right) = 2$

$P_1 = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$

$= 1 + 2\left(x - \frac{\pi}{4}\right)$

$P_1(x) = 2x + 1 - \frac{\pi}{2}$



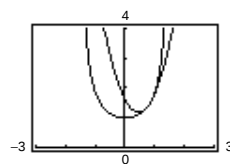
10. $f(x) = \sec x$ $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$ $f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x$ $f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$

$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$



x	-2.15	0.585	0.685	$\pi/4$	0.885	0.985	1.785
$f(x)$	-1.8270	1.1995	1.2913	1.4142	1.5791	1.8088	-4.7043
$P_2(x)$	15.5414	1.2160	1.2936	1.4142	1.5761	1.7810	4.9475

12. $f(x) = x^2 e^x$, $f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x$ $f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x$ $f''(0) = 2$

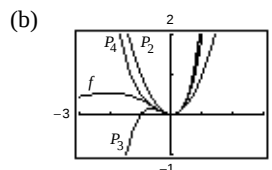
$f'''(x) = (x^2 + 6x + 6)e^x$ $f'''(0) = 6$

$f^{(4)}(x) = (x^2 + 8x + 12)e^x$ $f^{(4)}(0) = 12$

$P_2(x) = \frac{2x^2}{2!} = x^2$

$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$

$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$



(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

14. $f(x) = e^{-x}$ $f(0) = 1$

$f'(x) = -e^{-x}$ $f'(0) = -1$

$f''(x) = e^{-x}$ $f''(0) = 1$

$f'''(x) = -e^{-x}$ $f'''(0) = -1$

$f^{(4)}(x) = e^{-x}$ $f^{(4)}(0) = 1$

$f^{(5)}(x) = -e^{-x}$ $f^{(5)}(0) = -1$

$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$

$$16. \quad f(x) = e^{3x} \quad f(0) = 1$$

$$f'(x) = 3e^{3x} \quad f'(0) = 3$$

$$f''(x) = 9e^{3x} \quad f''(0) = 9$$

$$f'''(x) = 27e^{3x} \quad f'''(0) = 27$$

$$f^{(4)}(x) = 81e^{3x} \quad f^{(4)}(0) = 81$$

$$P_4(x) = 1 + 3x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$$

$$18. \quad f(x) = \sin \pi x \quad f(0) = 0$$

$$f'(x) = \pi \cos \pi x \quad f'(0) = \pi$$

$$f''(x) = -\pi^2 \sin \pi x \quad f''(0) = 0$$

$$f'''(x) = -\pi^3 \cos \pi x \quad f'''(0) = -\pi^3$$

$$P_3(x) = 0 + \pi x + \frac{0}{2!}x^2 + \frac{-\pi^3}{3!}x^3 = \pi x - \frac{\pi^3}{6}x^3$$

$$20. \quad f(x) = x^2 e^{-x} \quad f(0) = 0$$

$$f'(x) = 2xe^{-x} - x^2 e^{-x} \quad f'(0) = 0$$

$$f''(x) = 2e^{-x} - 4xe^{-x} + x^2 e^{-x} \quad f''(0) = 2$$

$$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2 e^{-x} \quad f'''(0) = -6$$

$$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2 e^{-x} \quad f^{(4)}(0) = 12$$

$$\begin{aligned} P_4(x) &= 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4 \\ &= x^2 - x^3 + \frac{1}{2}x^4 \end{aligned}$$

$$22. \quad f(x) = \frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - (x+1)^{-1} \quad f(0) = 0$$

$$f'(x) = (x+1)^{-2} \quad f'(0) = 1$$

$$f''(x) = -2(x+1)^{-3} \quad f''(0) = -2$$

$$f'''(x) = 6(x+1)^{-4} \quad f'''(0) = 6$$

$$f^{(4)}(x) = -24(x+1)^{-5} \quad f^{(4)}(0) = -24$$

$$P_4(x) = 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 = x - x^2 + x^3 - x^4$$

$$24. \quad f(x) = \tan x \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad f'(0) = 1$$

$$f''(x) = 2 \sec^2 x \tan x \quad f''(0) = 0$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \quad f'''(0) = 2$$

$$P_3(x) = 0 + 1(x) + 0 + \frac{2}{6}x^3 = x + \frac{1}{3}x^3$$

$$26. \quad f(x) = 2x^{-2} \quad f(2) = \frac{1}{2}$$

$$f'(x) = -4x^{-3} \quad f'(2) = -\frac{1}{2}$$

$$f''(x) = 12x^{-4} \quad f''(2) = \frac{3}{4}$$

$$f'''(x) = -48x^{-5} \quad f'''(2) = -\frac{3}{2}$$

$$f^{(4)}(x) = 240x^{-6} \quad f^{(4)}(2) = \frac{15}{4}$$

$$P_4(x) = \frac{1}{2} - \frac{1}{2}(x-2) + \frac{3}{8}(x-2)^2 - \frac{1}{4}(x-2)^3 + \frac{5}{32}(x-2)^4$$

28. $f(x) = x^{1/3} \quad f(8) = 2$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3} \quad f'''(8) = \frac{10}{27} \cdot \frac{1}{2^8} = \frac{5}{3456}$$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20,736}(x-8)^3$$

32. $f(x) = \frac{1}{x^2 + 1}$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{2(3x^2 - 1)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x(1 - x^2)}{(x^2 + 1)^4}$$

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

(a) $n = 2, c = 0$

$$P_2(x) = 1 + 0x + \frac{-2}{2!}x^2 = 1 - x^2$$

(c) $n = 4, c = 1$

$$Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x-1) + \frac{1/2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \frac{-3}{4!}(x-1)^4 = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4$$

34. $f(x) = \ln x$

$$P_1(x) = x - 1$$

$$P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

(a)

x	1.00	1.25	1.50	1.75	2.00
$\ln x$	0.0000	0.2231	0.4055	0.5596	0.6931
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_4(x)$	0.0000	0.2230	0.4010	0.5303	0.5833

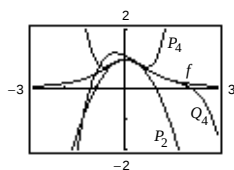
30. $f(x) = x^2 \cos x$

$$f(\pi) = -\pi^2$$

$$f'(x) = \cos x - x^2 \sin x \quad f'(\pi) = -2\pi$$

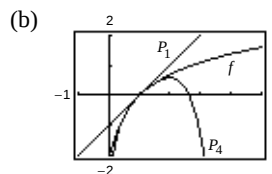
$$f''(x) = 2 \cos x - 4x \sin x - x^2 \cos x \quad f''(\pi) = -2 + \pi^2$$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + \frac{(\pi^2-2)}{2}(x-\pi)^2$$



(b) $n = 4, c = 0$

$$P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$$



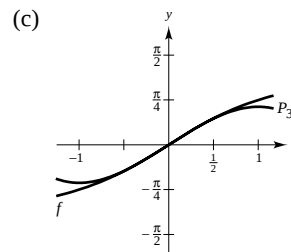
(c) As the distance increases, the accuracy decreases.

36. (a) $f(x) = \arctan x$

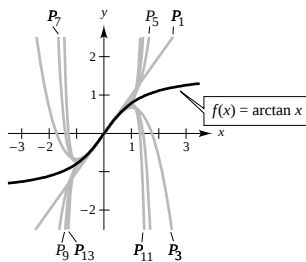
$$P_3(x) = x - \frac{x^3}{3}$$

(b)

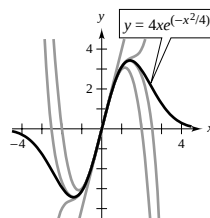
x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.6435	-0.4636	-0.2450	0	0.2450	0.4636	0.6435
$P_3(x)$	-0.6094	-0.4583	-0.2448	0	0.2448	0.4583	0.6094



38. $f(x) = \arctan x$



40. $f(x) = 4xe^{-x^2/4}$



42. $f(x) = x^2 e^{-x} \approx x^2 - x^3 + \frac{1}{2}x^4$

$$f\left(\frac{1}{5}\right) \approx 0.0328$$

44. $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\frac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$$f\left(\frac{7\pi}{8}\right) \approx -6.7954$$

46. $f(x) = e^x$; $f^{(6)}(x) = e^x \Rightarrow \text{Max on } [0, 1] \text{ is } e^1$.

$$R_5(x) \leq \frac{e^1}{6!} (1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$$

48. $f(x) = \arctan x$; $f^{(4)}(x) = \frac{24x(x^2 + 1)}{(1 - x^2)^4}$
 $\Rightarrow \text{Max on } [0, 0.4] \text{ is } f^{(4)}(0.4) \approx 22.3672$.

$$R_3(x) \leq \frac{22.3672}{4!} (0.4)^4 \approx 0.0239$$

50. $f(x) = e^x$

$$f^{(n+1)}(x) = e^x$$

$$\text{Max on } [0, 0.6] \text{ is } e^{0.6} \approx 1.8221$$

$$R_n \leq \frac{1.8221}{(n+1)!} (0.6)^{n+1} < 0.001$$

By trial and error, $n = 5$.

52. $f(x) = \cos(\pi x^2)$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$f(x) = g(\pi x^2)$$

$$= 1 - \frac{(\pi x^2)^2}{2!} + \frac{(\pi x^2)^4}{4!} - \frac{(\pi x^2)^6}{6!} + \dots$$

$$= 1 - \frac{\pi^2 x^4}{2!} + \frac{\pi^4 x^8}{4!} - \frac{\pi^6 x^{12}}{6!} + \dots$$

$$f(0.6) = 1 - \frac{\pi^2}{2!} (0.6)^4 + \frac{\pi^4}{4!} (0.6)^8 - \frac{\pi^6}{6!} (0.6)^{12} + \dots$$

Since this is an alternating series,

$$R_n \leq a_{n+1} = \frac{\pi^{2n}}{(2n)!} (0.6)^{4n} < 0.0001.$$

By trial and error, $n = 4$. Using 4 terms $f(0.6) \approx 0.4257$.

54. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$$|R_3(x)| = \left| \frac{\sin z}{4!} x^4 \right| \leq \frac{|x^4|}{4!} < 0.001$$

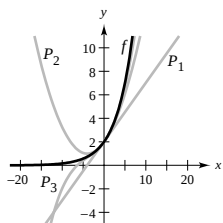
$$x^4 < 0.024$$

$$|x| < 0.3936$$

$$-0.3936 < x < 0.3936$$

56. $f(c) = P_2(c)$, $f'(c) = P_2'(c)$, and $f''(c) = P_2''(c)$

60.



58. See Theorem 8.19, page 611.

62. (a) $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$ for $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c) $R(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

64. Let f be an odd function and P_n be the n^{th} Maclaurin polynomial for f . Since f is odd, f' is even:

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+(-h)) - f(x)}{-h} = f'(x).$$

Similarly, f'' is odd, f''' is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \dots = 0$. Hence, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots \text{ all the coefficients of the even power of } x \text{ are zero.}$$

66. Let $P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n$ where $a_i = \frac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

$$\text{For } 1 \leq k \leq n, \quad P_n^{(k)}(c) = a_n k! = \left(\frac{f^{(k)}(c)}{k!} \right) k! = f^{(k)}(c).$$

Section 8.8 Power Series

2. Centered at 0

4. Centered at π

6. $\sum_{n=0}^{\infty} (2x)^n$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right| = 2|x|$$

$$2|x| < 1 \Rightarrow R = \frac{1}{2}$$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n x^n} \right|$$

$$= \frac{1}{2} |x|$$

$$\frac{1}{2} |x| < 1 \Rightarrow R = 2$$

$$10. \sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x^2}{(n+1)} \right| = \infty$$

The series only converges at $x = 0$. $R = 0$.

$$12. \sum_{n=0}^{\infty} \left(\frac{x}{k} \right)^n$$

Since the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

$$14. \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^{n+1}(n+1)x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+2)x}{n+1} \right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\sum_{n=0}^{\infty} -(n+1)$ diverges.

Therefore, the interval of convergence is $-1 < x < 1$.

$$16. \sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(2n+1)!} \cdot \frac{(2n)!}{(3x)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{3x}{(2n+2)(2n+1)} \right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$18. \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{(-1)^n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{n+3} \right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore, the interval of convergence is $-1 \leq x \leq 1$.

$$20. \sum_{n=0}^{\infty} \frac{(-1)^n n!(x-4)^n}{3^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(n+1)!(x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n!(x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-4)}{3} \right| = \infty$$

$R = 0$

Center: $x = 4$

Therefore, the series converges only for $x = 4$.

$$22. \sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-2)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)(n+1)}{4(n+2)} \right| = \frac{1}{4}|x-2|$$

$$R = 4$$

$$\text{Center: } x = 2$$

$$\text{Interval: } -4 < x - 2 < 4 \text{ or } -2 < x < 6$$

When $x = -2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges.

When $x = 6$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

Therefore, the interval of convergence is $-2 \leq x < 6$.

$$24. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}} \cdot \frac{nc^n}{(-1)^{n+1}(x-c)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-c)}{c(n+1)} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When $x = 0$, the p -series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Therefore, the interval of convergence is $0 < x \leq 2c$.

$$26. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)} \cdot \frac{(2n+1)}{(-1)^n x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+1)}{(2n+3)} x^2 \right| = |x^2|$$

$$R = 1$$

$$\text{Interval: } -1 < x < 1$$

When $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $-1 \leq x \leq 1$.

$$28. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{n+1} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$30. \sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n! x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$32. \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} (x^{2n+1})$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 4 \cdots (2n)(2n+2)x^{2n+3}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)x^2}{(2n+3)} \right| = |x^2|$$

$$R = 1$$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

which diverges. Therefore, the interval of convergence is $-1 < x < 1$.

$$34. \sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-c)^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!(x-c)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-c)}{2n+1} \right| = \frac{1}{2} |x-c|$$

$$R = 2$$

$$\text{Interval: } -2 < x - c < 2 \text{ or } c - 2 < x < c + 2$$

The series diverges at the endpoints. Therefore, the interval of convergence is $c - 2 < x < c + 2$.

$$\left[\frac{n!(c+2-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n!2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} > 1 \right]$$

$$36. (a) f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}, 0 < x \leq 10$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}, 0 < x < 10$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}, 0 < x < 10$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}, 0 \leq x \leq 10$$

$$38. (a) f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}, 1 < x \leq 3$$

$$(b) f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}, 1 < x < 3$$

$$(c) f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}, 1 < x < 3$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}, 1 \leq x \leq 3$$

$$40. g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \cdots$$

$$S_1 = 1, S_2 = 1.67. \text{ Matches (a)}$$

$$42. g(-2) = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n \text{ alternating. Matches (d)}$$

44. The set of all values of x for which the power series converges is the interval of convergence.

If the power series converges for all x , then the radius of convergence is $R = \infty$. If the power series converges at only c , then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

46. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

$$48. (a) f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, -\infty < x < \infty \quad (\text{See Exercise 11})$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

$$(c) f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$f(0) = 1$$

$$(d) f(x) = e^x$$

$$\begin{aligned}
50. \quad y &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
y' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n x^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \\
y'' &= \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4n x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} \\
y'' + x^2 y &= -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4n x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2 \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0
\end{aligned}$$

$$52. J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!}$$

$$(a) \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3} (k+1)! (k+2)!} \cdot \frac{2^{2k+1} k! (k+1)!}{(-1)^k x^{2k+1}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2 (k+2)(k+1)} \right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

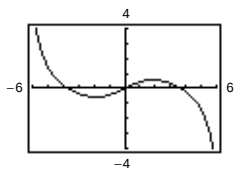
$$(b) J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!}$$

$$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k}}{2^{2k+1} k! (k+1)!}$$

$$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k-1}}{2^{2k+1} k! (k+1)!}$$

$$\begin{aligned}
x^2 J_1'' + x J_1' + (x^2 - 1) J_1 &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)x^{2k+1}}{2^{2k+1} k! (k+1)!} \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!} \\
&= \left[\sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k)x^{2k+1}}{2^{2k+1} k! (k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)x^{2k+1}}{2^{2k+1} k! (k+1)!} \right. \\
&\quad \left. - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!} \right] + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} [(2k+1)(2k) + (2k+1) - 1]}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1} (k-1)! k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1} k! (k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k! (k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3} [(-1) + 1]}{2^{2k+1} k! (k+1)!} = 0
\end{aligned}$$

$$(c) P_7(x) = \frac{x}{2} - \frac{1}{16} x^3 + \frac{1}{384} x^5 - \frac{1}{18,432} x^7$$

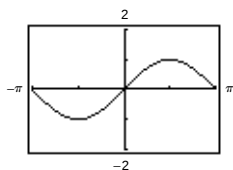


$$\begin{aligned}
(d) \quad J_0'(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1)x^{2k+1}}{2^{2k+2} (k+1)! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k! (k+1)!} \\
&= -J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k! (k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k! (k+1)!}
\end{aligned}$$

Note: $J_0'(x) = -J_1(x)$

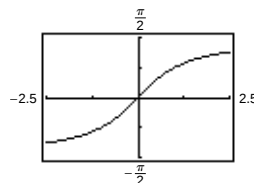
$$54. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

(See Exercise 47.)



$$56. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x, -1 \leq x \leq 1$$

(See Exercise 38 in Section 8.7.)



$$58. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

Replace n with $n-1$.

60. True; if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for $x = 2$, then we know that it must converge on $(-2, 2]$.

62. True

$$\int_0^1 f(x) dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

Section 8.9 Representation of Functions by Power Series

$$2. (a) f(x) = \frac{4}{5-x} = \frac{4/5}{1-x/5} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{4}{5} \left(\frac{x}{5} \right)^n = \sum_{n=0}^{\infty} \frac{4x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

$$\begin{array}{r} \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \frac{4x^3}{625} + \cdots \\ (b) \quad 5-x \overline{) 4} \\ \underline{4 - \frac{4}{5}x} \\ \frac{4}{5}x \\ \underline{\frac{4}{5}x - \frac{4}{25}x^2} \\ \frac{4}{25}x^2 \\ \underline{\frac{4}{25}x^2 - \frac{4x^3}{125}} \\ \frac{4x^3}{125} \\ \underline{\frac{4x^3}{125} - \frac{4x^4}{625}} \\ \vdots \end{array}$$

$$4. (a) \frac{1}{1+x} = \frac{1}{1-(-x)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

This series converges on $(-1, 1)$.

$$(b) \quad \begin{array}{r} 1-x+x^2-x^3+\cdots \\ 1+x \overline{) 1} \\ \underline{1+x} \\ -x \\ \underline{-x-x^2} \\ x^2 \\ \underline{x^2+x^3} \\ -x^3 \\ \underline{-x^3-x^4} \\ \vdots \end{array}$$

6. Writing $f(x)$ in the form $\frac{a}{1-r}$, we have

$$\frac{4}{5-x} = \frac{4}{7-(x+2)} = \frac{4/7}{1-1/7(x+2)} = \frac{a}{1-r}.$$

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}\frac{4}{5-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{4}{7} \left(\frac{1}{7}(x+2) \right)^n \\ &= \sum_{n=0}^{\infty} \frac{4(x+2)^n}{7^{n+1}}.\end{aligned}$$

$$|x+2| < 7 \text{ or } -5 < x < 9$$

8. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{3}{3+2(x-2)} = \frac{1}{1+(2/3)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = (-2/3)(x-2)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}\frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left[-\frac{2}{3}(x-2) \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n (x-2)^n}{3^n},\end{aligned}$$

$$|x-2| < \frac{3}{2} \text{ or } \frac{1}{2} < x < \frac{7}{2}.$$

10. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2x-5} = \frac{1}{-5+2x} = \frac{-1/5}{1-(2/5)x} = \frac{a}{1-r}$$

which implies that $a = -1/5$ and $r = (2/5)x$. Therefore, the power series for $f(x)$ is given by

$$\frac{1}{2x-5} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(-\frac{1}{5} \right) \left(\frac{2}{5}x \right)^n = - \sum_{n=0}^{\infty} \frac{2^n x^n}{5^{n+1}},$$

$$|x| < \frac{5}{2} \text{ or } -\frac{5}{2} < x < \frac{5}{2}.$$

12. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{4}{3x+2} = \frac{4}{8+3(x-2)} = \frac{1/2}{1+(3/8)(x-2)} = \frac{a}{1-r}$$

which implies that $a = 1/2$ and $r = (-3/8)(x-2)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned}\frac{4}{3x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{1}{2} \left[-\frac{3}{8}(x-2) \right]^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-3)^n (x-2)^n}{8^n},\end{aligned}$$

$$|x-2| < \frac{8}{3} \text{ or } -\frac{2}{3} < x < \frac{14}{3}.$$

$$14. \frac{4x-7}{2x^2+3x-2} = \frac{3}{x+2} - \frac{2}{2x-1} = \frac{3}{2+x} - \frac{2}{-1+2x} = \frac{3/2}{1+(1/2)x} + \frac{2}{1-2x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{4x-7}{2x^2+3x-2} = \sum_{n=0}^{\infty} \left(\frac{3}{2} \right) \left(-\frac{1}{2}x \right)^n + \sum_{n=0}^{\infty} 2(2x)^n = \sum_{n=0}^{\infty} \left[\frac{3(-1)^n}{2^{n+1}} + 2^{n+1} \right] x^n, \quad |x| < \frac{1}{2} \text{ or } -\frac{1}{2} < x < \frac{1}{2}.$$

16. First finding the power series for $4/(4+x)$, we have

$$\frac{1}{1+(1/4)x} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}x \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n}$$

Now replace x with x^2 .

$$\frac{4}{4+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}.$$

The interval of convergence is $|x^2| < 4$ or $-2 < x < 2$ since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(-1)^n x^{2n}} \right| = \left| -\frac{x^2}{4} \right| = \frac{|x^2|}{4}.$$

$$\begin{aligned}18. h(x) &= \frac{x}{x^2-1} = \frac{1}{2(1+x)} - \frac{1}{2(1-x)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} [(-1)^n - 1] x^n = \frac{1}{2} [0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-2)x^{2n+1} = - \sum_{n=0}^{\infty} x^{2n+1}, \quad 1 < x < 1\end{aligned}$$

20. By taking the second derivative, we have $\frac{d^2}{dx^2} \left[\frac{1}{x+1} \right] = \frac{2}{(x+1)^3}$. Therefore,

$$\begin{aligned} \frac{2}{(x+1)^3} &= \frac{d^2}{dx^2} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] \\ &= \frac{d}{dx} \left[\sum_{n=1}^{\infty} (-1)^n n x^{n-1} \right] = \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n, \quad -1 < x < 1. \end{aligned}$$

22. By integrating, we have

$$\begin{aligned} \int \frac{1}{1+x} dx &= \ln(1+x) + C_1 \text{ and } \int \frac{1}{1-x} dx = -\ln(1-x) + C_2. \\ f(x) &= \ln(1-x^2) = \ln(1+x) - [-\ln(1-x)]. \text{ Therefore,} \\ \ln(1-x^2) &= \int \frac{1}{1+x} dx - \int \frac{1}{1-x} dx \\ &= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx - \int \left[\sum_{n=0}^{\infty} x^n \right] dx = \left[C_1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right] - \left[C_2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right] \\ &= C + \sum_{n=0}^{\infty} \frac{[(-1)^n - 1] x^{n+1}}{n+1} = C + \sum_{n=0}^{\infty} \frac{-2x^{2n+2}}{2n+2} = C + \sum_{n=0}^{\infty} \frac{(-1)x^{2n+2}}{n+1} \end{aligned}$$

To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1-x^2) = - \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n+1}, \quad -1 < x < 1$$

24. $\frac{2x}{x^2+1} = 2x \sum_{n=0}^{\infty} (-1)^n x^{2n}$ (See Exercise 23.)

$$= \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

Since $\frac{d}{dx} (\ln(x^2+1)) = \frac{2x}{x^2+1}$, we have

$$\ln(x^2+1) = \int \left[\sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad -1 \leq x \leq 1.$$

To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x^2+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, \quad -1 \leq x \leq 1.$$

26. Since $\int \frac{1}{4x^2+1} dx = \frac{1}{2} \arctan(2x)$, we can use the result of Exercise 25 to obtain

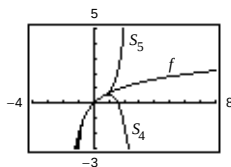
$$\arctan(2x) = 2 \int \frac{1}{4x^2+1} dx = 2 \int \left[\sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} \right] dx = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\arctan(2x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \quad -\frac{1}{2} < x \leq \frac{1}{2}.$$

28. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \leq \ln(x + 1)$

$$\leq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x + 1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

In Exercise 35–38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

30. $g(x) = x - \frac{x^3}{3}$, cubic with 3 zeros.

Matches (d)

32. $g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$,

Matches (b)

34. The approximations of degree 3, 7, 11, . . . ($4n - 1$, $n = 1, 2, \dots$) have relative extrema.

In Exercises 36 and 38, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

36.
$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\int_0^{3/4} \arctan x^2 dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344,064} + \frac{177,147}{230,686,720}$$

Since $177,147/230,686,720 < 0.001$, we can approximate the series by its first two terms: 0.13427

38.
$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \dots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x dx \approx 0.015625$.

In Exercises 40 and 42, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

40. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

$$42. (a) \frac{1}{3} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n = \frac{2}{9} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^{n-1} = \frac{2}{9} \frac{1}{[1 - (2/3)]^2} = 2$$

$$(b) \frac{1}{10} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^n = \frac{9}{100} \sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^{n-1} \\ = \frac{9}{100} \cdot \frac{1}{[1 - (9/10)]^2} = 9$$

44. Replace x with x^2 .

46. Integrate the series and multiply by (-1) .

48. (a) From Exercise 47, we have

$$\arctan \frac{120}{119} - \arctan \frac{1}{239} = \arctan \frac{120}{119} + \arctan \left(-\frac{1}{239}\right) \\ = \arctan \left[\frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)} \right] = \arctan \left(\frac{28,561}{28,561} \right) = \arctan 1 = \frac{\pi}{4}$$

$$(b) 2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan \left[\frac{2(1/5)}{1 - (1/5)^2} \right] = \arctan \frac{10}{24} = \arctan \frac{5}{12}$$

$$4 \arctan \frac{1}{5} = 2 \arctan \frac{1}{5} + 2 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12} = \arctan \left[\frac{2(5/12)}{1 - (5/12)^2} \right] = \arctan \frac{120}{119}$$

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

$$50. (a) \arctan \frac{1}{2} + \arctan \frac{1}{3} = \arctan \left[\frac{(1/2) + (1/3)}{1 - (1/2)(1/3)} \right] = \arctan \left(\frac{5/6}{5/6} \right) = \frac{\pi}{4}$$

$$(b) \pi = 4 \left[\arctan \frac{1}{2} + \arctan \frac{1}{3} \right] \\ = 4 \left[\frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} \right] + 4 \left[\frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} \right] \\ \approx 4(0.4635) + 4(0.3217) \approx 3.14$$

52. From Exercise 51, we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/3)^n}{n} \\ = \ln \left(\frac{1}{3} + 1 \right) = \ln \frac{4}{3} \approx 0.2877.$$

54. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} \\ = \arctan 1 = \frac{\pi}{4} \approx 0.7854$$

56. From Exercise 54, we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^{2n-1}(2n-1)} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} \\ = \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1} \\ = \arctan \frac{1}{3} \approx 0.3218.$$

58. From Example 5, we have $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n}(2n+1)} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n (1/\sqrt{3})^{2n+1}}{2n+1} \\ &= \sqrt{3} \arctan \frac{1}{\sqrt{3}} \\ &= \sqrt{3} \left(\frac{\pi}{6} \right) = \frac{\pi}{2\sqrt{3}} \end{aligned}$$

Section 8.10 Taylor and Maclaurin Series

2. For $c = 0$, we have

$$\begin{aligned} f(x) &= e^{3x} \\ f^{(n)}(x) &= 3^n e^{3x} \Rightarrow f^{(n)}(0) = 3^n \\ e^{3x} &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} \end{aligned}$$

4. For $c = \pi/4$, we have:

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \sin x & f^{(4)}\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} \end{aligned}$$

and so on. Therefore we have:

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[1 + \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} - \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} + \cdots \right] \\ &= \frac{\sqrt{2}}{2} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^{n+1}}{(n+1)!} + 1 \right\} \end{aligned}$$

6. For $c = 1$, we have:

$$\begin{aligned} f(x) &= e^x \\ f^{(n)}(x) &= e^x \Rightarrow f^{(n)}(1) = e \\ e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = e \left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots \right] = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \end{aligned}$$

8. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \ln(x^2 + 1) & f(0) &= 0 \\
 f'(x) &= \frac{2x}{x^2 + 1} & f'(0) &= 0 \\
 f''(x) &= \frac{2 - 2x^2}{(x^2 + 1)^2} & f''(0) &= 2 \\
 f'''(x) &= \frac{4x(x^2 - 3)}{(x^2 + 1)^3} & f'''(0) &= 0 \\
 f^{(4)}(x) &= \frac{12(-x^4 + 6x^2 - 1)}{(x^2 + 1)^4} & f^{(4)}(0) &= -12 \\
 f^{(5)}(x) &= \frac{48x(x^4 - 10x^2 + 5)}{(x^2 + 1)^5} & f^{(5)}(0) &= 0 \\
 f^{(6)}(x) &= \frac{-240(5x^6 - 15x^4 + 15x^2 - 1)}{(x^2 + 1)^6} & f^{(6)}(0) &= 240
 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned}
 \ln(x^2 + 1) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 0x + \frac{2x^2}{2!} + \frac{0x^3}{3!} - \frac{12x^4}{4!} + \frac{0x^5}{5!} + \frac{240x^6}{6!} + \cdots \\
 &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}
 \end{aligned}$$

10. For $c = 0$, we have;

$$\begin{aligned}
 f(x) &= \tan(x) & f(0) &= 0 \\
 f'(x) &= \sec^2(x) & f'(0) &= 1 \\
 f''(x) &= 2 \sec^2(x)\tan(x) & f''(0) &= 0 \\
 f'''(x) &= 2[\sec^4(x) + 2 \sec^2(x)\tan^2(x)] & f'''(0) &= 2 \\
 f^{(4)}(x) &= 8[\sec^4(x)\tan(x) + \sec^2(x)\tan^3(x)] & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= 8[2 \sec^6(x) + 11 \sec^4(x)\tan^2(x) + 2 \sec^2(x)\tan^4(x)] & f^{(5)}(0) &= 16 \\
 \tan(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots
 \end{aligned}$$

12. The Maclaurin Series for $f(x) = e^{-2x}$ is $\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$.

$f^{(n+1)}(x) = (-2)^{n+1}e^{-2x}$. Hence, by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{(-2)^{n+1}e^{-2z}}{(n+1)!} x^{n+1} \right|.$$

Since $\lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}x^{n+1}}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \right| = 0$, it follows that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$.

Hence, the Maclaurin Series for e^{-2x} converges to e^{-2x} for all x .

14. Since $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \dots$, we have

$$\begin{aligned} \left[1 + (-x)\right]^{-1/2} &= 1 + \left(\frac{1}{2}\right)x + \frac{(1/2)(3/2)x^2}{2!} + \frac{(1/2)(3/2)(5/2)x^3}{3!} + \dots \\ &= 1 + \frac{x}{2} + \frac{(1)(3)x^2}{2^2 2!} + \frac{(1)(3)(5)x^3}{2^3 3!} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)x^n}{2^n n!} \end{aligned}$$

16. Since $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \dots$, we have

$$\begin{aligned} (1+x)^{1/3} &= 1 + \left(\frac{1}{3}\right)x + \frac{(1/3)(-2/3)x^2}{2!} + \frac{(1/3)(-2/3)(-5/3)x^3}{3!} + \dots \\ &= 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \dots \\ &= 1 + \frac{x}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n+2} \cdot 5 \cdot 8 \cdot \dots \cdot (3n-4)}{3^n n!}. \end{aligned}$$

18. Since $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)x^n}{2^n n!}$ (Exercise 14)

$$\text{we have } (1+x^3)^{1/2} = 1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)x^{3n}}{2^n n!}.$$

20. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \dots$$

22. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\begin{aligned} \cos 4x &= \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!} \\ &= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \dots \end{aligned}$$

24. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\begin{aligned} 2 \sin x^3 &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} \\ &= 2 \left[x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \dots \right] \\ &= 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \dots \end{aligned}$$

26. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots$$

$$2 \cosh(x) = e^x + e^{-x} = \sum_{n=0}^{\infty} 2 \frac{x^{2n}}{(2n)!}$$

28. The formula for the binomial series gives $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, which implies that

$$\begin{aligned}(1+x^2)^{-1/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^n n!} \\ \ln(x + \sqrt{x^2 + 1}) &= \int \frac{1}{\sqrt{x^2 + 1}} dx \\ &= x + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n+1}}{2^n (2n+1)n!} \\ &= x - \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots\end{aligned}$$

$$\begin{aligned}30. \quad x \cos x &= x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) \\ &= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}\end{aligned}$$

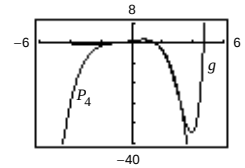
$$\begin{aligned}32. \quad \frac{\arcsin x}{x} &= \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} \cdot \frac{1}{x} \\ &= \sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{(2^n n!)^2 (2n+1)}, \quad x \neq 0\end{aligned}$$

$$34. \quad e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots \quad (\text{See Exercise 33.})$$

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$$

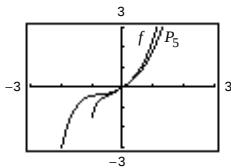
$$36. \quad g(x) = e^x \cos x$$

$$\begin{aligned}&= \left(1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^4}{24} + \cdots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \right) \\ &= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2} \right) + \left(\frac{x^3}{6} - \frac{x^3}{2} \right) + \left(\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24} \right) + \cdots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots\end{aligned}$$



$$38. \quad f(x) = e^x \ln(1+x)$$

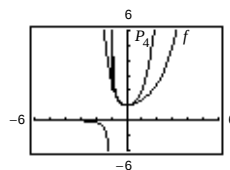
$$\begin{aligned}&= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \right) \\ &= x + \left(x^2 - \frac{x^2}{2} \right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2} \right) + \left(-\frac{x^4}{4} + \frac{x^4}{3} - \frac{x^4}{4} + \frac{x^4}{6} \right) + \left(\frac{x^5}{5} - \frac{x^5}{4} + \frac{x^5}{6} - \frac{x^5}{12} + \frac{x^5}{24} \right) + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots\end{aligned}$$



40. $f(x) = \frac{e^x}{1+x}$. Divide the series for e^x by $(1+x)$.

$$\begin{array}{r}
 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} + \cdots \\
 \hline
 1+x \overline{) 1+x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 \underline{1+x} \phantom{+ \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 0 + \frac{x^2}{2} + \frac{x^3}{6} \phantom{+ \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 \underline{ \frac{x^2}{2} + \frac{x^3}{6}} \phantom{+ \frac{x^4}{24} + \frac{x^5}{120} + \cdots} \\
 -\frac{x^3}{3} + \frac{x^4}{24} \phantom{+ \frac{x^5}{120} + \cdots} \\
 \underline{ -\frac{x^3}{3} - \frac{x^4}{3}} \phantom{+ \frac{x^5}{120} + \cdots} \\
 \frac{3x^4}{8} + \frac{x^5}{120} \\
 \underline{ \frac{3x^4}{8} + \frac{3x^5}{8}} \\
 \vdots
 \end{array}$$

$$f(x) = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$$



42. $y = x - \frac{x^3}{2!} + \frac{x^5}{4!} = x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \right) \approx x \cos x$.

Matches (b)

44. $y = x^2 - x^3 + x^4 = x^2(1 - x + x^2) \approx x^2 \left(\frac{1}{1+x} \right)$.

Matches (d)

$$\begin{aligned}
 46. \int_0^x \sqrt{1+t^3} dt &= \int_0^x \left[1 + \frac{t^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)t^{3n}}{2^n n!} \right] dt \\
 &= \left[t + \frac{t^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)t^{3n+1}}{(3n+1)2^n n!} \right]_0^x \\
 &= x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{3n+1}}{(3n+1)2^n n!}
 \end{aligned}$$

48. Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, we have

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots \approx 0.8415. \quad (4 \text{ terms})$$

50. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$, we have $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$

$$\text{and } \frac{e-1}{e} = 1 - e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \approx 0.6321. \quad (6 \text{ terms})$$

52. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1.$$

$$54. \int_0^{1/2} \frac{\arctan x}{x} dx = \int_0^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right) dx = \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots\right]_0^{1/2}$$

Since $1/(9^2 2^9) < 0.0001$, we have

$$\int_0^{1/2} \frac{\arctan x}{x} dx \approx \left(\frac{1}{2} - \frac{1}{3^2 2^3} + \frac{1}{5^2 2^5} - \frac{1}{7^2 2^7} + \frac{1}{9^2 2^9}\right) \approx 0.4872.$$

Note: We are using $\lim_{x \rightarrow 0^+} \frac{\arctan x}{x} = 1$.

$$56. \int_{0.5}^1 \cos \sqrt{x} dx = \int_{0.5}^1 \left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots\right) dx = \left[x - \frac{x^2}{2(2!)} + \frac{x^3}{3(4!)} - \frac{x^4}{4(6!)} + \frac{x^5}{5(8!)} - \cdots\right]_{0.5}^1$$

Since $\frac{1}{210,600} (1 - 0.5^5) < 0.0001$, we have

$$\int_{0.5}^1 \cos \sqrt{x} dx \approx \left[(1 - 0.5) - \frac{1}{4}(1 - 0.5^2) + \frac{1}{72}(1 - 0.5^3) - \frac{1}{2880}(1 - 0.5^4) + \frac{1}{201,600}(1 - 0.5^5)\right] \approx 0.3243.$$

$$58. \int_0^{1/4} x \ln(x+1) dx = \int_0^{1/4} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4 \cdot 2} + \frac{x^5}{5 \cdot 3} - \frac{x^6}{6 \cdot 4} + \cdots\right]_0^{1/4}$$

Since $\frac{(1/4)^5}{15} < 0.0001$,

$$\int_0^{1/4} x \ln(x+1) dx \approx \frac{(1/4)^3}{3} - \frac{(1/4)^4}{8} \approx 0.00472.$$

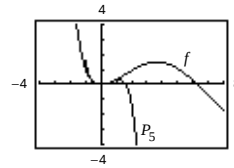
60. From Exercise 19, we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_1^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_1^2 \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (2^{n+1} - 1)}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[1 - \frac{7}{2 \cdot 1 \cdot 3} + \frac{31}{2^2 \cdot 2! \cdot 5} - \frac{127}{2^3 \cdot 3! \cdot 7} + \frac{511}{2^4 \cdot 4! \cdot 9} - \frac{2047}{2^5 \cdot 5! \cdot 11} \right. \\ &\quad \left. + \frac{8191}{2^6 \cdot 6! \cdot 13} - \frac{32,767}{2^7 \cdot 7! \cdot 15} + \frac{131,071}{2^8 \cdot 8! \cdot 17} - \frac{524,287}{2^9 \cdot 9! \cdot 19} \right] \approx 0.1359. \end{aligned}$$

$$62. f(x) = \sin \frac{x}{2} \ln(1+x)$$

$$P_5(x) = \frac{x^2}{2} - \frac{x^3}{4} + \frac{7x^4}{48} - \frac{11x^5}{96}$$

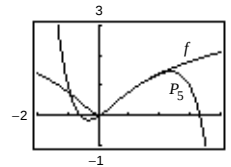
The polynomial is a reasonable approximation on the interval $(-0.60, 0.73)$.



$$64. f(x) = \sqrt[3]{x} \cdot \arctan x, c = 1$$

$$\begin{aligned} P_5(x) &\approx 0.7854 + 0.7618(x-1) - 0.3412 \left[\frac{(x-1)^2}{2!} \right] - 0.0424 \left[\frac{(x-1)^3}{3!} \right] \\ &\quad + 1.3025 \left[\frac{(x-1)^4}{4!} \right] - 5.5913 \left[\frac{(x-1)^5}{5!} \right] \end{aligned}$$

The polynomial is a reasonable approximation on the interval $(0.48, 1.75)$.



66. $a_{2n+1} = 0$ (odd coefficients are zero)

68. Answers will vary.

70. $\theta = 60^\circ, v_0 = 64, k = \frac{1}{16}, g = -32$

$$\begin{aligned}
 y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots \\
 &= \sqrt{3}x - 32 \left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^316} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots \right] \\
 &= \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{x^n}{n(32)^n(16)^{n-2}}
 \end{aligned}$$

72. (a) $f(x) = \frac{\ln(x^2 + 1)}{x^2}$.

From Exercise 8, you obtain

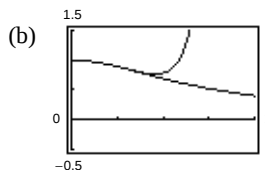
$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

(c) $F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt$

$$G(x) = \int_0^x P_8(t) dt$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6920	0.8805	5.3064	652.21

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.74. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!} \right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N! \left[e - \left(1 + 1 + \cdots + \frac{1}{N!} \right) \right]$, a positive integer. But,

$$\begin{aligned}
 a &= N! \left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots \right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \\
 &= \frac{1}{N+1} \left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \right] = \frac{1}{N+1} \left[\frac{1}{1 - \left(\frac{1}{N+1} \right)} \right] = \frac{1}{N}, \text{ a contradiction.}
 \end{aligned}$$

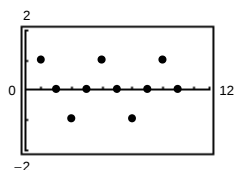
Review Exercises for Chapter 8

2. $a_n = \frac{n}{n^2 + 1}$

4. $a_n = 4 - \frac{n}{2}$: 3.5, 3, . . .
Matches (c)

6. $a_n = 6 \left(-\frac{2}{3} \right)^{n-1}$: 6, -4, . . .
Matches (b)

8. $a_n = \sin \frac{n\pi}{2}$



The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: 1, 0, -1, 0, 1, 0, \dots$$

10. $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

Converges

12. $\lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$

Diverges

14. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{k \rightarrow \infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges; $k = 2n$

16. Let $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{1}{b^n + c^n} (b^n \ln b + c^n \ln c)$$

Assume $b \geq c$ and note that the terms

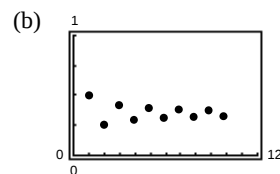
$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \rightarrow \infty$. Hence a_n converges.

20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

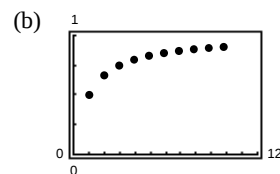
(c) The series converges by the Alternating Series Test.



22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

(c) The series converges, by the limit comparison test with $\sum \frac{1}{n^2}$.



24. Diverges. Geometric series, $r = 1.82 > 1$.

26. Diverges. n th Term Test, $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$.

28. $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

30.
$$\sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)} \right] = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{1 - (2/3)} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \right] = 3 - 1 = 2$$

$$32. 0.\overline{923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923,076}{999,999} = \frac{12(76,923)}{13(76,923)} = \frac{12}{13}$$

$$34. S = \sum_{n=0}^{39} 32,000(1.055)^n = \frac{32,000(1 - 1.055^{40})}{1 - 1.055}$$

$$\approx \$4,371,379.65$$

36. See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$$\approx \$16,840.32$$

$$38. \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

Divergent p -series, $p = \frac{3}{4} < 1$

$$40. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The first series is a convergent p -series and the second series is a convergent geometric series. Therefore, their difference converges.

$$42. \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)/n(n+2)}{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3 \ln(n+1)}{n+1} < \frac{3 \ln n}{n} = a_n, \lim_{n \rightarrow \infty} \frac{3 \ln n}{n} = 0$$

$$50. \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

$$52. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

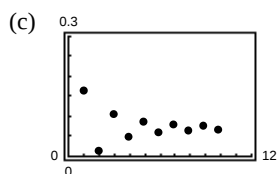
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721



(d) The sum is approximately 0.0714.

56. No. Let $a_n = \frac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$ is a convergent p -series.

58. $f(x) = \tan x$ $f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x$ $f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2 \sec^2 x \tan x$ $f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$ $f'''\left(-\frac{\pi}{4}\right) = 16$

$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$

60. $\cos(0.75) \approx 1 - \frac{(0.75)^2}{2!} + \frac{(0.75)^4}{4!} - \frac{(0.75)^6}{6!} \approx 0.7317$

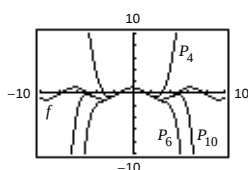
62. $e^{-0.25} \approx 1 - 0.25 + \frac{(0.25)^2}{2!} - \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!} \approx 0.779$

64. $f(x) = \cos x$

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$



66. $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

68. $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n(x-2)^n} \right|$$

$$= 3|x-2|$$

$R = \frac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \leq x < \frac{7}{3}$.

70. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left| \frac{x-2}{2} \right| < 1 \quad \text{or} \quad 0 < x < 4.$$

$$\begin{aligned}
72. \quad y &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!} \\
y' &= \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!} \\
y'' &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) (2n+1) x^{2n}}{2^{n+1} (n+1)!} \\
y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) (2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-(2n+1) + 1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-2n + 2n] = 0
\end{aligned}$$

$$74. \quad \frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

$$76. \text{ Integral: } \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$$

$$\begin{aligned}
78. \quad 8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots &= \sum_{n=0}^{\infty} 8 \left[\frac{-(x-3)}{4} \right]^n = \frac{8}{1 - [-(x-3)/4]} \\
&= \frac{32}{4 + (x-3)} = \frac{32}{1+x}, \quad -1 < x < 7
\end{aligned}$$

$$80. \quad f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x + (\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!} \left(x + \frac{\pi}{4}\right)^4 + \cdots \\
&= \frac{\sqrt{2}}{2} \left[1 + \left(x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2} [x + (\pi/4)]^{n+1}}{(n+1)!} \right]
\end{aligned}$$

$$82. \quad f(x) = \csc(x)$$

$$f'(x) = -\csc(x) \cot(x)$$

$$f''(x) = \csc^3(x) + \csc(x) \cot^2(x)$$

$$f'''(x) = -5 \csc^3(x) \cot(x) - \csc(x) \cot^3(x)$$

$$f^{(4)}(x) = 5 \csc^5(x) + 15 \csc^3(x) \cot^2(x) + \csc(x) \cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!} \left(x - \frac{\pi}{2}\right)^4 + \cdots$$

84. $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \dots$$

$$\begin{aligned}\sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} \\ &= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \dots \\ &= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}\end{aligned}$$

86. $h(x) = (1+x)^{-3}$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)x}{2}$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$

$$\begin{aligned}\ln\left(\frac{6}{5}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823\end{aligned}$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

94. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \dots$$

$$\int_0^1 xe^x dx = \left[xe^x - e^x \right]_0^1 = (e - e) - (0 - 1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

$$\begin{aligned}
96. \quad (a) \quad & f(x) = \sin 2x & f(0) &= 0 \\
& f'(x) = 2 \cos 2x & f'(0) &= 2 \\
& f''(x) = -4 \sin 2x & f''(0) &= 0 \\
& f'''(x) = -8 \cos 2x & f'''(0) &= -8 \\
& f^{(4)}(x) = 16 \sin 2x & f^{(4)}(0) &= 0 \\
& f^{(5)}(x) = 32 \cos 2x & f^{(5)}(0) &= 32 \\
& f^{(6)}(x) = -64 \sin 2x & f^{(6)}(0) &= 0 \\
& f^{(7)}(x) = -128 \cos 2x & f^{(7)}(0) &= -128
\end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

$$(b) \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \\
&= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$(c) \quad \sin 2x = 2 \sin x \cos x$$

$$\begin{aligned}
&= 2 \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\
&= 2 \left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6} \right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} \right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040} \right) + \cdots \right] \\
&= 2 \left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots \right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$98. \quad \cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\cos \frac{\sqrt{t}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!}$$

$$\begin{aligned}
\int_0^x \cos \frac{\sqrt{t}}{2} dt &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)} \right]_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}
\end{aligned}$$

$$100. \quad e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$$

$$\frac{e^t - 1}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$$

$$\int_0^x \frac{e^t - 1}{t} dt = \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} \right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

$$102. \quad \arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sqrt{1-x^2}} \right)}{1} = 1.$$

66. $a_{2n+1} = 0$ (odd coefficients are zero)

68. Answers will vary.

70. $\theta = 60^\circ, v_0 = 64, k = \frac{1}{16}, g = -32$

$$\begin{aligned}
 y &= \sqrt{3}x - \frac{32x^2}{2(64)^2(1/2)^2} - \frac{(1/16)(32)x^3}{3(64)^3(1/2)^3} - \frac{(1/16)^2(32)x^4}{4(64)^4(1/2)^4} - \cdots \\
 &= \sqrt{3}x - 32 \left[\frac{2^2x^2}{2(64)^2} + \frac{2^3x^3}{3(64)^316} + \frac{2^4x^4}{4(64)^4(16)^2} + \cdots \right] \\
 &= \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{2^n x^n}{n(64)^n(16)^{n-2}} = \sqrt{3}x - 32 \sum_{n=2}^{\infty} \frac{x^n}{n(32)^n(16)^{n-2}}
 \end{aligned}$$

72. (a) $f(x) = \frac{\ln(x^2 + 1)}{x^2}$.

From Exercise 8, you obtain

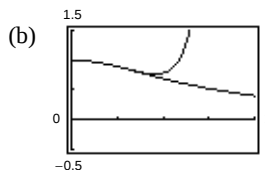
$$P = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$$

$$P_8 = 1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5}$$

$$(c) F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2} dt$$

$$G(x) = \int_0^x P_8(t) dt$$

x	0.25	0.50	0.75	1.00	1.50	2.00
$F(x)$	0.2475	0.4810	0.6920	0.8776	1.1798	1.4096
$G(x)$	0.2475	0.4810	0.6920	0.8805	5.3064	652.21

(d) The curves are nearly identical for $0 < x < 1$. Hence, the integrals nearly agree on that interval.74. Assume $e = p/q$ is rational. Let $N > q$ and form the following.

$$e - \left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{N!} \right] = \frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots$$

Set $a = N! \left[e - \left(1 + 1 + \cdots + \frac{1}{N!} \right) \right]$, a positive integer. But,

$$\begin{aligned}
 a &= N! \left[\frac{1}{(N+1)!} + \frac{1}{(N+2)!} + \cdots \right] = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \\
 &= \frac{1}{N+1} \left[1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \cdots \right] = \frac{1}{N+1} \left[\frac{1}{1 - \left(\frac{1}{N+1} \right)} \right] = \frac{1}{N}, \text{ a contradiction.}
 \end{aligned}$$

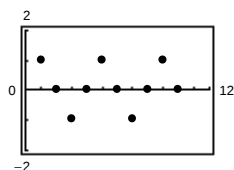
Review Exercises for Chapter 8

2. $a_n = \frac{n}{n^2 + 1}$

4. $a_n = 4 - \frac{n}{2}$: 3.5, 3, . . .
Matches (c)

6. $a_n = 6 \left(-\frac{2}{3} \right)^{n-1}$: 6, -4, . . .
Matches (b)

8. $a_n = \sin \frac{n\pi}{2}$



The sequence seems to diverge (oscillates).

$$\sin \frac{n\pi}{2}: 1, 0, -1, 0, 1, 0, \dots$$

10. $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

Converges

12. $\lim_{n \rightarrow \infty} \frac{n}{\ln(n)} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$

Diverges

14. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = \lim_{k \rightarrow \infty} \left[\left(1 + \frac{1}{k}\right)^k\right]^{1/2} = e^{1/2}$

Converges; $k = 2n$

16. Let $y = (b^n + c^n)^{1/n}$

$$\ln y = \frac{\ln(b^n + c^n)}{n}$$

$$\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{1}{b^n + c^n} (b^n \ln b + c^n \ln c)$$

Assume $b \geq c$ and note that the terms

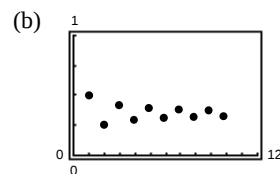
$$\frac{b^n \ln b + c^n \ln c}{b^n + c^n} = \frac{b^n \ln b}{b^n + c^n} + \frac{c^n \ln c}{b^n + c^n}$$

converge as $n \rightarrow \infty$. Hence a_n converges.

20. (a)

k	5	10	15	20	25
S_k	0.3917	0.3228	0.3627	0.3344	0.3564

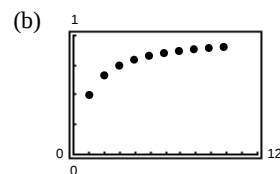
(c) The series converges by the Alternating Series Test.



22. (a)

k	5	10	15	20	25
S_k	0.8333	0.9091	0.9375	0.9524	0.9615

(c) The series converges, by the limit comparison test with $\sum \frac{1}{n^2}$.



24. Diverges. Geometric series, $r = 1.82 > 1$.

26. Diverges. n th Term Test, $\lim_{n \rightarrow \infty} a_n = \frac{2}{3}$.

28. $\sum_{n=0}^{\infty} \frac{2^{n+2}}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 4(3) = 12$

See Exercise 27.

30.
$$\sum_{n=0}^{\infty} \left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)} \right] = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{1 - (2/3)} - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \right] = 3 - 1 = 2$$

$$32. 0.\overline{923076} = 0.923076[1 + 0.000001 + (0.000001)^2 + \cdots]$$

$$= \sum_{n=0}^{\infty} (0.923076)(0.000001)^n = \frac{0.923076}{1 - 0.000001} = \frac{923,076}{999,999} = \frac{12(76,923)}{13(76,923)} = \frac{12}{13}$$

$$34. S = \sum_{n=0}^{39} 32,000(1.055)^n = \frac{32,000(1 - 1.055^{40})}{1 - 1.055}$$

$$\approx \$4,371,379.65$$

36. See Exercise 86 in Section 8.2.

$$A = P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]$$

$$= 100\left(\frac{12}{0.065}\right)\left[\left(1 + \frac{0.065}{12}\right)^{120} - 1\right]$$

$$\approx \$16,840.32$$

$$38. \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

Divergent p -series, $p = \frac{3}{4} < 1$

$$40. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The first series is a convergent p -series and the second series is a convergent geometric series. Therefore, their difference converges.

$$42. \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)/n(n+2)}{1/n} = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$$

By a limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$, the series diverges.

44. Since $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$ converges by the Limit Comparison Test.

$$46. \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$$

By the Alternating Series Test, the series converges.

48. Converges by the Alternating Series Test.

$$a_{n+1} = \frac{3 \ln(n+1)}{n+1} < \frac{3 \ln n}{n} = a_n, \lim_{n \rightarrow \infty} \frac{3 \ln n}{n} = 0$$

$$50. \sum_{n=1}^{\infty} \frac{n!}{e^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n+1}{e} = \infty$$

By the Ratio Test, the series diverges.

$$52. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

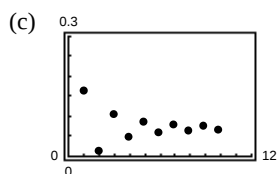
$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 5 \cdots (3n-1)(3n+2)} \cdot \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 3 \cdots (2n-1)} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$$

By the Ratio Test, the series converges.

54. (a) The series converges by the Alternating Series Test.

(b)

x	5	10	15	20	25
S_n	0.0871	0.0669	0.0734	0.0702	0.0721



(d) The sum is approximately 0.0714.

56. No. Let $a_n = \frac{3937.5}{n^2}$, then $a_{75} = 0.7$. The series $\sum_{n=1}^{\infty} \frac{3937.5}{n^2}$ is a convergent p -series.

58. $f(x) = \tan x$ $f\left(-\frac{\pi}{4}\right) = -1$

$f'(x) = \sec^2 x$ $f'\left(-\frac{\pi}{4}\right) = 2$

$f''(x) = 2 \sec^2 x \tan x$ $f''\left(-\frac{\pi}{4}\right) = -4$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$ $f'''\left(-\frac{\pi}{4}\right) = 16$

$P_3(x) = -1 + 2\left(x + \frac{\pi}{4}\right) - 2\left(x + \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{\pi}{4}\right)^3$

60. $\cos(0.75) \approx 1 - \frac{(0.75)^2}{2!} + \frac{(0.75)^4}{4!} - \frac{(0.75)^6}{6!} \approx 0.7317$

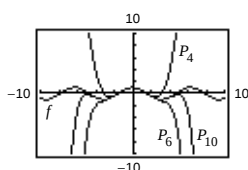
62. $e^{-0.25} \approx 1 - 0.25 + \frac{(0.25)^2}{2!} - \frac{(0.25)^3}{3!} + \frac{(0.25)^4}{4!} \approx 0.779$

64. $f(x) = \cos x$

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$P_{10}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$



66. $\sum_{n=0}^{\infty} (2x)^n$

Geometric series which converges only if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

68. $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{3^n(x-2)^n} \right|$$

$$= 3|x-2|$$

$R = \frac{1}{3}$

Center: 2

Since the series converges at $\frac{5}{3}$ and diverges at $\frac{7}{3}$, the interval of convergence is $\frac{5}{3} \leq x < \frac{7}{3}$.

70. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n$

Geometric series which converges only if

$$\left| \frac{x-2}{2} \right| < 1 \quad \text{or} \quad 0 < x < 4.$$

$$\begin{aligned}
72. \quad y &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!} \\
y' &= \sum_{n=1}^{\infty} \frac{(-3)^n (2n) x^{2n-1}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) x^{2n+1}}{2^{n+1} (n+1)!} \\
y'' &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) (2n+1) x^{2n}}{2^{n+1} (n+1)!} \\
y'' + 3xy' + 3y &= \sum_{n=0}^{\infty} \frac{(-3)^{n+1} (2n+2) (2n+1) x^{2n}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} (2n+2) x^{2n+2}}{2^{n+1} (n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+1} (2n+2) x^{2n}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-(2n+1) + 1] + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} (-2n) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^{n+2} x^{2n+2}}{2^n n!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1} x^{2n}}{2^n n!} (2n) + \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^{n-1} (n-1)!} \cdot \frac{2n}{2n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1} x^{2n}}{2^n n!} [-2n + 2n] = 0
\end{aligned}$$

$$74. \quad \frac{3}{2+x} = \frac{3/2}{1+(x/2)} = \frac{3/2}{1-(-x/2)} = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^n}{2^{n+1}}$$

$$76. \text{ Integral: } \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{n+1}}{(n+1)2^{n+1}}$$

$$\begin{aligned}
78. \quad 8 - 2(x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots &= \sum_{n=0}^{\infty} 8 \left[\frac{-(x-3)}{4} \right]^n = \frac{8}{1 - [-(x-3)/4]} \\
&= \frac{32}{4 + (x-3)} = \frac{32}{1+x}, \quad -1 < x < 7
\end{aligned}$$

$$80. \quad f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-\pi/4)[x + (\pi/4)]^n}{n!} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!} \left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!} \left(x + \frac{\pi}{4}\right)^4 + \cdots \\
&= \frac{\sqrt{2}}{2} \left[1 + \left(x + \frac{\pi}{4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{[n(n+1)]/2} [x + (\pi/4)]^{n+1}}{(n+1)!} \right]
\end{aligned}$$

$$82. \quad f(x) = \csc(x)$$

$$f'(x) = -\csc(x) \cot(x)$$

$$f''(x) = \csc^3(x) + \csc(x) \cot^2(x)$$

$$f'''(x) = -5 \csc^3(x) \cot(x) - \csc(x) \cot^3(x)$$

$$f^{(4)}(x) = 5 \csc^5(x) + 15 \csc^3(x) \cot^2(x) + \csc(x) \cot^4(x)$$

$$\csc(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)[x - (\pi/2)]^n}{n!} = 1 + \frac{1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{5}{4!} \left(x - \frac{\pi}{2}\right)^4 + \cdots$$

84. $f(x) = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-3/2}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$$

$$f^{(4)}(x) = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)x^{-7/2}, \dots$$

$$\begin{aligned}\sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(4)(x-4)^n}{n!} \\ &= 2 + \frac{(x-4)}{2^2} - \frac{(x-4)^2}{2^5 2!} + \frac{1 \cdot 3(x-4)^3}{2^8 3!} - \frac{1 \cdot 3 \cdot 5(x-4)^4}{2^{11} 4!} + \dots \\ &= 2 + \frac{(x-4)}{2^2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)(x-4)^n}{2^{3n-1} n!}\end{aligned}$$

86. $h(x) = (1+x)^{-3}$

$$h'(x) = -3(1+x)^{-4}$$

$$h''(x) = 12(1+x)^{-5}$$

$$h'''(x) = -60(1+x)^{-6}$$

$$h^{(4)}(x) = 360(1+x)^{-7}$$

$$h^{(5)}(x) = -2520(1+x)^{-8}$$

$$\frac{1}{(1+x)^3} = 1 - 3x + \frac{12x^2}{2!} - \frac{60x^3}{3!} + \frac{360x^4}{4!} - \frac{2520x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!x^n}{2n!} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)x}{2}$$

88. $\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$

$$\begin{aligned}\ln\left(\frac{6}{5}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(6/5)-1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n} \approx 0.1823\end{aligned}$$

90. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$

$$e^{2/3} = \sum_{n=0}^{\infty} \frac{(2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{3^n n!} \approx 1.9477$$

92. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$

$$\sin\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)!} \approx 0.3272$$

94. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \dots$$

$$\int_0^1 xe^x dx = \left[xe^x - e^x \right]_0^1 = (e - e) - (0 - 1) = 1$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} dx = \sum_{n=0}^{\infty} \left[\frac{x^{n+2}}{(n+2)n!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = 1$$

$$\begin{aligned}
96. \quad (a) \quad & f(x) = \sin 2x & f(0) &= 0 \\
& f'(x) = 2 \cos 2x & f'(0) &= 2 \\
& f''(x) = -4 \sin 2x & f''(0) &= 0 \\
& f'''(x) = -8 \cos 2x & f'''(0) &= -8 \\
& f^{(4)}(x) = 16 \sin 2x & f^{(4)}(0) &= 0 \\
& f^{(5)}(x) = 32 \cos 2x & f^{(5)}(0) &= 32 \\
& f^{(6)}(x) = -64 \sin 2x & f^{(6)}(0) &= 0 \\
& f^{(7)}(x) = -128 \cos 2x & f^{(7)}(0) &= -128
\end{aligned}$$

$$\sin 2x = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots$$

$$(b) \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}
\sin 2x &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots \\
&= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040} + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$(c) \quad \sin 2x = 2 \sin x \cos x$$

$$\begin{aligned}
&= 2 \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right) \\
&= 2 \left[x + \left(-\frac{x^3}{2} - \frac{x^3}{6} \right) + \left(\frac{x^5}{24} + \frac{x^5}{12} + \frac{x^5}{120} \right) + \left(-\frac{x^7}{720} - \frac{x^7}{144} - \frac{x^7}{240} - \frac{x^7}{5040} \right) + \cdots \right] \\
&= 2 \left[x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots \right] = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots
\end{aligned}$$

$$\begin{aligned}
98. \quad \cos t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \\
\cos \frac{\sqrt{t}}{2} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(2n)!} \\
\int_0^x \cos \frac{\sqrt{t}}{2} dt &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{2^{2n}(2n)!(n+1)} \right]_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{2^{2n}(2n)!(n+1)}
\end{aligned}$$

$$\begin{aligned}
100. \quad e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
e^t - 1 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \\
\frac{e^t - 1}{t} &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \\
\int_0^x \frac{e^t - 1}{t} dt &= \left[\sum_{n=1}^{\infty} \frac{t^n}{n \cdot n!} \right]_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}
\end{aligned}$$

$$102. \quad \arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\frac{\arcsin x}{x} = 1 + \frac{x^2}{2 \cdot 3} + \frac{1 \cdot 3x^4}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sqrt{1-x^2}} \right)}{1} = 1.$$

Problem Solving for Chapter 8

2. Let $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Then $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$$= S + \frac{1}{2^2} + \frac{1}{4^2} + \cdots$$

$$= S + \frac{1}{2^2} \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right]$$

$$= S + \frac{1}{2^2} \left(\frac{\pi^2}{6} \right).$$

Thus, $S = \frac{\pi^2}{6} - \frac{1}{4} \frac{\pi^2}{6} = \frac{\pi^2}{6} \left(\frac{3}{4} \right) = \frac{\pi^2}{8}.$

4. (a) Position the three blocks as indicated in the figure. The bottom block extends $1/6$ over the edge of the table, the middle block extends $1/4$ over the edge of the bottom block, and the top block extends $1/2$ over the edge of the middle block.

The centers of gravity are located at

bottom block: $\frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$

middle block: $\frac{1}{6} + \frac{1}{4} - \frac{1}{2} = -\frac{1}{12}$

top block: $\frac{1}{6} + \frac{1}{4} + \frac{1}{2} - \frac{1}{2} = \frac{5}{12}.$

The center of gravity of the top 2 blocks is

$$\left(-\frac{1}{12} + \frac{5}{12} \right) / 2 = \frac{1}{6},$$

which lies over the bottom block. The center of gravity of the 3 blocks is

$$\left(-\frac{1}{3} - \frac{1}{12} + \frac{5}{12} \right) / 3 = 0$$

which lies over the table. Hence, the far edge of the top block lies

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \frac{11}{12}$$

beyond the edge of the table.

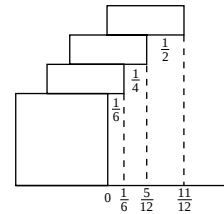
- (b) Yes. If there are n blocks, then the edge of the top block lies $\sum_{i=1}^n \frac{1}{2^i}$ from the edge of the table. Using 4 blocks,

$$\sum_{i=1}^4 \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

which shows that the top block extends beyond the table.

- (c) The blocks can extend any distance beyond the table because the series diverges:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \infty.$$



$$6. a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b) + (a-b)}{2n}$$

$$\text{If } a = b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2a)}{2n} = a \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges conditionally.}$$

$$\text{If } a \neq b, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(a+b)}{2n} + \sum_{n=1}^{\infty} \frac{a-b}{2n} \text{ diverges.}$$

No values of a and b give absolute convergence. $a = b$ implies conditional convergence.

$$8. e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{12}}{6!} + \cdots$$

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{6!} \Rightarrow f^{(12)}(0) = \frac{12!}{6!} = 665,280$$

$$10. (a) \text{ If } p = 1, \int_2^{\infty} \frac{1}{x \ln x} dx = \ln \ln x \Big|_2^{\infty} \text{ diverges.}$$

$$\text{If } p > 1, \int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \lim_{b \rightarrow \infty} \left[\frac{(\ln b)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right] \text{ converges.}$$

If $p < 1$, diverges.

$$(b) \sum_{n=4}^{\infty} \frac{1}{n \ln(n^2)} = \frac{1}{2} \sum_{n=4}^{\infty} \frac{1}{n \ln n} \text{ diverges by part (a).}$$

$$12. \text{ Let } b_n = a_n r^n.$$

$$(bn)^{1/n} = (a_n r^n)^{1/n} = a_n^{1/n} \cdot r \rightarrow Lr \text{ as } n \rightarrow \infty.$$

$$Lr < \frac{1}{r} = 1.$$

By the Root Test, $\sum b_n$ converges $\Rightarrow \sum a_n r^n$ converges.

$$14. (a) \frac{1}{0.99} = \frac{1}{1-0.01} = \sum_{n=0}^{\infty} (0.01)^n$$

$$= 1 + 0.01 + (0.01)^2 + \cdots$$

$$= 1.010101 \dots$$

$$(b) \frac{1}{0.98} = \frac{1}{1-0.02} = \sum_{n=0}^{\infty} (0.02)^n$$

$$= 1 + 0.02 + (0.02)^2 + \cdots$$

$$= 1 + 0.02 + 0.0004 + \cdots$$

$$= 1.0204081632 \dots$$

$$16. (a) \text{ Height} = 2 \left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \right]$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty \left(p\text{-series, } p = \frac{1}{2} < 1 \right)$$

$$(b) S = 4\pi \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots \right] 4\pi \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$(c) W = \frac{4}{3}\pi \left[1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots \right]$$

$$= \frac{4}{3}\pi \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges.}$$

CHAPTER 8

Infinite Series

Section 8.1	Sequences	121
Section 8.2	Series and Convergence	126
Section 8.3	The Integral Test and p-Series	131
Section 8.4	Comparisons of Series	135
Section 8.5	Alternating Series	138
Section 8.6	The Ratio and Root Tests	142
Section 8.7	Taylor Polynomials and Approximations	147
Section 8.8	Power Series	152
Section 8.9	Representation of Functions by Power Series	157
Section 8.10	Taylor and Maclaurin Series	160
Review Exercises		167
Problem Solving		172

CHAPTER 8

Infinite Series

Section 8.1 Sequences

Solutions to Odd-Numbered Exercises

1. $a_n = 2^n$

$$a_1 = 2^1 = 2$$

$$a_2 = 2^2 = 4$$

$$a_3 = 2^3 = 8$$

$$a_4 = 2^4 = 16$$

$$a_5 = 2^5 = 32$$

3. $a_n = \left(-\frac{1}{2}\right)^n$

$$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

$$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

$$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$$

5. $a_n = \sin \frac{n\pi}{2}$

$$a_1 = \sin \frac{\pi}{2} = 1$$

$$a_2 = \sin \pi = 0$$

$$a_3 = \sin \frac{3\pi}{2} = -1$$

$$a_4 = \sin 2\pi = 0$$

$$a_5 = \sin \frac{5\pi}{2} = 1$$

7. $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$$a_1 = \frac{(-1)^1}{1^2} = -1$$

$$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$$

$$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$$

$$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$$

$$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$$

9. $a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$

$$a_1 = 5 - 1 + 1 = 5$$

$$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$$

$$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$$

$$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$$

$$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$$

11. $a_n = \frac{3^n}{n!}$

$$a_1 = \frac{3}{1!} = 3$$

$$a_2 = \frac{3^2}{2!} = \frac{9}{2}$$

$$a_3 = \frac{3^3}{3!} = \frac{27}{6}$$

$$a_4 = \frac{3^4}{4!} = \frac{81}{24}$$

$$a_5 = \frac{3^5}{5!} = \frac{243}{120}$$

13. $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$$a_2 = 2(a_1 - 1)$$

$$= 2(3 - 1) = 4$$

$$a_3 = 2(a_2 - 1)$$

$$= 2(4 - 1) = 6$$

$$a_4 = 2(a_3 - 1)$$

$$= 2(6 - 1) = 10$$

$$a_5 = 2(a_4 - 1)$$

$$= 2(10 - 1) = 18$$

15. $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$

$$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$$

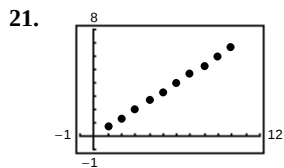
$$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$$

$$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$$

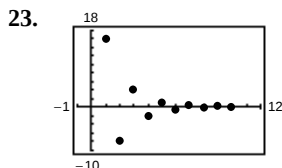
$$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$$

17. Because $a_1 = 8/(1 + 1) = 4$ and $a_2 = 8/(2 + 1) = \frac{8}{3}$, the sequence matches graph (d).

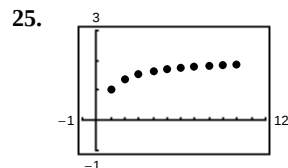
19. This sequence decreases and $a_1 = 4$, $a_2 = 4(0.5) = 2$. Matches (c).



$$a_n = \frac{2}{3}n, n = 1, \dots, 10$$



$$a_n = 16(-0.5)^{n-1}, n = 1, \dots, 10$$



$$a_n = \frac{2n}{n+1}, n = 1, 2, \dots, 10$$

27. $a_n = 3n - 1$

$$a_5 = 3(5) - 1 = 14$$

$$a_6 = 3(6) - 1 = 17$$

Add 3 to preceeding term.

29. $a_n = \frac{3}{(-2)^{n-1}}$

$$a_n = \frac{3}{(-2)^4} = \frac{3}{16}$$

$$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$$

Multiply the preceeding term by $-\frac{1}{2}$.

31. $\frac{10!}{8!} = \frac{8!(9)(10)}{8!}$
 $= (9)(10) = 90$

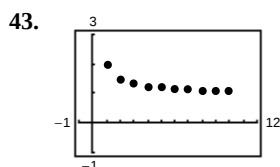
33. $\frac{(n+1)!}{n!} = \frac{n!(n+1)}{n!}$
 $= n+1$

35. $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)}$
 $= \frac{1}{2n(2n+1)}$

37. $\lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + 2} = 5$

39. $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{1 + (1/n^2)}}$
 $= \frac{2}{1} = 2$

41. $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$



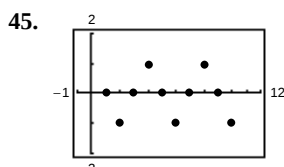
The graph seems to indicate that the sequence converges to 1. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = \lim_{x \rightarrow \infty} 1 = 1.$$

47. $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{n+1} \right)$

does not exist (oscillates between -1 and 1), diverges.

51. $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n} = 0$, converges



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, \dots\}.$$

Hence, $\lim_{n \rightarrow \infty} a_n$ does not exist.

49. $\lim_{n \rightarrow \infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2}$, converges

53. $\lim_{n \rightarrow \infty} \frac{\ln(n^3)}{2n} = \lim_{n \rightarrow \infty} \frac{3 \ln(n)}{2n}$
 $= \lim_{n \rightarrow \infty} \frac{3 \left(\frac{1}{n} \right)}{2} = 0$, converges

(L'Hôpital's Rule)

$$55. \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0, \text{ converges}$$

$$57. \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty, \text{ diverges}$$

$$59. \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} - \frac{n}{n-1}\right) = \lim_{n \rightarrow \infty} \frac{(n-1)^2 - n^2}{n(n-1)} \\ = \lim_{n \rightarrow \infty} \frac{1-2n}{n^2-n} = 0, \text{ converges}$$

$$61. \lim_{n \rightarrow \infty} \frac{n^p}{e^n} = 0, \text{ converges} \\ (p > 0, n \geq 2)$$

$$63. a_n = \left(1 + \frac{k}{n}\right)^n \\ \lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = \lim_{u \rightarrow 0} [(1+u)^{1/u}]^k = e^k$$

where $u = \frac{k}{n}$, converges

$$65. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} (\sin n) \frac{1}{n} = 0, \text{ converges}$$

$$67. a_n = 3n - 2$$

$$69. a_n = n^2 - 2$$

$$71. a_n = \frac{n+1}{n+2}$$

$$73. a_n = \frac{(-1)^{n-1}}{2^{n-2}}$$

$$75. a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$$

$$77. a_n = \frac{n}{(n+1)(n+2)}$$

$$79. a_n = \frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{(-1)^{n-1} 2^n n!}{(2n)!}$$

$$81. a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1}, \\ \text{monotonic; } |a_n| < 4 \text{ bounded.}$$

$$83. \frac{n}{2^{n+2}} \stackrel{?}{\geq} \frac{n+1}{2^{(n+1)+2}}$$

$$2^{n+3}n \stackrel{?}{\geq} 2^{n+2}(n+1)$$

$$2n \stackrel{?}{\geq} n+1$$

$$n \geq 1$$

Hence, $n \geq 1$

$$2n \geq n+1$$

$$2^{n+3}n \geq 2^{n+2}(n+1)$$

$$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$$

$$a_n \geq a_{n+1}$$

True; monotonic; $|a_n| \leq \frac{1}{8}$, bounded

$$85. a_n = (-1)^n \left(\frac{1}{n}\right)$$

$$a_1 = -1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

Not monotonic; $|a_n| \leq 1$, bounded

$$87. a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$$

Monotonic; $|a_n| \leq \frac{2}{3}$, bounded

$$89. a_n = \sin\left(\frac{n\pi}{6}\right)$$

$$a_1 = 0.500$$

$$a_2 = 0.8660$$

$$a_3 = 1.000$$

$$a_4 = 0.8660$$

Not monotonic; $|a_n| \leq 1$, bounded

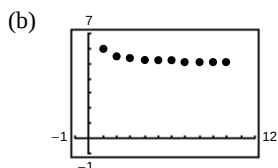
91. (a) $a_n = 5 + \frac{1}{n}$

$$\left| 5 + \frac{1}{n} \right| \leq 6 \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(5 + \frac{1}{n} \right) = 5$$

95. $A_n = P \left[1 + \frac{r}{12} \right]^n$

(a) $\lim_{n \rightarrow \infty} A_n = \infty$, divergent. The amount will grow arbitrarily large over time.

(b) $A_n = 9000 \left[1 + \frac{0.115}{12} \right]^n$

$$A_1 = \$9086.25 \quad A_6 = \$9530.06$$

$$A_2 = \$9173.33 \quad A_7 = \$9621.39$$

$$A_3 = \$9261.24 \quad A_8 = \$9713.59$$

$$A_4 = \$9349.99 \quad A_9 = \$9806.68$$

$$A_5 = \$9439.60 \quad A_{10} = \$9900.66$$

99. $a_n = 10 - \frac{1}{n}$

103. (a) $A_n = (0.8)^n (2.5)$ billion

(b) $A_1 = \$2$ billion

$$A_2 = \$1.6 \text{ billion}$$

$$A_3 = \$1.28 \text{ billion}$$

$$A_4 = \$1.024 \text{ billion}$$

(c) $\lim_{n \rightarrow \infty} (0.8)^n (2.5) = 0$

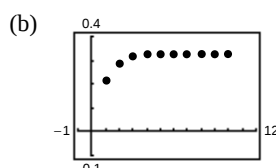
93. (a) $a_n = \frac{1}{3} \left(1 - \frac{1}{3^n} \right)$

$$\left| \frac{1}{3} \left(1 - \frac{1}{3^n} \right) \right| < \frac{1}{3} \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = \frac{1}{3} \left(1 - \frac{1}{3^n} \right) < \frac{1}{3} \left(1 - \frac{1}{3^{n+1}} \right)$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left[\frac{1}{3} \left(1 - \frac{1}{3^n} \right) \right] = \frac{1}{3}$$

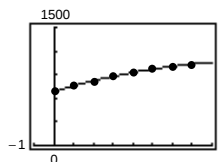
97. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit.

(c) A bounded monotonic sequence is a sequence that has nondecreasing or nonincreasing terms, and an upper and lower bound.

101. $a_n = \frac{3n}{4n+1}$

105. (a) $a_n = -3.7262n^2 + 75.9167n + 684.25$



(b) For 2004, $n = 14$ and $a_{14} \approx 1017$, or \$1017.

107. $a_n = \frac{10^n}{n!}$

$$\begin{aligned} \text{(a) } a_9 &= a_{10} = \frac{10^9}{9!} \\ &= \frac{1,000,000,000}{362,880} \\ &= \frac{1,562,500}{567} \end{aligned}$$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

109. $\{a_n\} = \{\sqrt[n]{n}\} = \{n^{1/n}\}$

$$a_1 = 1^{1/1} = 1$$

$$a_2 = \sqrt{2} \approx 1.4142$$

$$a_3 = \sqrt[3]{3} \approx 1.4422$$

$$a_4 = \sqrt[4]{4} \approx 1.4142$$

$$a_5 = \sqrt[5]{5} \approx 1.3797$$

$$a_6 = \sqrt[6]{6} \approx 1.3480$$

$$\text{Let } y = \lim_{n \rightarrow \infty} n^{1/n}.$$

$$\begin{aligned} \ln y &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \ln n \right) \\ &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0 \end{aligned}$$

Since $\ln y = 0$, we have $y = e^0 = 1$. Therefore,
 $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

111. $a_{n+2} = a_n + a_{n+1}$

$$\begin{array}{ll} \text{(a) } a_1 = 1 & a_7 = 8 + 5 = 13 \\ a_2 = 1 & a_8 = 13 + 8 = 21 \\ a_3 = 1 + 1 = 2 & a_9 = 21 + 13 = 34 \\ a_4 = 2 + 1 = 3 & a_{10} = 34 + 21 = 55 \\ a_5 = 3 + 2 = 5 & a_{11} = 55 + 34 = 89 \\ a_6 = 5 + 3 = 8 & a_{12} = 89 + 55 = 144 \end{array}$$

(b) $b_n = \frac{a_{n+1}}{a_n}, n \geq 1$

$$\begin{array}{ll} b_1 = \frac{1}{1} = 1 & b_6 = \frac{13}{8} \\ b_2 = \frac{2}{1} = 2 & b_7 = \frac{21}{13} \\ b_3 = \frac{3}{2} & b_8 = \frac{34}{21} \\ b_4 = \frac{5}{3} & b_9 = \frac{55}{34} \\ b_5 = \frac{8}{5} & b_{10} = \frac{89}{55} \end{array}$$

$$\begin{aligned} \text{(c) } 1 + \frac{1}{b_{n-1}} &= 1 + \frac{1}{a_n/a_{n-1}} \\ &= 1 + \frac{a_{n-1}}{a_n} \\ &= \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n \end{aligned}$$

(d) If $\lim_{n \rightarrow \infty} b_n = \rho$, then $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_{n-1}} \right) = \rho$.

Since $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} b_{n-1}$ we have,

$$1 + (1/\rho) = \rho.$$

$$\rho + 1 = \rho^2$$

$$0 = \rho^2 - \rho - 1$$

$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since a_n , and thus b_n , is positive,

$$\rho = (1 + \sqrt{5})/2 \approx 1.6180.$$

113. True

115. True

117. $a_1 = \sqrt{2} \approx 1.4142$

$$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$$

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$$

$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$$

$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$$

$\{a_n\}$ is increasing and bounded by 2, and hence converges to L . Letting $\lim_{n \rightarrow \infty} a_n = L$ implies that $\sqrt{2 + L} = L \Rightarrow L = 2$. Hence, $\lim_{n \rightarrow \infty} a_n = 2$.

Section 8.2 Series and Convergence

1. $S_1 = 1$

$$S_2 = 1 + \frac{1}{4} = 1.2500$$

$$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$$

$$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$$

$$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$$

3. $S_1 = 3$

$$S_2 = 3 - \frac{9}{2} = -1.5$$

$$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$$

$$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$$

$$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$$

5. $S_1 = 3$

$$S_2 = 3 + \frac{3}{2} = 4.5$$

$$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$$

$$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$$

$$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$$

7. $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$ Geometric series

$$r = \frac{3}{2} > 1$$

Diverges by Theorem 8.6

9. $\sum_{n=0}^{\infty} 1000(1.055)^n$ Geometric series

$$r = 1.055 > 1$$

Diverges by Theorem 8.6

11. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Diverges by Theorem 8.9

13. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

Diverges by Theorem 8.9

15. $\sum_{n=0}^{\infty} \frac{2^n+1}{2^{n+1}}$

$$\lim_{n \rightarrow \infty} \frac{2^n+1}{2^{n+1}} = \lim_{n \rightarrow \infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

17. $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right]$

$$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \dots$$

Matches graph (c).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

19. $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15}{4}\left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$

$$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \dots$$

Matches graph (a).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$$

21. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

23. $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Geometric series with $r = \frac{3}{4} < 1$.

Converges by Theorem 8.6

25. $\sum_{n=0}^{\infty} (0.9)^n$

Geometric series with $r = 0.9 < 1$.

Converges by Theorem 8.6

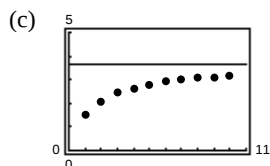
$$27. (a) \sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

$$= 2 \left[\left(1 - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \cdots \right]$$

$$= 2 \left[1 + \frac{1}{2} + \frac{1}{3} \right] = \frac{11}{3} \approx 3.667$$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

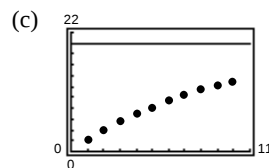


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$29. (a) \sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$$

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

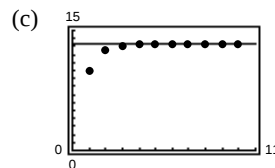


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$31. (a) \sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

$$33. \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1/2}{n-1} - \frac{1/2}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots \right]$$

$$= \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}$$

$$35. \sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8 \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 8 \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \cdots \right] = 8 \left(\frac{1}{2} \right) = 4$$

$$37. \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{1-(1/2)} = 2$$

$$39. \sum_{n=0}^{\infty} \left(-\frac{1}{2} \right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$$

$$41. \sum_{n=0}^{\infty} \left(\frac{1}{10} \right)^n = \frac{1}{1-(1/10)} = \frac{10}{9}$$

$$43. \sum_{n=0}^{\infty} 3 \left(-\frac{1}{3} \right)^n = \frac{3}{1-(-1/3)} = \frac{9}{4}$$

$$\begin{aligned}
 45. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n \\
 &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} \\
 &= 2 - \frac{3}{2} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 49. 0.075\overline{75} &= \sum_{n=0}^{\infty} \frac{3}{40} \left(\frac{1}{100} \right)^n \\
 \text{Geometric series with } a &= \frac{3}{40} \text{ and } r = \frac{1}{100} \\
 S &= \frac{a}{1 - r} = \frac{3/40}{99/100} = \frac{5}{66}
 \end{aligned}$$

$$53. \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}, \text{ converges}$$

$$\begin{aligned}
 55. \sum_{n=1}^{\infty} \frac{3n-1}{2n+1} \\
 \lim_{n \rightarrow \infty} \frac{3n-1}{2n+1} &= \frac{3}{2} \neq 0 \\
 \text{Diverges by Theorem 8.9}
 \end{aligned}$$

$$\begin{aligned}
 61. \sum_{n=2}^{\infty} \frac{n}{\ln n} \\
 \lim_{n \rightarrow \infty} \frac{n}{\ln n} &= \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty \\
 \text{(by L'Hôpital's Rule) Diverges by Theorem 8.9}
 \end{aligned}$$

65. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$$

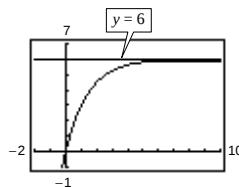
is a geometric series with ratio r . When $0 < |r| < 1$, the series converges to $\frac{a}{1-r}$. The series diverges if $|r| \geq 1$.

$$69. f(x) = 3 \left[\frac{1 - 0.5^x}{1 - 0.5} \right]$$

Horizontal asymptote: $y = 6$

$$\begin{aligned}
 \sum_{n=0}^{\infty} 3 \left(\frac{1}{2} \right)^n \\
 S = \frac{3}{1 - (1/2)} = 6
 \end{aligned}$$

The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.



$$\begin{aligned}
 47. 0.\overline{4} &= \sum_{n=0}^{\infty} \frac{4}{10} \left(\frac{1}{10} \right)^n \\
 \text{Geometric series with } a &= \frac{4}{10} \text{ and } r = \frac{1}{10} \\
 S &= \frac{a}{1 - r} = \frac{4/10}{1 - (1/10)} = \frac{4}{9}
 \end{aligned}$$

$$\begin{aligned}
 51. \sum_{n=1}^{\infty} \frac{n+10}{10n+1} \\
 \lim_{n \rightarrow \infty} \frac{n+10}{10n+1} &= \frac{1}{10} \neq 0 \\
 \text{Diverges by Theorem 8.9}
 \end{aligned}$$

$$\begin{aligned}
 57. \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \\
 \text{Geometric series with } r &= \frac{1}{2} \\
 \text{Converges by Theorem 8.6}
 \end{aligned}$$

$$\begin{aligned}
 59. \sum_{n=0}^{\infty} (1.075)^n \\
 \text{Geometric series with } r &= 1.075 \\
 \text{Diverges by Theorem 8.6}
 \end{aligned}$$

63. See definition, page 567.

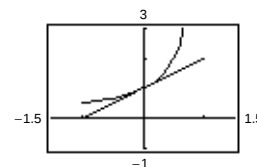
67. (a) x is the common ratio.

$$(b) 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$$

Geometric series: $a = 1, r = x, |x| < 1$

$$(c) y_1 = \frac{1}{1-x}$$

$$y_2 = 1 + x$$



$$71. \frac{1}{n(n+1)} < 0.001$$

$$10,000 < n^2 + n$$

$$0 < n^2 + n - 10,000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10,000)}}{2}$$

Choosing the positive value for n we have $n \approx 99.5012$. The first *term* that is less than 0.001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.001$$

$$10,000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

$$73. \sum_{i=0}^{n-1} 8000(0.9)^i = \frac{8000[1 - (0.9)^{(n-1)+1}]}{1 - 0.9}$$

$$= 80,000(1 - 0.9^n), \quad n > 0$$

$$75. \sum_{i=0}^{n-1} 100(0.75)^i = \frac{100[1 - 0.75^{(n-1)+1}]}{1 - 0.75}$$

$$= 400(1 - 0.75^n) \text{ million dollars.}$$

Sum = 400 million dollars

$$77. D_1 = 16$$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

$\vdots$

$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots = -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81} = 152.42 \text{ ft}$$

$$79. P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$$

$$P(2) = \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$$

$$81. (a) \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n = \frac{1}{2} \frac{1}{1 - (1/2)} = 1$$

(b) No, the series is not geometric.

$$(c) \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$$

$$83. \text{ Present Value} = \sum_{n=1}^{19} 50,000 \left(\frac{1}{1.06}\right)^n$$

$$= \sum_{n=0}^{18} \frac{50,000}{1.06} \left(\frac{1}{1.06}\right)^n, \quad r = \frac{1}{1.06}$$

$$= \frac{50,000}{1.06} \left(\frac{1 - 1.06^{-19}}{1 - 1.06^{-1}}\right)$$

$$\approx \$557,905.82$$

The present value is less than \$1,000,000. After accruing interest over 20 years, it attains its full value.

$$85. w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$$

(a) When $n = 29$: $w = \$5,368,709.11$

(b) When $n = 30$: $w = \$10,737,418.23$

(c) When $n = 31$: $w = \$21,474,836.47$

87. $P = 50, r = 0.03, t = 20$

(a) $A = 50 \left(\frac{12}{0.03} \right) \left[\left(1 + \frac{0.03}{12} \right)^{12(20)} - 1 \right] \approx \$16,415.10$

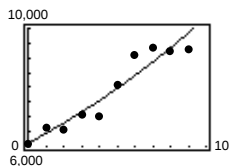
(b) $A = \frac{50 - (e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$16,421.83$

89. $P = 100, r = 0.04, t = 40$

(a) $A = 100 \left(\frac{12}{0.04} \right) \left[\left(1 + \frac{0.04}{12} \right)^{12(40)} - 1 \right] \approx \$118,196.13$

(b) $A = \frac{100(e^{0.04(40)} - 1)}{e^{0.04/12} - 1} \approx \$118,393.43$

91. (a) $a_n = 6110.1832(1.0544)^x = 6110.1832e^{0.05297n}$



(b) 78,530 or \$78,530,000,000

(c) $\text{Total} = \sum_{n=0}^9 a_n \approx 78,449 \text{ or } \$78,449,000,000$

95. By letting $S_0 = 0$, we have $a_n = \sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$. Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_n - S_{n-1}) = \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) = \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)].$$

97. Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

99. False. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

101. False

$$\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r} \right) - a$$

The formula requires that the geometric series begins with $n = 0$.103. Let H represent the half-life of the drug. If a patient receives n equal doses of P units each of this drug, administered at equal time interval of length t , the total amount of the drug in the patient's system at the time the last dose is administered is given by

$$T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$$

where $k = -(\ln 2)/H$. One time interval *after* the last dose is administered is given by

$$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}.$$

Two time intervals *after* the last dose is administered is given by

$$T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \cdots + Pe^{(n+1)kt}$$

and so on. Since $k < 0$, $T_{n+s} \rightarrow 0$ as $s \rightarrow \infty$, where s is an integer.

Section 8.3 The Integral Test and p -Series

1. $\sum_{n=1}^{\infty} \frac{1}{n+1}$

Let $f(x) = \frac{1}{x+1}$.

 f is positive, continuous and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x+1} dx = \left[\ln(x+1) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \left[\arctan x \right]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 8.10

9. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$

Let $f(x) = \frac{x^{k-1}}{x^k + c}$.

 f is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1) - x^k]}{(x^k + c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k + c} dx = \left[\frac{1}{k} \ln(x^k + c) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

13. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent p -series with $p = \frac{1}{5} < 1$

17. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p -series with $p = \frac{3}{2} > 1$

3. $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x} dx = \left[-e^{-x} \right]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 8.10

7. $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \frac{\ln(x+1)}{x+1}$.

 f is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{1 - \ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1} dx = \left[\frac{\ln^2(x+1)}{2} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

11. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 8.10

15. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent p -series with $p = \frac{1}{2} < 1$

19. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p -series with $p = 1.04 > 1$

$$21. \sum_{n=1}^{\infty} \frac{2}{\sqrt[4]{n^3}} = \frac{2}{1} + \frac{2}{2^{3/4}} + \frac{2}{3^{3/4}} + \cdots$$

$$S_1 = 2$$

$$S_2 \approx 3.189$$

$$S_3 \approx 4.067$$

Matches (a)

Diverges— p -series with $p = \frac{3}{4} < 1$

$$23. \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = 2 + 2/2^{3/2} + 2/3^{3/2} + \cdots$$

$$S_1 = 2$$

$$S_2 \approx 2.707$$

$$S_3 \approx 3.092$$

Matches (b)

Converges— p -series with $p = 3/2 > 1$

25. No. Theorem 8.9 says that if the series converges, then the terms a_n tend to zero. Some of the series in Exercises 21–24 converge because the terms tend to 0 very rapidly.

$$27. \sum_{n=1}^N \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$$

(a)

M	2	4	6	8
N	4	31	227	1674

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

$$29. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x} dx = \left[\frac{(\ln x)^{-p+1}}{-p+1} \right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$.

31. Let f be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x) dx$$

either both converge or both diverge (Theorem 8.10).

See Example 1, page 578.

33. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and hence diverges.

35. Since f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

$$\text{Also, } R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x) dx \leq \int_N^{\infty} f(x) dx. \text{ Thus,}$$

$$0 \leq R_N \leq \int_N^{\infty} f(x) dx.$$

$$37. S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

$$39. S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan x \right]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^{\infty} \frac{1}{n^5} \leq 0.9818 + 0.0997 = 1.0815$$

$$41. S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$$

$$R_4 \leq \int_4^{\infty} x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_4^{\infty} = 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

$$43. 0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_N^{\infty} = \frac{1}{3N^3} < 0.001$$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

$$45. R_N \leq \int_N^{\infty} e^{-5x} dx = \left[-\frac{1}{5} e^{-5x} \right]_N^{\infty} = \frac{e^{-5N}}{5} < 0.001$$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

$$47. R_N \leq \int_N^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan x \right]_N^{\infty} = \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < -1.5698$$

$$\arctan N > 1.5698$$

$$N > \tan 1.5698$$

$$N \geq 1004$$

$$49. (a) \sum_{n=2}^{\infty} \frac{1}{n^{1.1}}. \text{ This is a convergent } p\text{-series with } p = 1.1 > 1.$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ is a divergent series. Use the Integral Test.}$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \left[\ln |\ln x| \right]_2^{\infty} = \infty$$

$$(b) \sum_{n=2}^6 \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$$

$$\sum_{n=2}^6 \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$$

The terms of the convergent series **seem** to be larger than those of the divergent series!

$$(c) \frac{1}{n^{1.1}} < \frac{1}{n \ln n}$$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$.

51. (a) Let
- $f(x) = 1/x$
- .
- f
- is positive, continuous, and decreasing on
- $[1, \infty)$
- .

$$S_n - 1 \leq \int_1^n \frac{1}{x} dx$$

$$S_n - 1 \leq \ln n$$

Hence, $S_n \leq 1 + \ln n$. Similarly,

$$S_n \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

Thus, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

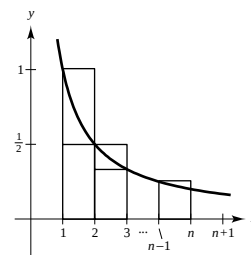
- (b) Since
- $\ln(n+1) \leq S_n \leq 1 + \ln n$
- , we have
- $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$
- . Also, since
- $\ln x$
- is an increasing function,
- $\ln(n+1) - \ln n > 0$
- for
- $n \geq 1$
- . Thus,
- $0 \leq S_n - \ln n \leq 1$
- and the sequence
- $\{a_n\}$
- is bounded.

$$(c) a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x} dx - \frac{1}{n+1} \geq 0$$

Thus, $a_n \geq a_{n+1}$ and the sequence is decreasing.

- (d) Since the sequence is bounded and monotonic, it converges to a limit,
- γ
- .

$$(e) a_{100} = S_{100} - \ln 100 \approx 0.5822 \text{ (Actually } \gamma \approx 0.577216.)$$



53. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Let $f(x) = \frac{1}{2x-1}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x-1} dx = \left[\ln \sqrt{2x-1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

55. $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$

 p -series with $p = \frac{5}{4}$

Converges by Theorem 8.11

57. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 8.6

59. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 8.9

61. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

Fails n th Term Test

Diverges by Theorem 8.9

63. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

Let $f(x) = \frac{1}{x(\ln x)^3}$.

 f is positive, continuous and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx = \int_2^{\infty} (\ln x)^{-3} \frac{1}{x} dx = \left[\frac{(\ln x)^{-2}}{-2} \right]_2^{\infty} = \left[-\frac{1}{2(\ln x)^2} \right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

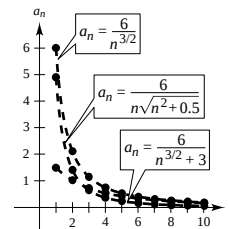
Converges by Theorem 8.10. See Exercise 13.

Section 8.4 Comparisons of Series

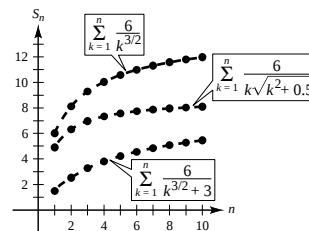
$$1. (a) \sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots \quad S_1 = 6$$

$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \cdots \quad S_1 = \frac{3}{2}$$

$$\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots \quad S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$$



- (b) The first series is a p -series. It converges ($p = 3/2 > 1$).
- (c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent p -series. Hence, the other two series converge.
- (d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



$$3. \frac{1}{n^2 + 1} < \frac{1}{n^2}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$7. \frac{1}{3^n + 1} < \frac{1}{3^n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

$$11. \text{ For } n > 3, \frac{1}{n^2} > \frac{1}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$5. \frac{1}{n-1} > \frac{1}{n} \text{ for } n \geq 2$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n-1}$$

diverges by comparison with the divergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

$$9. \text{ For } n \geq 3, \frac{\ln n}{n+1} > \frac{1}{n+1}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Note: $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

$$13. \frac{1}{e^{n^2}} \leq \frac{1}{e^n}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n.$$

$$15. \lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$19. \lim_{n \rightarrow \infty} \frac{2n^2 - 1}{3n^5 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} = \frac{2}{3}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$23. \lim_{n \rightarrow \infty} \frac{1/(n\sqrt{n^2 + 1})}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$27. \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2) \cos(1/n)}{-1/n^2} \\ = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$31. \sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

Converges

Direct comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

$$33. \sum_{n=1}^{\infty} \frac{n}{2n + 3}$$

Diverges; n th Term Test

$$\lim_{n \rightarrow \infty} \frac{n}{2n + 3} = \frac{1}{2} \neq 0$$

$$17. \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$21. \lim_{n \rightarrow \infty} \frac{n + 3}{n(n + 2)} = \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{n^2 + 2n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n + 3}{n(n + 2)}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$25. \lim_{n \rightarrow \infty} \frac{(n^{k-1})/(n^k + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^k}{n^k + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$29. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Diverges

p -series with $p = \frac{1}{2}$

$$35. \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

Converges; integral test

37. $\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = \lim_{n \rightarrow \infty} na_n$ by given conditions $\lim_{n \rightarrow \infty} na_n$ is finite and nonzero.

Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

41. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

converges since the degree of the numerator is three less than the degree of the denominator.

45. See Theorem 8.12, page 583. One example is $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges because

$$\frac{1}{n^2 + 1} < \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (p -series).

49. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$, diverges

53. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

57. True

59. Since $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{n \rightarrow \infty} b_n = 0$. There exists N such that $b_n < 1$ for $n > N$. Thus,

$$a_n b_n < a_n \text{ for } n > N \text{ and } \sum_{n=1}^{\infty} a_n b_n$$

converges by comparison to the convergent series $\sum_{i=1}^{\infty} a_n$.

61. $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, and hence so does $\sum \left(\frac{1}{n^2} \right) \left(\frac{1}{n^3} \right) = \sum \frac{1}{n^5}$.

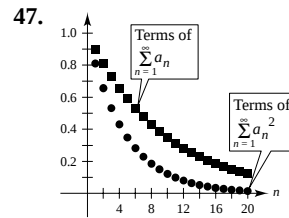
39. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$,

which diverges since the degree of the numerator is only one less than the degree of the denominator.

43. $\lim_{n \rightarrow \infty} n \left(\frac{n^3}{5n^4 + 3} \right) = \lim_{n \rightarrow \infty} \frac{n^4}{5n^4 + 3} = \frac{1}{5} \neq 0$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3} \text{ diverges.}$$



For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$. Hence, the lower terms are those of $\sum a_n^2$.

51. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200 + n^2}$, converges

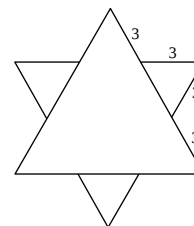
55. False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$. $0 < a_n \leq b_n$ and both

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge.

63. (a) Suppose $\sum b_n$ converges and $\sum a_n$ diverges. Then there exists N such that $0 < b_n < a_n$ for $n \geq N$. This means that $1 < a_n/b_n$ for $n \geq N$. Therefore, $\lim_{n \rightarrow \infty} a_n/b_n \neq 0$. Thus, $\sum a_n$ must also converge.
- (b) Suppose $\sum b_n$ diverges and $\sum a_n$ converges. Then there exists N such that $0 < a_n < b_n$ for $n \geq N$. This means that $0 < a_n/b_n < 1$ for $n \geq N$. Therefore, $\lim_{n \rightarrow \infty} a_n/b_n \neq \infty$. Thus, $\sum a_n$ must also diverge.
65. Start with one triangle whose sides have length 9. At the n th step, each side is replaced by four smaller line segments each having $\frac{1}{3}$ the length of the original side.

#Sides	Length of sides
3	9
$3 \cdot 4$	$9(\frac{1}{3})$
$3 \cdot 4^2$	$9(\frac{1}{3})^2$
$\vdots$	
$3 \cdot 4^n$	$9(\frac{1}{3})^n$



At the n th step there are $3 \cdot 4^n$ sides, each of length $9(\frac{1}{3})^n$. At the next step, there are $3 \cdot 4^{n+1}$ new triangles of side $9(\frac{1}{3})^{n+1}$. The area of an equilateral triangle of side x is $\frac{1}{4}\sqrt{3}x^2$. Thus, the new triangles each have area

$$9 \frac{\sqrt{3}}{4} \left(\frac{1}{3^{n+1}} \right)^2 = \frac{\sqrt{3}}{4} \frac{1}{3^{2n+2}}.$$

The area of the $3 \cdot 4^n$ new triangles is

$$(3 \cdot 4^n) \left(\frac{\sqrt{3}}{4} \frac{1}{3^{2n+2}} \right) = \frac{3\sqrt{3}}{4} \left(\frac{4}{9} \right)^n.$$

The total area is the infinite sum

$$\frac{9\sqrt{3}}{4} + \sum_{n=0}^{\infty} \frac{3\sqrt{3}}{4} \left(\frac{4}{9} \right)^n = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \left(\frac{1}{1 - 4/9} \right) = \frac{9\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \left(\frac{9}{5} \right) = \frac{18\sqrt{3}}{5}.$$

The perimeter is infinite, since at step n there are $3 \cdot 4^n$ sides of length $9(\frac{1}{3})^n$. Thus, the perimeter at step n is $27(\frac{4}{3})^n \rightarrow \infty$.

Section 8.5 Alternating Series

$$1. \sum_{n=1}^{\infty} \frac{6}{n^2} = \frac{6}{1} + \frac{6}{4} + \frac{6}{9} + \dots$$

$$S_1 = 6, S_2 = 7.5$$

Matches (b)

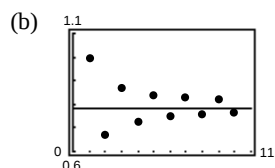
$$3. \sum_{n=1}^{\infty} \frac{10}{n2^n} = \frac{10}{2} + \frac{10}{8} + \dots$$

$$S_1 = 5, S_2 = 6.25$$

Matches (c)

$$5. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$$

(a)	n	1	2	3	4	5	6	7	8	9	10
	S_n	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605



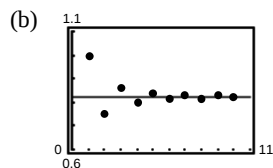
(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

$$7. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

$$9. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Converges by Theorem 8.14.

$$11. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$$

Converges by Theorem 8.14

$$13. \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Diverges by the n th Term Test

$$15. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 8.14

$$17. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{1/(n+1)} = \lim_{n \rightarrow \infty} (n+1) = \infty$$

Diverges by the n th Term Test

$$19. \sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Diverges by the n th Term Test

$$21. \sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$$

Diverges by the n th Term Test

$$23. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Converges by Theorem 8.14

$$25. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n+2}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 8.14

$$27. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$$

Let $f(x) = \frac{2e^x}{e^{2x} - 1}$. Then

$$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$$

Thus, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 8.14.

$$29. S_6 = \sum_{n=1}^6 \frac{3(-1)^{n+1}}{n^2} = 2.4325$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{3}{49} \approx 0.0612; 2.3713 \leq S \leq 2.4937$$

$$31. S_6 = \sum_{n=0}^5 \frac{2(-1)^n}{n!} \approx 0.7333$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{2}{6!} = 0.002778; 0.7305 \leq S \leq 0.7361$$

$$33. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when $N = 6$.

(b) We may approximate the series by

$$\begin{aligned} \sum_{n=0}^6 \frac{(-1)^n}{n!} &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \\ &\approx 0.368. \end{aligned}$$

(7 terms. Note that the sum begins with $n = 0$.)

$$35. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when $N = 2$.

(b) We may approximate the series by

$$\sum_{n=0}^2 \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with $n = 0$.)

$$37. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when $N = 1000$.

(b) We may approximate the series by

$$\begin{aligned} \sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \\ &\approx 0.693. \end{aligned}$$

(1000 terms)

$$39. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$$

By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when $N = 7$.

$$41. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \text{ converges by comparison to the } p\text{-series}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converge absolutely.

$$45. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the n th Term Test.

$$49. \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 1}$$

$$\sum_{n=2}^{\infty} \frac{n}{n^3 - 1}$$

converges by a limit comparison to the convergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converges absolutely.

$$53. \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1, \text{ therefore the series}$$

converges conditionally.

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p -series. Therefore, the series converges conditionally.

$$47. \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges by comparison to the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, the series converges conditionally.

$$51. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

$$55. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p -series. Therefore, the given series converges absolutely.

57. An alternating series is a series whose terms alternate in sign. See Theorem 8.14.
59. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
 $\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.
61. (b). The partial sums alternate above and below the horizontal line representing the sum.
63. Since $\sum_{n=1}^{\infty} |a_n|$ converges we have

$$\lim_{n \rightarrow \infty} |a_n| = 0.$$
 Thus, there must exist an $N > 0$ such that $|a_n| < 1$ for all $n > N$ and it follows that $a_n^2 \leq |a_n|$ for all $n > N$. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$
 converges. Let $a_n = 1/n$ to see that the converse is false.
67. False
 Let $a_n = \frac{(-1)^n}{n}$.
69. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent p -series
71. Diverges by n th Term Test. $\lim_{n \rightarrow \infty} a_n = \infty$
73. Convergent Geometric Series ($r = \frac{7}{8} < 1$)
75. Convergent Geometric Series ($r = \frac{1}{\sqrt{e}}$) or Integral Test
77. Converges (absolutely) by Alternating Series Test
79. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

Section 8.6 The Ratio and Root Tests

1.
$$\frac{(n+1)!}{(n-2)!} = \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!}$$

$$= (n+1)(n)(n-1)$$
3. Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid since $1 = \frac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}$$

—CONTINUED—

3. —CONTINUED—

To do this, note that:

$$\begin{aligned}
 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) &= [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1) \\
 &= \frac{(2n)!}{2^n n!} \cdot (2n+1) \\
 &= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)} \\
 &= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n! (n+1)} \\
 &= \frac{(2n+2)!}{2^{n+1} (n+1)}
 \end{aligned}$$

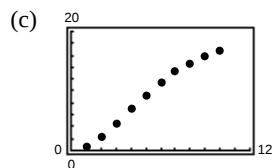
The formula is valid for all $n \geq 1$.

$$\begin{array}{lll}
 5. \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n = 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots & 7. \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} = 9 - \frac{3^3}{2} + \cdots & 9. \sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n = \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots \\
 S_1 = \frac{3}{4}, S_2 \approx 1.875 & S_1 = 9 & S_1 = 2 \\
 \text{Matches (d)} & \text{Matches (f)} & \text{Matches (a)}
 \end{array}$$

11. (a) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2(5/8)^{n+1}}{n^2(5/8)^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 \frac{5}{8} = \frac{5}{8} < 1$. Converges

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491



(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

13. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{3} = \infty
 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

15. $\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(3/4)^{n+1}}{n(3/4)^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{3(n+1)}{4n} \right| = \frac{3}{4}
 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

17. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2}
 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

19. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{2n^2}{(n+1)^2} = 2
 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$21. \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$23. \sum_{n=1}^{\infty} \frac{n!}{n3^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n}{3} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$25. \sum_{n=0}^{\infty} \frac{4^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$27. \sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n \rightarrow \infty} \frac{3}{n+2} \left(\frac{n+1}{n+2} \right)^n = (0) \left(\frac{1}{e} \right) = 0$$

To find $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^n$, let $y = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^n$. Then,

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(\frac{n+1}{n+2} \right) = \lim_{n \rightarrow \infty} \frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}.$$

Therefore, by the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \frac{4^n}{3^n + 1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{3^{n+1} + 1} \cdot \frac{3^n + 1}{4^n} \right| = \lim_{n \rightarrow \infty} \frac{4(3^n + 1)}{3^{n+1} + 1} = \lim_{n \rightarrow \infty} \frac{4(1 + 1/3^n)}{3 + 1/3^n} = \frac{4}{3}$$

Therefore, by the Ratio Test, the series diverges.

$$31. \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$

$$33. (a) \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{3/2} = 1$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{1/2} = 1$$

$$35. \sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{2n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$37. \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n}{(\ln n)^n} \right|} \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\ln n|} = 0 \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$39. \sum_{n=1}^{\infty} (2\sqrt[n]{n} + 1)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{(2\sqrt[n]{n} + 1)^n} = \lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1)$$

To find $\lim_{n \rightarrow \infty} \sqrt[n]{n}$, let $y = \lim_{n \rightarrow \infty} \sqrt[n]{x}$. Then

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{x}) = \lim_{n \rightarrow \infty} \frac{1}{x} \ln x = \lim_{n \rightarrow \infty} \frac{\ln x}{x} = \lim_{n \rightarrow \infty} \frac{1/x}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1) = 2(1) + 1 = 3$. Therefore, by the Root Test, the series diverges.

$$41. \sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

$$45. \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is convergent p -series.

$$47. \sum_{n=1}^{\infty} \frac{2n}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the n th Term Test for Divergence.

$$49. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9} \left(-\frac{3}{2} \right)^n$$

Since $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

$$51. \sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$$

$$\lim_{n \rightarrow \infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n \rightarrow \infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a limit comparison test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n.$$

$$53. \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$$

$$\left| \frac{\cos(n)}{2^n} \right| \leq \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{2^n} \right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n.$$

$$57. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

$$59. \sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

61. (a) and (c)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n5^n}{n!} &= \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!} \\ &= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots \end{aligned}$$

$$55. \sum_{n=1}^{\infty} \frac{n7^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n} \right| = \lim_{n \rightarrow \infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

63. (a) and (b) are the same.

65. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

67. Since

$$\frac{3^{10}}{2^{10}10!} = 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^9 \frac{(-3)^k}{2^k k!} \approx -0.7769$$

69. See Theorem 8.17, page 597.

71. No. Let $a_n = \frac{1}{n+10,000}$.

The series $\sum_{n=1}^{\infty} \frac{1}{n+10,000}$ diverges.

73. The series converges absolutely. See Theorem 8.17.

75. First, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r < 1$$

and choose R such that $0 \leq r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. Thus, for $n > N$, we $|a_n| < R^n$ and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for all $n > M$. Thus, for $n > M$, we have $|a_n| > R^n > 1$ which implies that $\lim_{n \rightarrow \infty} a_n \neq 0$ which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Section 8.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

3. $y = e^{-1/2}[(x+1) + 1]$

Linear

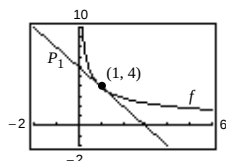
Matches (a)

5. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$\begin{aligned} P_1(x) &= f(1) + f'(1)(x-1) \\ &= 4 + (-2)(x-1) \end{aligned}$$

$$P_1(x) = -2x + 6$$

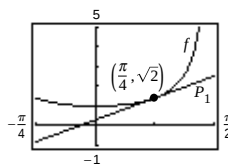


7. $f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$

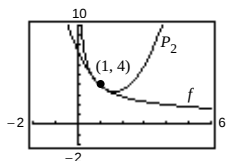


9. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$f''(x) = 3x^{-5/2} \quad f''(1) = 3$$

$$\begin{aligned} P_2 &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= 4 - 2(x-1) + \frac{3}{2}(x-1)^2 \end{aligned}$$



x	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

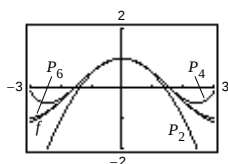
11. $f(x) = \cos x$

$$P_2(x) = 1 - \frac{1}{2}x^2$$

$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$

(a)



13. $f(x) = e^{-x}$ $f(0) = 1$

$$f'(x) = -e^{-x} \quad f'(0) = -1$$

$$f''(x) = e^{-x} \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \quad f'''(0) = -1$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

17. $f(x) = \sin x$ $f(0) = 0$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

21. $f(x) = \frac{1}{x+1}$ $f(0) = 1$

$$f'(x) = -\frac{1}{(x+1)^2} \quad f'(0) = -1$$

$$f''(x) = \frac{2}{(x+1)^3} \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad f'''(0) = -6$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5} \quad f^{(4)}(0) = 24$$

$$P_4(x) = 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 - x + x^2 - x^3 + x^4$$

(b) $f'(x) = -\sin x$ $P_2'(x) = -x$

$$f''(x) = -\cos x \quad P_2''(x) = -1$$

$$f'''(0) = P_2'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad P_4^{(4)}(x) = x$$

$$f^{(4)}(0) = 1 = P_4^{(4)}(0)$$

$$f^{(5)}(x) = -\sin x \quad P_6^{(5)}(x) = -x$$

$$f^{(6)}(x) = -\cos x \quad P_6^{(6)}(x) = -1$$

$$f^{(6)}(0) = -1 = P_6^{(6)}(0)$$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all n .

15. $f(x) = e^{2x}$ $f(0) = 1$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$f^{(4)}(x) = 16e^{2x} \quad f^{(4)}(0) = 16$$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

19. $f(x) = xe^x$ $f(0) = 0$

$$f'(x) = xe^x + e^x \quad f'(0) = 1$$

$$f''(x) = xe^x + 2e^x \quad f''(0) = 2$$

$$f'''(x) = xe^x + 3e^x \quad f'''(0) = 3$$

$$f^{(4)}(x) = xe^x + 4e^x \quad f^{(4)}(0) = 4$$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

23. $f(x) = \sec x$ $f(0) = 1$

$$f'(x) = \sec x \tan x \quad f'(0) = 0$$

$$f''(x) = \sec^3 x + \sec x \tan^2 x \quad f''(0) = 1$$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

25. $f(x) = \frac{1}{x} \quad f(1) = 1$

$f'(x) = -\frac{1}{x^2} \quad f'(1) = -1$

$f''(x) = \frac{2}{x^3} \quad f''(1) = 2$

$f'''(x) = -\frac{6}{x^4} \quad f'''(1) = -6$

$f^{(4)}(x) = \frac{24}{x^5} \quad f^{(4)}(1) = 24$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

27. $f(x) = \sqrt{x} \quad f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4x\sqrt{x}} \quad f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8x^2\sqrt{x}} \quad f'''(1) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16x^3\sqrt{x}} \quad f^{(4)}(1) = -\frac{15}{16}$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

29. $f(x) = \ln x \quad f(1) = 0$

$f'(x) = \frac{1}{x} \quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

31. $f(x) = \tan x$

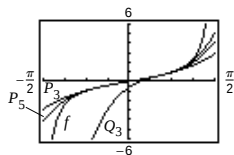
$f'(x) = \sec^2 x$

$f''(x) = 2 \sec^2 x \tan x$

$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$

$f^{(4)}(x) = 8 \sec^2 x \tan^3 x + 16 \sec^4 x \tan x$

$f^{(5)}(x) = 16 \sec^2 x \tan^4 x + 88 \sec^4 x \tan^2 x + 16 \sec^6 x$



(a) $n = 3, c = 0$

$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

(b) $n = 5, c = 0$

$$P_5(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5 = x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

(c) $n = 3, c = \frac{\pi}{4}$

$$Q_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

33. $f(x) = \sin x$

$P_1(x) = x$

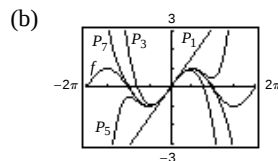
$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

(a)

x	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417
$P_7(x)$	0.0000	0.2474	0.4794	0.6816	0.8415



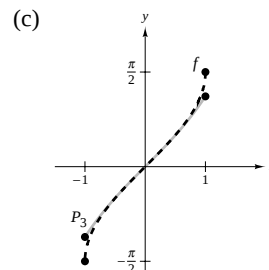
(c) As the distance increases, the accuracy decreases

35. $f(x) = \arcsin x$

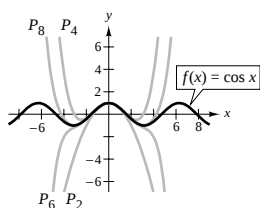
(a) $P_3(x) = x + \frac{x^3}{6}$

(b)

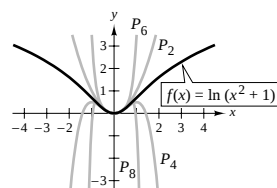
x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.848	-0.524	-0.253	0	0.253	0.524	0.848
$P_3(x)$	-0.820	-0.521	-0.253	0	0.253	0.521	0.820



37. $f(x) = \cos x$



39. $f(x) = \ln(x^2 + 1)$



41. $f(x) = e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

$f\left(\frac{1}{2}\right) \approx 0.6042$

43. $f(x) = \ln x \approx (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

$f(1.2) \approx 0.1823$

45. $f(x) = \cos x; f^{(5)}(x) = -\sin x \Rightarrow \text{Max on } [0, 0.3] \text{ is } 1.$

$R_4(x) \leq \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

47. $f(x) = \arcsin x$; $f^{(4)}(x) = \frac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Rightarrow \text{Max on } [0, 0.4] \text{ is } f^{(4)}(0.4) \approx 7.3340.$

$$R_3(x) \leq \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$$

49. $g(x) = \sin x$

$$g^{(n+1)}(x) \leq 1 \text{ for all } x$$

$$R_n(x) \leq \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error, $n = 3$.

51. $f(x) = \ln(x + 1)$

$$f^{(n+1)}(x) = \frac{(-1)^{n+1}n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \leq \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

53. $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, x < 0$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$xe^{z/4} < 0.3936$$

$$x < \frac{0.3936}{e^{z/4}} < 0.3936, z < 0$$

$$-0.3936 < x < 0$$

57. See definition on page 607.

55. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$ and the slopes of P and f agree at $(c, f(c))$. Depending on the degree of P , the n th derivatives of P and f agree at $(c, f(c))$.

59. The accuracy increases as the degree increases (for values within the interval of convergence).

61. (a) $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = xP_4(x)$$

(b) $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = xP_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c) $g(x) = \frac{\sin x}{x} = \frac{1}{x}P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$

63. (a) $Q_2(x) = -1 + \frac{\pi^2(x+2)^2}{32}$

(b) $R_2(x) = -1 + \frac{\pi^2(x-6)^2}{32}$

(c) No. The polynomial will be linear.
Translations are possible at $x = -2 + 8n$.

65. Let f be an even function and P_n be the n th Maclaurin polynomial for f . Since f is even, f' is odd, f'' is even, f''' is odd, etc. (see Exercise 45). All of the odd derivatives of f are odd and thus, all of the odd powers of x will have coefficients of zero. P_n will only have terms with even powers of x .

67. As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

Section 8.8 Power Series

1. Centered at 0

$$\begin{aligned}
 5. \quad & \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1} \\
 L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| |x| = |x| \\
 |x| < 1 &\Rightarrow R = 1
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} \\
 L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{(2x)^2}{(2n+2)(2n+1)} \right| = 0
 \end{aligned}$$

Thus, the series converges for all x . $R = \infty$.

$$\begin{aligned}
 13. \quad & \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \\
 \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{nx}{n+1} \right| = |x|
 \end{aligned}$$

Interval: $-1 < x < 1$ When $x = 1$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.When $x = -1$, the p -series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.Therefore, the interval of convergence is $-1 < x \leq 1$.

$$\begin{aligned}
 17. \quad & \sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2} \right)^n \\
 \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x}{2} \right| = \infty
 \end{aligned}$$

Therefore, the series converges only for $x = 0$.

$$19. \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Since the series is geometric, it converges only if $|x/4| < 1$ or $-4 < x < 4$.

3. Centered at 2

$$\begin{aligned}
 7. \quad & \sum_{n=1}^{\infty} \frac{(2x)^n}{n^2} \\
 L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{2n^2 x}{(n+1)^2} \right| = 2|x| \\
 2|x| < 1 &\Rightarrow R = \frac{1}{2}
 \end{aligned}$$

$$11. \quad \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n$$

Since the series is geometric, it converges only if $|x/2| < 1$ or $-2 < x < 2$.

$$\begin{aligned}
 15. \quad & \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
 \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0
 \end{aligned}$$

The series converges for all x . Therefore, the interval of convergence is $-\infty < x < \infty$.

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-5)}{5(n+1)} \right| = \frac{1}{5}|x-5|$$

$$R = 5$$

$$\text{Center: } x = 5$$

$$\text{Interval: } -5 < x - 5 < 5 \text{ or } 0 < x < 10$$

When $x = 0$, the p -series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 10$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $0 < x \leq 10$.

23. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-1)}{n+2} \right| = |x-1|$$

$$R = 1$$

$$\text{Center: } x = 1$$

$$\text{Interval: } -1 < x - 1 < 1 \text{ or } 0 < x < 2$$

When $x = 0$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $0 < x \leq 2$.

25. $\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When $x = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

When $x = 2c$, the series $\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $0 < x < 2c$.

27. $\sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-2x)(n+1)^2}{n(n+2)} \right| = 2|x| \end{aligned}$$

$$R = \frac{1}{2}$$

$$\text{Interval: } -\frac{1}{2} < x < \frac{1}{2}$$

When $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the n th Term Test.

When $x = \frac{1}{2}$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

$$29. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$31. \sum_{n=1}^{\infty} \frac{k(k+1) \cdots (k+n-1)x^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{k(k+1) \cdots (k+n-1)(k+n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k+1) \cdots (k+n-1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(k+n)x}{n+1} \right| = |x|$$

$$R = 1$$

When $x = \pm 1$, the series diverges and the interval of convergence is $-1 < x < 1$.

$$\left[\frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \geq 1 \right]$$

$$33. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n+1} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(4n+3)(x-3)}{4} \right| = \infty \end{aligned}$$

$$R = 0$$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

$$35. (a) f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n, -2 < x < 2 \quad (\text{Geometric})$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right) \left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \left(\frac{n}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$$

$$(d) \int f(x) dx = \sum_{n=0}^{\infty} \frac{2}{n+1} \left(\frac{x}{2}\right)^{n+1}, -2 \leq x < 2$$

$$37. (a) f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \leq 2$$

$$(b) f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, 0 < x < 2$$

$$(c) f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1}n(x-1)^{n-1}, 0 < x < 2$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \leq x \leq 2$$

$$39. g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

$$S_1 = 1, S_2 = 1.33. \text{ Matches (c)}$$

$$41. g(3.1) = \sum_{n=0}^{\infty} \left(\frac{3.1}{3}\right)^n \text{ diverges. Matches (b)}$$

43. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at c .

45. A single point, a_n interval, or the entire real line.

47. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty$ (See Exercise 29.)

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty$$

(b) $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d) $f(x) = \sin x$ and $g(x) = \cos x$

49. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

$$y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$$

$$y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$$

$$\begin{aligned} y'' - xy' - y &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\ &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right] \\ &= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n} [(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0 \end{aligned}$$

51. $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}$

(a) $\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2} [(k+1)!]^2} \cdot \frac{2^{2k} (k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2(k+1)^2} \right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2}$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1} [(k+1)!]^2}$$

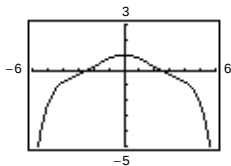
$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1} [(k+1)!]^2}$$

$$\begin{aligned} x^2 J_0'' + x J_0' + J_0 &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[(-1) \frac{2(2k+1)}{4(k+1)} + (-1) \frac{2}{4(k+1)} + 1 \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0 \end{aligned}$$

—CONTINUED—

51. —CONTINUED—

$$(c) P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

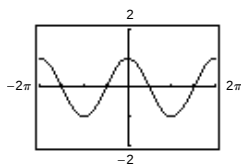


$$\begin{aligned} (d) \int_0^1 J_0 dx &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx \\ &= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} \\ &= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92 \end{aligned}$$

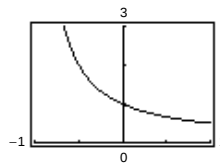
(exact integral is 0.9197304101)

$$53. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

(See Exercise 47.)

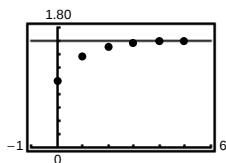


$$\begin{aligned} 55. f(x) &= \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n \\ &= \frac{1}{1 - (-x)} = \frac{1}{1+x} \text{ for } -1 < x < 1 \end{aligned}$$

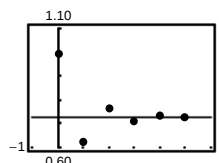


$$57. \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$\begin{aligned} (a) \sum_{n=0}^{\infty} \left(\frac{3/4}{2}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n \\ &= \frac{1}{1 - (3/8)} = \frac{8}{5} = 1.6 \end{aligned}$$



$$\begin{aligned} (b) \sum_{n=0}^{\infty} \left(\frac{-3/4}{2}\right)^n &= \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n \\ &= \frac{1}{1 - (-3/8)} = \frac{8}{11} \approx 0.7272 \end{aligned}$$



(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

$$(d) \sum_{n=0}^N \left(\frac{3}{2}\right)^n > M$$

M	10	100	1000	10,000
N	4	9	15	21

59. False;

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n 2^n}$$

converges for $x = 2$ but diverges for $x = -2$.61. True; the radius of convergence is $R = 1$ for both series.

Section 8.9 Representation of Functions by Power Series

$$\begin{aligned} 1. \text{ (a) } \frac{1}{2-x} &= \frac{1/2}{1-(x/2)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \end{aligned}$$

This series converges on $(-2, 2)$.

$$\frac{\frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots}{(b) \quad 2-x \Big) 1}$$

$$\begin{array}{r} 1 - \frac{x}{2} \\ \hline \frac{x}{2} \\ \frac{x}{2} - \frac{x^2}{4} \\ \hline \frac{x^2}{4} \\ \frac{x^2}{4} - \frac{x^3}{8} \\ \hline \frac{x^3}{8} \\ \frac{x^3}{8} - \frac{x^4}{16} \\ \hline \vdots \end{array}$$

$$\begin{aligned} 3. \text{ (a) } \frac{1}{2+x} &= \frac{1/2}{1-(-x/2)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} \end{aligned}$$

This series converges on $(-2, 2)$.

$$\frac{\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots}{(b) \quad 2+x \Big) 1}$$

$$\begin{array}{r} 1 + \frac{x}{2} \\ \hline -\frac{x}{2} \\ -\frac{x}{2} - \frac{x^2}{4} \\ \hline \frac{x^2}{4} \\ \frac{x^2}{4} + \frac{x^3}{8} \\ \hline -\frac{x^3}{8} \\ -\frac{x^3}{8} - \frac{x^4}{16} \\ \hline \vdots \end{array}$$

5. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (-1/3)(x-5)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{1}{2-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3} \left[-\frac{1}{3}(x-5)\right]^n \\ &= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}}, \quad |x-5| < 3 \text{ or } 2 < x < 8. \end{aligned}$$

9. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\begin{aligned} \frac{1}{2x-5} &= \frac{-1}{11-2(x+3)} \\ &= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r} \end{aligned}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{1}{2x-5} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(-\frac{1}{11}\right) \left[\frac{2}{11}(x+3)\right]^n \\ &= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}}, \end{aligned}$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

7. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n \\ &= -3 \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}. \end{aligned}$$

11. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{3}{x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{1}{2}x\right)^n \\ &= 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n, \end{aligned}$$

$$|x| < 2 \text{ or } -2 < x < 2.$$

$$13. \frac{3x}{x^2 + x - 2} = \frac{2}{x + 2} + \frac{1}{x - 1} = \frac{2}{2 + x} + \frac{1}{-1 + x} = \frac{1}{1 + (1/2)x} + \frac{-1}{1 - x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2 + x - 2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} (-1)(x)^n = \sum_{n=0}^{\infty} \left[\frac{1}{(-2)^n} - 1\right] x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1 - (-2)^n)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x}{-2 - (-2)^{n+1}} \right| = |x|.$$

$$15. \frac{2}{1 - x^2} = \frac{1}{1 - x} + \frac{1}{1 + x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1 - x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (1 + (-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^{2n+2}}{2x^{2n}} \right| = |x^2|.$

$$17. \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^{2n} x^n = \sum_{n=0}^{\infty} x^n$$

$$h(x) = \frac{-2}{x^2 - 1} = \frac{1}{1 + x} + \frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} [(-1)^n + 1] x^n$$

$$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty} 2x^{2n}, \quad -1 < x < 1 \text{ (See Exercise 15.)}$$

19. By taking the first derivative, we have $\frac{d}{dx} \left[\frac{1}{x + 1} \right] = \frac{-1}{(x + 1)^2}$. Therefore,

$$\begin{aligned} \frac{-1}{(x + 1)^2} &= \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} (n + 1) x^n, \quad -1 < x < 1. \end{aligned}$$

21. By integrating, we have $\int \frac{1}{x + 1} dx = \ln(x + 1)$. Therefore,

$$\ln(x + 1) = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}, \quad -1 < x \leq 1.$$

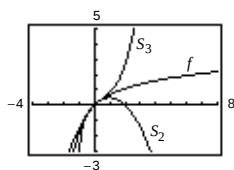
To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}, \quad -1 < x \leq 1.$$

$$23. \frac{1}{x^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1$$

$$25. \text{ Since, } \frac{1}{x + 1} = \sum_{n=0}^{\infty} (-1)^n x^n, \text{ we have } \frac{1}{4x^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

$$27. x - \frac{x^2}{2} \leq \ln(x+1) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.180	0.336	0.470	0.588	0.693
$x - \frac{x^2}{2} + \frac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

$$29. g(x) = x, \text{ line, Matches (c)}$$

$$31. g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}, \text{ Matches (a)}$$

$$33. f(x) = \arctan x \text{ is an odd function (symmetric to the origin)}$$

$$\text{In Exercises 35 and 37, } \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$35. \arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \dots$$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

$$37. \frac{\arctan x^2}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)}$$

$$\int_0^{1/2} \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \dots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x} dx \approx 0.125$

$$\text{In Exercises 39 and 41, use } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1.$$

$$39. (a) \frac{1}{(1-x)^2} = \frac{d}{dx} \left[\frac{1}{1-x} \right] = \frac{d}{dx} \left[\sum_{n=0}^{\infty} x^n \right] = \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$$

$$(b) \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, |x| < 1$$

$$(c) \frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n(x^{n-1} + x^n), |x| < 1$$

$$= \sum_{n=0}^{\infty} (2n+1)x^n, |x| < 1$$

$$(d) \frac{x(1+x)}{(1-x)^2} = x \sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, |x| < 1$$

$$41. P(n) = \left(\frac{1}{2}\right)^n$$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{2} \frac{1}{[1 - (1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

43. Replace x with $(-x)$.45. Replace x with $(-x)$ and multiply the series by 5.47. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan \theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \tan \theta$$

$$\frac{x + y}{1 - xy} = \tan \theta$$

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right) \text{ for } xy \neq 1.$$

$$49. (a) 2 \arctan \frac{1}{2} = \arctan \frac{1}{2} + \arctan \frac{1}{2} = \arctan \left[\frac{2(1/2)}{1 - (1/2)^2} \right] = \arctan \frac{4}{3}$$

$$2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \arctan \frac{4}{3} + \arctan \left(-\frac{1}{7} \right) = \arctan \left[\frac{(4/3) - (1/7)}{1 + (4/3)(1/7)} \right] = \arctan \frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

$$(b) \pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7} \approx 8 \left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7} \right] - 4 \left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7} \right] \approx 3.14$$

51. From Exercise 21, we have

$$\begin{aligned} \ln(x + 1) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}. \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/2)^n}{n} \\ &= \ln\left(\frac{1}{2} + 1\right) = \ln \frac{3}{2} \approx 0.4055 \end{aligned}$$

53. From Exercise 51, we have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2/5)^n}{n} \\ &= \ln\left(\frac{2}{5} + 1\right) = \ln \frac{7}{5} \approx 0.3365. \end{aligned}$$

55. From Exercise 54, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} = \arctan \frac{1}{2} \approx 0.4636.$$

57. The series in Exercise 54 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

$$59. f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$f(0.5) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-0.5)^n}{n} = \sum_{n=1}^{\infty} -\frac{(1/2)^n}{n}$$

$$\sum_{n=1}^{\infty} -\frac{(1/2)^n}{n} = -0.6931$$

Section 8.10 Taylor and Maclaurin Series

1. For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

3. For $c = \pi/4$, we have:

$$f(x) = \cos(x) \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \quad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \quad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \quad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, we have:

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \dots \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (\pi/4)]^n}{n!}. \end{aligned}$$

[Note: $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \dots$]

5. For $c = 1$, we have,

$$f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \quad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \end{aligned}$$

7. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \sin 2x & f(0) &= 0 \\
 f'(x) &= 2 \cos 2x & f'(0) &= 2 \\
 f''(x) &= -4 \sin 2x & f''(0) &= 0 \\
 f'''(x) &= -8 \cos 2x & f'''(0) &= -8 \\
 f^{(4)}(x) &= 16 \sin 2x & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= 32 \cos 2x & f^{(5)}(0) &= 32 \\
 f^{(6)}(x) &= -64 \sin 2x & f^{(6)}(0) &= 0 \\
 f^{(7)}(x) &= -128 \cos 2x & f^{(7)}(0) &= -128
 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned}
 \sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots \\
 &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}
 \end{aligned}$$

9. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \sec(x) & f(0) &= 1 \\
 f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\
 f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\
 f'''(x) &= 5 \sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\
 f^{(4)}(x) &= 5 \sec^5(x) + 18 \sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \\
 \sec(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots
 \end{aligned}$$

11. The Maclaurin series for $f(x) = \cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, we have $|f^{(n+1)}(z)| \leq 1$ for all z . Hence by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all x .

13. Since $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\begin{aligned}
 (1+x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{5!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\
 &= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n.
 \end{aligned}$$

15. $\frac{1}{\sqrt{4+x^2}} = \left(\frac{1}{2}\right) \left[1 + \left(\frac{x}{2}\right)^2\right]^{-1/2}$ and since $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!} \right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

17. Since $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$ (Exercise 14)

we have $(1+x^2)^{1/2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{2n}}{2^n n!}.$

19. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{x^2/2} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

21. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

23. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!} = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

25. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

27. $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$$= \frac{1}{2} \left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right]$$

$$= \frac{1}{2} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right]$$

29. $x \sin x = x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

31. $\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$$

$$33. \quad e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots$$

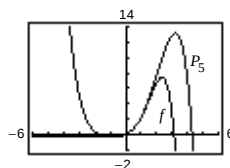
$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots$$

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$$

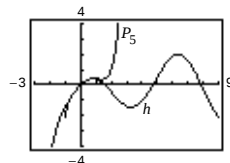
$$35. \quad f(x) = e^x \sin x$$

$$\begin{aligned} &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\ &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots \end{aligned}$$



$$37. \quad h(x) = \cos x \ln(1+x)$$

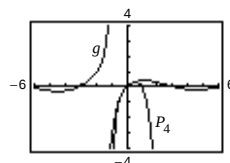
$$\begin{aligned} &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\ &= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots \\ &= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \end{aligned}$$



$$39. \quad g(x) = \frac{\sin x}{1+x}. \text{ Divide the series for } \sin x \text{ by } (1+x).$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$

$$\begin{array}{r} x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots \\ 1+x \overline{) \quad x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots} \\ \underline{x + x^2} \\ -x^2 - \frac{x^3}{6} \\ \underline{-x^2 - x^3} \\ \frac{5x^3}{6} + 0x^4 \\ \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\ -\frac{5x^4}{6} + \frac{x^5}{120} \\ \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\ \vdots \end{array}$$



$$41. \quad y = x^2 - \frac{x^4}{3!} = x \left(x - \frac{x^3}{3!} \right) \approx x \sin x.$$

Matches (a)

$$43. \quad y = x + x^2 + \frac{x^3}{2!} = x \left(1 + x + \frac{x^2}{2!} \right) \approx x e^x.$$

Matches (c)

$$\begin{aligned}
 45. \int_0^x (e^{-t^2} - 1) dt &= \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) - 1 \right] dt \\
 &= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!} \right] dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}
 \end{aligned}$$

$$47. \text{ Since } \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$\text{we have } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931. \quad (10,001 \text{ terms})$$

$$49. \text{ Since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

$$\text{we have } e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891. \quad (12 \text{ terms})$$

51. Since

$$\begin{aligned}
 \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \\
 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\
 \frac{1 - \cos x}{x} &= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}
 \end{aligned}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!} = 0.$$

$$53. \int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

Since $1/(7 \cdot 7!) < 0.0001$, we have

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \dots \approx 0.9461.$$

Note: We are using $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$.

$$55. \int_0^{\pi/2} \sqrt{x} \cos x dx = \int_0^{\pi/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!} \right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!} \right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!} \right]_0^{\pi/2}$$

Since $(\pi/2)^{19/2}/766,080 < 0.0001$, we have

$$\int_0^1 \sqrt{x} \cos x dx = 2 \left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10,800} + \frac{(\pi/2)^{19/2}}{766,080} \right] \approx 0.7040.$$

$$57. \int_{0.1}^{0.3} \sqrt{1+x^3} dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \dots \right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \dots \right]_{0.1}^{0.3}$$

Since $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, we have

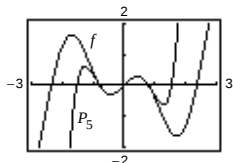
$$\int_{0.1}^{0.3} \sqrt{1+x^3} dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) - \frac{1}{56}(0.3^7 - 0.1^7) \right] \approx 0.2010.$$

59. From Exercise 19, we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_0^1 = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[1 - \frac{1}{2 \cdot 1 \cdot 3} + \frac{1}{2^2 \cdot 2! \cdot 5} - \frac{1}{2^3 \cdot 3! \cdot 7} \right] \approx 0.3414.\end{aligned}$$

61. $f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

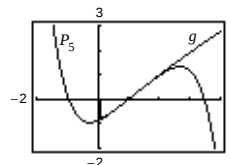
$$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$$



The polynomial is a reasonable approximation on the interval $[-\frac{3}{4}, \frac{3}{4}]$.

63. $f(x) = \sqrt{x} \ln x, c = 1$

$$P_5(x) = (x-1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$$



The polynomial is a reasonable approximation on the interval $[\frac{1}{4}, 2]$.

65. See Guidelines, page 636.

67. (a) Replace x with $(-x)$.

(b) Replace x with $3x$.

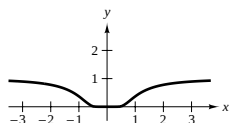
(c) Multiply series by x .

(d) Replace x with $2x$, then replace x with $-2x$, and add the two together.

$$\begin{aligned}69. y &= \left(\tan \theta - \frac{g}{kv_0 \cos \theta} \right) x - \frac{g}{k^2} \ln \left(1 - \frac{kx}{v_0 \cos \theta} \right) \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} - \frac{g}{k^2} \left[-\frac{kx}{v_0 \cos \theta} - \frac{1}{2} \left(\frac{kx}{v_0 \cos \theta} \right)^2 - \frac{1}{3} \left(\frac{kx}{v_0 \cos \theta} \right)^3 - \frac{1}{4} \left(\frac{kx}{v_0 \cos \theta} \right)^4 - \dots \right] \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} + \frac{gx}{kv_0 \cos \theta} + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{gk^2x^4}{4v_0^4 \cos^4 \theta} + \dots \\ &= (\tan \theta)x + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{k^2gx^4}{4v_0^4 \cos^4 \theta} + \dots\end{aligned}$$

71. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b) $f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x}$

Let $y = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x \rightarrow 0} \ln \left(\frac{e^{-1/x^2}}{x} \right) = \lim_{x \rightarrow 0^+} \left[-\frac{1}{x^2} - \ln x \right] = \lim_{x \rightarrow 0^+} \left[\frac{-1 - x^2 \ln x}{x^2} \right] = -\infty.$$

Thus, $y = e^{-\infty} = 0$ and we have $f'(0) = 0$.

(c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots = 0 \neq f(x)$

This series converges to f at $x = 0$ only.

73. By the Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$ which shows that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x .

Review Exercises for Chapter 8

1. $a_n = \frac{1}{n!}$

3. $a_n = 4 + \frac{2}{n}$: 6, 5, 4.67, . . .

5. $a_n = 10(0.3)^{n-1}$: 10, 3, . . .

Matches (a)

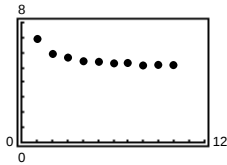
Matches (d)

7. $a_n = \frac{5n+2}{n}$

9. $\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0$

11. $\lim_{n \rightarrow \infty} \frac{n^3}{n^2+1} = \infty$

Converges



The sequence seems to converge to 5.

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{5n+2}{n} \\ &= \lim_{n \rightarrow \infty} \left(5 + \frac{2}{n} \right) = 5\end{aligned}$$

$$13. \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \quad \text{Converges}$$

15. $\lim_{n \rightarrow \infty} \frac{\sin(n)}{\sqrt{n}} = 0$

Converges

$$17. A_n = 5000 \left(1 + \frac{0.05}{4} \right)^n = 5000(1.0125)^n$$

$$n = 1, 2, 3$$

(a) $A_1 \approx 5062.50$ $A_5 \approx 5320.41$

$A_2 \approx 5125.78$ $A_6 \approx 5386.92$

$A_3 \approx 5189.85$ $A_7 \approx 5454.25$

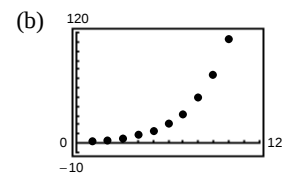
$A_4 \approx 5254.73$ $A_8 \approx 5522.43$

(b) $A_{40} \approx 8218.10$

19. (a)

k	5	10	15	20	25
S_k	13.2	113.3	873.8	6448.5	50,500.3

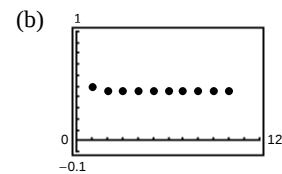
(c) The series diverges (geometric $r = \frac{3}{2} > 1$)



21. (a)

k	5	10	15	20	25
S_k	0.4597	0.4597	0.4597	0.4597	0.4597

(c) The series converges by the Alternating Series Test.



23. Converges. Geometric series, $r = 0.82$, $|r| < 1$.

25. Diverges. n th Term Test. $\lim_{n \rightarrow \infty} a_n \neq 0$.

$$27. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

Geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

$$31. 0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty} (0.09)(0.01)^n = \frac{0.09}{1-0.01} = \frac{1}{11}$$

$$33. D_1 = 8$$

$$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$$

$\vdots$

$$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$$

$$= -8 + \sum_{n=0}^{\infty} 16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\frac{1}{3} \text{ meters}$$

$$37. \int_1^{\infty} x^{-4} \ln(x) dx = \lim_{b \rightarrow \infty} \left[-\frac{\ln x}{3x^3} - \frac{1}{9x^3} \right]_1^b$$

$$= 0 + \frac{1}{9} = \frac{1}{9}$$

By the Integral Test, the series converges.

$$41. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$$

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^3 + 2n}}{1/(n^{3/2})} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 + 2n}} = 1$$

By a limit comparison test with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \text{ the series converges.}$$

45. Converges by the Alternating Series Test (Conditional convergence)

$$49. \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n} \right|$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{e^{2n+1}} \right) \left(\frac{n+1}{n} \right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n$$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

35. See Exercise 86 in Section 8.2.

$$A = \frac{P(e^{rt} - 1)}{e^{r/12} - 1}$$

$$= \frac{200(e^{(0.06)(2)} - 1)}{e^{0.06/12} - 1}$$

$$\approx \$5087.14$$

$$39. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the second series is a divergent p -series while the first series is a convergent p -series, the difference diverges.

$$43. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$= \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \right) \frac{1}{2n} > \frac{1}{2n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does the original series.

47. Diverges by the n th Term Test

$$51. \sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

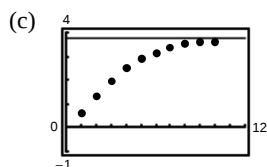
53. (a) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n}$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) \left(\frac{3}{5} \right) = \frac{3}{5} < 1$$

Converges

(b)

x	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499



(d) The sum is approximately 3.75.

55. (a) $\int_N^\infty \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_N^\infty = \frac{1}{N}$

N	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^\infty \frac{1}{x^2} dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b) $\int_N^\infty \frac{1}{x^5} dx = \left[-\frac{1}{4x^4} \right]_N^\infty = \frac{1}{4N^4}$

N	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^\infty \frac{1}{x^5} dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

57. $f(x) = e^{-x/2} \quad f(0) = 1$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad f'''(0) = -\frac{1}{8}$$

$$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{4}\frac{x^2}{2!} - \frac{1}{8}\frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$$

59. $\sin(95^\circ) = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} + \frac{(95\pi)^9}{180^9 9!} \approx 0.996$

61. $\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \cdots + \frac{(0.75)^{15}}{15} \approx 0.560$

63. $f(x) = \cos x$, $c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$$|f^{(n+1)}(z)| \leq 1 \Rightarrow R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$

(a) $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 4$.

(c) $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 5$.

(b) $R_n(x) \leq \frac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 6$.

(d) $R_n(x) \leq \frac{2^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 10$.

65. $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

67. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{(n+1)^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(-1)^n(x-2)^n} \right|$$

$$= |x-2|$$

$$R = 1$$

Center: 2

Since the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $1 \leq x \leq 3$.

69. $\sum_{n=0}^{\infty} n!(x-2)^n$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n} \right| = \infty$$

which implies that the series converges only at the center $x = 2$.

71.

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n(n!)^2}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n(2n)x^{2n-1}}{4^n(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{4^{n+1}[(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{4^{n+1}[(n+1)!]^2}$$

$$\begin{aligned} x^2 y'' + xy' + x^2 y &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+2}}{4^{n+1}[(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n(n!)^2} \\ &= \sum_{n=0}^{\infty} \left[(-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^{n+1}(2n+2)}{4^{n+1}[(n+1)!]^2} + \frac{(-1)^n}{4^n(n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}(2n+2)(2n+1+1)}{4^{n+1}[(n+1)!]^2} + (-1)^n \frac{1}{4^n(n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}4(n+1)^2}{4^{n+1}[(n+1)!]^2} + (-1)^n \frac{1}{4^n(n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}1}{4^n(n!)^2} + (-1)^n \frac{1}{4^n(n!)^2} \right] x^{2n+2} = 0 \end{aligned}$$

73. $\frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

75. Derivative: $\sum_{n=1}^{\infty} \frac{2nx^{n-1}}{3^{n+1}}$

$$77. 1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n = \frac{1}{1 - (2x/3)} = \frac{3}{3 - 2x}, \quad -\frac{3}{2} < x < \frac{3}{2}$$

$$79. f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x), \dots$$

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[x - (3\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (3\pi/4)]^n}{n!} \end{aligned}$$

$$81. 3^x = (e^{\ln 3})^x = e^{x \ln 3} \text{ and since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ we have}$$

$$\begin{aligned} 3^x &= \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} \\ &= 1 + x \ln 3 + \frac{x^2 \ln^2 3}{2!} + \frac{x^3 \ln^3 3}{3!} + \frac{x^4 \ln^4 3}{4!} + \cdots \end{aligned}$$

$$83. f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}, \dots$$

$$\begin{aligned} \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty} (x+1)^n \end{aligned}$$

$$85. (1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

$$\begin{aligned} (1+x)^{1/5} &= 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots \\ &= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \cdots \\ &= 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!} \\ &= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots \end{aligned}$$

$$87. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\begin{aligned} \ln\left(\frac{5}{4}\right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{5/4 - 1}{n}\right)^n \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n} \approx 0.2231 \end{aligned}$$

$$89. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \approx 1.6487$$

$$91. \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

$$95. \quad (a) \quad f(x) = e^{2x} \quad f(0) = 1$$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$(c) \quad e^{2x} = e^x \cdot e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$

$$= 1 + (x + x) + \left(x^2 + \frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^3}{6} + \frac{x^3}{6} + \frac{x^3}{2} + \frac{x^3}{2}\right) + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$97. \quad \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$\int_0^x \frac{\sin t}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

$$101. \quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \cdots$$

$$\lim_{x \rightarrow 0} \frac{\arctan x}{\sqrt{x}} = 0$$

By L'Hôpital's Rule, $\lim_{x \rightarrow 0} \frac{\arctan x}{\sqrt{x}} = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x \rightarrow 0} \frac{2\sqrt{x}}{1+x^2} = 0.$

93. The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

$$(b) \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$99. \quad \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\ln(1+t) = \int \frac{1}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$

$$\int_0^x \frac{\ln(t+1)}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

Problem Solving for Chapter 8

$$1. \quad (a) \quad 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$$

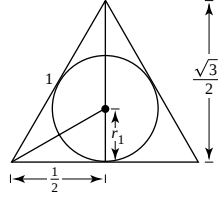
$$(b) \quad 0, \frac{1}{3}, \frac{2}{3}, 1, \text{ etc.}$$

$$(c) \quad \lim_{n \rightarrow \infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 1 - 1 = 0$$

3. If there are n rows, then $a_n = \frac{n(n+1)}{2}$.

For one circle,

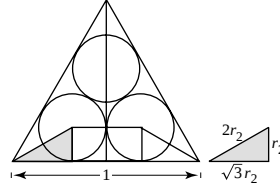
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$$



For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

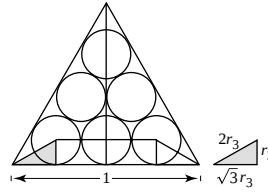
$$r_2 = \frac{1}{2 + 2\sqrt{3}}$$



For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}$$



Continuing this pattern, $r_n = \frac{1}{2\sqrt{3} + 2(n-1)}$.

$$\text{Total Area} = (\pi r_n^2) a_n = \pi \left(\frac{1}{2\sqrt{3} + 2(n-1)} \right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2} \frac{n(n+1)}{[2\sqrt{3} + 2(n+1)]^2}$$

$$\lim_{n \rightarrow \infty} A_n = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

5. (a) $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \dots$

$$= (1 + x^3 + x^6 + \dots) + 2(x + x^4 + x^7 + \dots) + 3(x^2 + x^5 + x^8 + \dots)$$

$$= (1 + x^3 + x^6 + \dots)[1 + 2x + 3x^2]$$

$$= (1 + 2x + 3x^2) \frac{1}{1 - x^3}$$

$R = 1$ because each series in the second line has $R = 1$.

- (b) $\sum a_n x^n = (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \dots) + \dots$

$$= a_0(1 + x^p + \dots) + a_1 x(1 + x^p + \dots) + \dots + a_{p-1} x^{p-1}(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1})(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) \frac{1}{1 - x^p}$$

$R = 1$

$$7. \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting $x = 0$, $C = 1$. Letting $x = 1$,

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

$$\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}.$$

$$9. \text{ Let } a_1 = \int_0^{\pi} \frac{\sin x}{x} dx, a_2 = -\int_{\pi}^{2\pi} \frac{\sin x}{x} dx, a_3 = \int_{2\pi}^{3\pi} \frac{\sin x}{x} dx, \text{ etc.}$$

Then,

$$\int_0^{\infty} \frac{\sin x}{x} dx = a_1 - a_2 + a_3 - a_4 + \cdots$$

Since $\lim_{n \rightarrow \infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

$$11. (a) \quad a_1 = 3.0$$

$$a_2 \approx 1.73205$$

$$a_3 \approx 2.17533$$

$$a_4 \approx 2.27493$$

$$a_5 \approx 2.29672$$

$$a_6 \approx 2.30146$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1 + \sqrt{13}}{2} \text{ [See part (b) for proof.]}$$

(b) Use mathematical induction to show the sequence is increasing. Clearly, $a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1$.

Now assume $a_n > a_{n-1}$. Then

$$\begin{aligned} a_n + a &> a_{n-1} + a \\ \sqrt{a_n + a} &> \sqrt{a_{n-1} + a} \\ a_{n+1} &> a_n. \end{aligned}$$

Use mathematical induction to show that the sequence is bounded above by a . Clearly, $a_1 = \sqrt{a} < a$.

Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$$\begin{aligned} a(a-1) &> a_n(1) \\ a^2 - a &> a_n \\ a^2 &> a_n + a \\ a &> \sqrt{a_n + a} = a_{n+1}. \end{aligned}$$

Hence, the sequence converges to some number L . To find L , assume $a_{n+1} \approx a_n \approx L$:

$$L = \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2}.$$

$$\text{Hence, } L = \frac{1 + \sqrt{1 + 4a}}{2}.$$

$$13. (a) \sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \dots$$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_2 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

$$(b) \frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \dots$ which diverges.

$$(c) \sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}} \right)^{1/n}$$

$$= \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1 \text{ converges because } \{2^{(-1)^n}\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \dots \text{ and } \sqrt[n]{1/2} \rightarrow 1 \text{ and } \sqrt[n]{2} \rightarrow 1.$$

$$15. S_6 = 130 + 70 + 40 = 240$$

$$S_7 = 240 + 130 + 70 = 440$$

$$S_8 = 440 + 240 + 130 = 810$$

$$S_9 = 810 + 440 + 240 = 1490$$

$$S_{10} = 1490 + 810 + 440 = 2740$$

CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1	Conics and Calculus	424
Section 9.2	Plane Curves and Parametric Equations	434
Section 9.3	Parametric Equations and Calculus	439
Section 9.4	Polar Coordinates and Polar Graphs	444
Section 9.5	Area and Arc Length in Polar Coordinates	452
Section 9.6	Polar Equations of Conics and Kepler's Laws	458
	Review Exercises	461
	Problem Solving	469

CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1 Conics and Calculus

Solutions to Even-Numbered Exercises

2. $x^2 = 8y$

Vertex: $(0, 0)$

$p = 2 > 0$

Opens upward

Matches graph (a).

6. $\frac{x^2}{9} + \frac{y^2}{9} = 1$

Circle radius 3.

Matches (g)

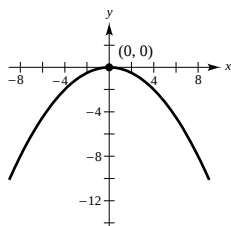
10. $x^2 + 8y = 0$

$x^2 = 4(-2)y$

Vertex: $(0, 0)$

Focus: $(0, -2)$

Directrix: $y = 2$



4. $\frac{(x - 2)^2}{16} + \frac{(y + 1)^2}{4} = 1$

Center: $(2, -1)$

Ellipse

Matches (b)

8. $\frac{(x - 2)^2}{9} - \frac{y^2}{4} = 1$

Hyperbola

Center: $(-2, 0)$

Horizontal transverse axis.

Matches (d)

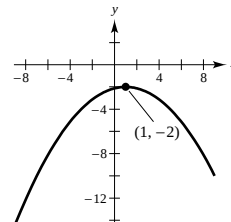
12. $(x - 1)^2 + 8(y + 2) = 0$

$(x - 1)^2 = 4(-2)(y + 2)$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$



14. $y^2 + 6y + 8x + 25 = 0$

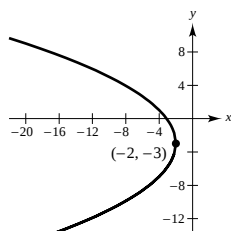
$y^2 + 6y + 9 = -8x - 25 + 9$

$(y + 3)^2 = 4(-2)(x + 2)$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



16. $y^2 + 4y + 8x - 12 = 0$

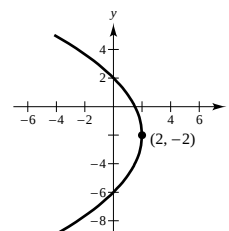
$y^2 + 4y + 4 = -8x + 12 + 4$

$(y + 2)^2 = 4(-2)(x - 2)$

Vertex: $(2, -2)$

Focus: $(0, -2)$

Directrix: $x = 4$



18. $y = -\frac{1}{6}(x^2 - 8x + 6) = -\frac{1}{6}(x^2 - 8x + 16 - 10)$

$$-6y = (x - 4)^2 - 10$$

$$-6y + 10 = (x - 4)^2$$

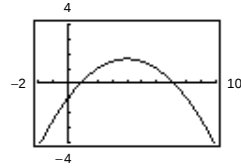
$$(x - 4)^2 = -6\left(y - \frac{5}{3}\right)$$

$$(x - 4)^2 = 4\left(-\frac{3}{2}\right)\left(y - \frac{5}{3}\right)$$

Vertex: $\left(4, \frac{5}{3}\right)$

Focus: $\left(4, \frac{1}{6}\right)$

Directrix: $y = \frac{19}{6}$



20. $x^2 - 2x + 8y + 9 = 0$

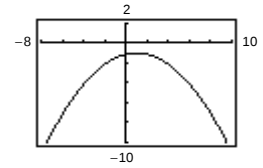
$$x^2 - 2x + 1 = -8y - 9 + 1$$

$$(x - 1)^2 = 4(-2)(y + 1)$$

Vertex: $(1, -1)$

Focus: $(1, -3)$

Directrix: $y = 1$



22. $(x + 1)^2 = 4(-2)(y - 2)$

$$x^2 + 2x + 8y - 15 = 0$$

24. Vertex: $(0, 2)$

$$(y - 2)^2 = 4(2)(x - 0)$$

$$y^2 - 8x - 4y + 4 = 0$$

26. $y = 4 - (x - 2)^2 = 4x - x^2$

$$x^2 - 4x + y = 0$$

28. From Example 2: $4p = 8$ or $p = 2$

Vertex: $(4, 0)$

$$(x - 4)^2 = 8(y - 0)$$

$$x^2 - 8x - 8y + 16 = 0$$

30. $5x^2 + 7y^2 = 70$

$$\frac{x^2}{14} + \frac{y^2}{10} = 1$$

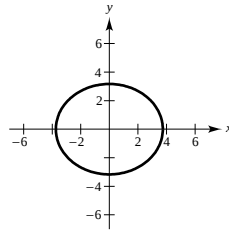
$$a^2 = 14, b^2 = 10, c^2 = 4$$

Center: $(0, 0)$

Foci: $(\pm 2, 0)$

Vertices: $(\pm \sqrt{14}, 0)$

$$e = \frac{2}{\sqrt{14}} = \frac{\sqrt{14}}{7}$$



32. $\frac{(x + 2)^2}{1} + \frac{(y + 4)^2}{1/4} = 1$

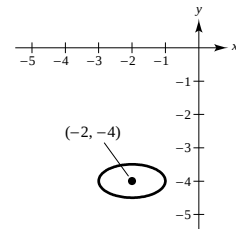
$$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$$

Center: $(-2, -4)$

Foci: $\left(-2 \pm \frac{\sqrt{3}}{2}, -4\right)$

Vertices: $(-1, -4), (-3, -4)$

$$e = \frac{\sqrt{3}}{2}$$



34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

$$16(x^2 - 4x + 4) + 25(y^2 + 6y + 0) = -279 + 64 + 225$$

$$= 10$$

$$\frac{(x - 2)^2}{(5/8)} + \frac{(y + 3)^2}{(2/5)} = 1$$

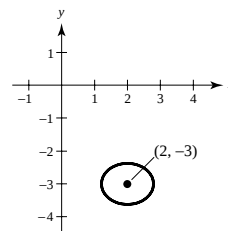
$$a^2 = \frac{5}{8}, b^2 = \frac{2}{5}, c^2 = a^2 - b^2 = \frac{9}{40}$$

Center: $(2, -3)$

Foci: $\left(2 \pm \frac{3\sqrt{10}}{20}, -3\right)$

Vertices: $\left(2 \pm \frac{\sqrt{10}}{4}, -3\right)$

$$e = \frac{c}{a} = \frac{3}{5}$$



36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = -43 + 16 + 36$$

$$= 9$$

$$\frac{[x + (2/3)]^2}{1/4} + \frac{(y - 2)^2}{1} = 1$$

$$a^2 = 1, b^2 = \frac{1}{4}, c^2 = \frac{3}{4}$$

Center: $\left(-\frac{2}{3}, 2\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{3}}{2}\right)$

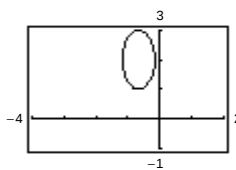
Vertices: $\left(-\frac{2}{3}, 3\right), \left(-\frac{2}{3}, 1\right)$

Solve for y:

$$9(y^2 - 4y + 4) = -36x^2 - 48x - 43 + 36$$

$$(y - 2)^2 = \frac{-(36x^2 + 48x + 7)}{9}$$

$$y = 2 \pm \frac{1}{3}\sqrt{-(36x^2 + 48x + 7)} \quad (\text{Graph each of these separately.})$$



38. $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

$$50x^2 + 25y^2 + 120x - 160y + 78 = 0$$

$$50\left(x^2 + \frac{12}{5}x + \frac{36}{25}\right) + 25\left(y^2 - \frac{32}{5}y + \frac{256}{25}\right) = -78 + 72 + 256 = 250$$

$$\frac{[x + (6/5)]^2}{5} + \frac{[y - (16/5)]^2}{10} = 1$$

$$a^2 = 10, b^2 = 5, c^2 = 5$$

Center: $\left(-\frac{6}{5}, \frac{16}{5}\right)$

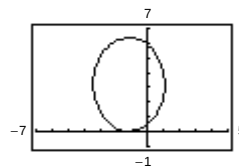
Foci: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{5}\right)$

Vertices: $\left(-\frac{6}{5}, \frac{16}{5} \pm \sqrt{10}\right)$

Solve for y: $(y^2 - 6.4y + 10.24) = -2x^2 - 4.8x - 3.12 + 10.24$

$$(y - 3.2)^2 = 7.12 - 4x - 2x^2$$

$$y = 3.2 \pm \sqrt{7.12 - 4x - 2x^2} \quad (\text{Graph each of these separately.})$$



40. Vertices: $(0, 2), (4, 2)$

Eccentricity: $\frac{1}{2}$

Horizontal major axis

Center: $(2, 2)$

$$a = 2, c = 1 \Rightarrow b = \sqrt{3}$$

$$\frac{(x - 2)^2}{4} + \frac{(y - 2)^2}{3} = 1$$

42 Foci: $(0, \pm 5)$

Major axis length: 14

Vertical major axis

Center: $(0, 0)$

$$c = 5, a = 7 \Rightarrow b = \sqrt{24}$$

$$\frac{x^2}{24} + \frac{y^2}{49} = 1$$

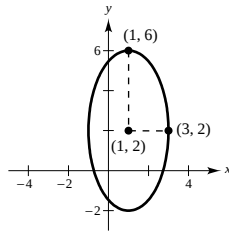
44. Center: (1, 2)

Vertical major axis

Points on ellipse: (1, 6), (3, 2)

From the sketch, we can see that
 $h = 1, k = 2, a = 4, b = 2$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{16} = 1.$$



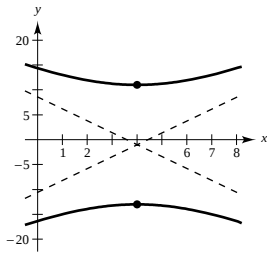
$$48. \frac{(y+1)^2}{12^2} - \frac{(x-4)^2}{5^2} = 1$$

$$a = 12, b = 5, c = \sqrt{a^2 + b^2} = 13$$

Center: (4, -1)

Vertices: (4, 11), (4, -13)

Foci: (4, -14), (4, 12)

Asymptotes: $y = -1 \pm \frac{12}{5}(x - 4)$ 

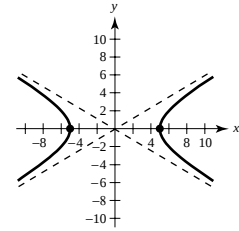
$$46. \frac{x^2}{25} - \frac{y^2}{9} = 1$$

$$a = 5, b = 3, c = \sqrt{a^2 + b^2} = \sqrt{34}$$

Center: (0, 0)

Vertices: (±5, 0)

Foci: (±√34, 0)

Asymptotes: $y = \pm \frac{3}{5}x$ 

$$50. y^2 - 9x^2 + 36x - 72 = 0$$

$$y^2 - 9(x^2 - 4x + 4) = 72 - 36 = 36$$

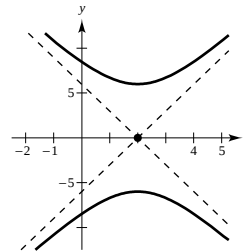
$$\frac{y^2}{36} - \frac{(x-2)^2}{4} = 1$$

$$a = 6, b = 2, c = \sqrt{a^2 + b^2} = 2\sqrt{10}$$

Center: (2, 0)

Vertices: (2, 6), (2, -6)

Foci: (2, 2√10), (2, -2√10)

Asymptotes: $y = \pm 3(x - 2)$ 

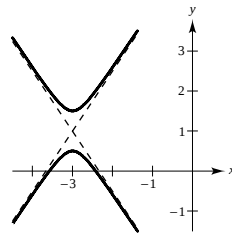
$$52. 9(x^2 + 6x + 9) - 4(y^2 - 2y + 1) = -78 + 81 - 4 = -1$$

$$9(x+3)^2 - 4(y-1)^2 = -1$$

$$\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/9} = 1$$

$$a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{\sqrt{13}}{6}$$

Center: (-3, 1)

Vertices: $(-3, \frac{1}{2}), (-3, \frac{3}{2})$ Foci: $(-3, 1 \pm \frac{\sqrt{13}}{6})$ Asymptotes: $y = 1 \pm \frac{3}{2}(x + 3)$ 

$$54. 9x^2 - y^2 + 54x + 10y + 55 = 0$$

$$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25 = 1$$

$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

$$a = \frac{1}{3}, b = 1, c = \frac{\sqrt{10}}{3}$$

Center: (-3, 5)

Vertices: $(-3 \pm \frac{1}{3}, 5)$ Foci: $(-3 \pm \frac{\sqrt{10}}{3}, 5)$

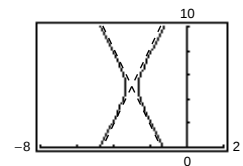
Solve for y:

$$y^2 - 10y + 25 = 9x^2 + 54x + 55 + 25$$

$$(y-5)^2 = 9x^2 + 54x + 80$$

$$y = 5 \pm \sqrt{9x^2 + 54x + 80}$$

(Graph each curve separately.)



56. $3y^2 - x^2 + 6x - 12y = 0$

$$3(y^2 - 4y + 4) - (x^2 - 6x + 9) = 0 + 12 - 9 = 3$$

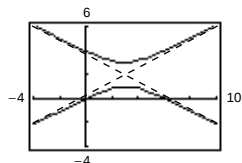
$$\frac{(y-2)^2}{1} - \frac{(x-3)^2}{3} = 1$$

$$a = 1, b = \sqrt{3}, c = 2$$

Center: (3, 2)

Vertices: (3, 1), (3, 3)

Foci: (3, 0), (3, 4)

Solve for y :

$$3(y^2 - 4y + 4) = x^2 - 6x + 12$$

$$(y-2)^2 = \frac{x^2 - 6x + 12}{3}$$

$$y = 2 \pm \sqrt{\frac{x^2 - 6x + 12}{3}}$$

(Graph each curve separately.)

60. Vertices: (2, ± 3)

Foci: (2, ± 5)

Vertical transverse axis

Center: (2, 0)

$$a = 3, c = 5, b^2 = c^2 - a^2 = 16$$

Therefore, $\frac{y^2}{9} - \frac{(x-2)^2}{16} = 1$.

64. Focus: (10, 0)

Asymptotes: $y = \pm \frac{3}{4}x$

Horizontal transverse axis

Center: (0, 0) since asymptotes intersect at the origin.

$$c = 10$$

Slopes of asymptotes: $\pm \frac{b}{a} = \pm \frac{3}{4}$ and $b = \frac{3}{4}a$

$$c^2 = a^2 + b^2 = 100$$

Solving these equations, we have $a^2 = 64$ and $b^2 = 36$.

Therefore, the equation is

$$\frac{x^2}{64} - \frac{y^2}{36} = 1.$$

68. $4x^2 - y^2 - 4x - 3 = 0$

$$A = 4, C = -1$$

$$AC < 0$$

Hyperbola

70. $25x^2 - 10x - 200y - 119 = 0$

$$A = 25, C = 0$$

Parabola

58. Vertices: (0, ± 3)

Asymptotes: $y = \pm 3x$

Vertical transverse axis

$$a = 3$$

Slopes of asymptotes: $\pm \frac{a}{b} = \pm 3$

Thus, $b = 1$. Therefore,

$$\frac{y^2}{9} - \frac{x^2}{1} = 1.$$

62. Center: (0, 0)

Vertex: (3, 0)

Focus: (5, 0)

Horizontal transverse axis

$$a = 3, c = 5, b^2 = c^2 - a^2 = 16$$

Therefore, $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

66. (a) $\frac{y^2}{4} - \frac{x^2}{2} = 1, y^2 - 2x^2 = 4, 2yy' - 4x = 0,$

$$y' = \frac{4x}{2y} = \frac{2x}{y}$$

At $x = 4$: $y = \pm 6, y' = \frac{\pm 2(4)}{6} = \pm \frac{4}{3}$

At (4, 6): $y - 6 = -\frac{4}{3}(x - 4)$ or $4x - 3y + 2 = 0$

At (4, -6): $y + 6 = -\frac{4}{3}(x - 4)$ or $4x + 3y + 2 = 0$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 3/4$.

At (4, 6): $y - 6 = -\frac{3}{4}(x - 4)$ or $3x + 4y - 36 = 0$

At (4, -6): $y + 6 = \frac{3}{4}(x - 4)$ or $3x - 4y - 36 = 0$

72. $y^2 - x - 4y - 5 = 0$

$$A = 0, C = 1$$

Parabola

$$74. \quad 2x^2 - 2xy = 3y - y^2 - 2xy$$

$$2x^2 + y^2 - 3y = 0$$

$$A = 2, C = 1, AC > 0$$

Ellipse

78. (a) An ellipse is the set of all points (x, y) , the sum of whose distance from two distinct fixed points (foci) is constant.

$$(b) \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

82. Assume that the vertex is at the origin.

$$(a) \quad x^2 = 4py$$

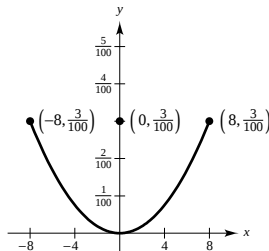
$$8^2 = 4p\left(\frac{3}{100}\right)$$

$$\frac{1600}{3} = p$$

$$x^2 = 4\left(\frac{1600}{3}\right)y = \frac{6400}{3}y$$

- (b) The deflection is 1 cm when

$$y = \frac{2}{100} \Rightarrow x = \pm \sqrt{\frac{128}{3}} \approx \pm 6.53 \text{ meters.}$$



$$76. \quad 9x^2 + 54x + 81 = 36 - 4(y^2 - 4y + 4)$$

$$9x^2 + 4y^2 + 54x - 16y + 61 = 0$$

$$A = 9, C = 4, AC > 0$$

Ellipse

$$80. \quad e = \frac{c}{a}, c = \sqrt{a^2 - b^2} \quad 0 < e < 1$$

For $e \approx 0$, the ellipse is nearly circular.

For $e \approx 1$, the ellipse is elongated.

84. (a) Without loss of generality, place the coordinate system so that the equation of the parabola is $x^2 = 4py$ and, hence,

$$y' = \left(\frac{1}{2p}\right)x.$$

Therefore, for distinct tangent lines, the slopes are unequal and the lines intersect.

$$(b) \quad x^2 - 4x - 4y = 0$$

$$2x - 4 - 4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(0, 0)$, the slope is -1 : $y = -x$. At $(6, 3)$, the slope is 2 : $y = 2x - 9$. Solving for x ,

$$-x = 2x - 9$$

$$-3x = -9$$

$$x = 3$$

$$y = -3.$$

Point of intersection: $(3, -3)$

86. The focus of $x^2 = 8y = 4(2)y$ is $(0, 2)$. The distance from a point on the parabola, $(x, x^2/8)$, and the focus, $(0, 2)$, is

$$d = \sqrt{(x-0)^2 + \left(\frac{x^2}{8} - 2\right)^2}.$$

Since d is minimized when d^2 is minimized, it is sufficient to minimize the function

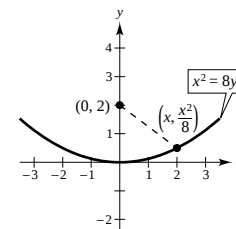
$$f(x) = x^2 + \left(\frac{x^2}{8} - 2\right)^2.$$

$$f'(x) = 2x + 2\left(\frac{x^2}{8} - 2\right)\left(\frac{x}{4}\right) = \frac{x^3}{16} + x.$$

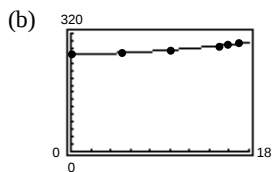
$f'(x) = 0$ implies that

$$\frac{x^3}{16} + x = x\left(\frac{x^2}{16} + 1\right) = 0 \Rightarrow x = 0.$$

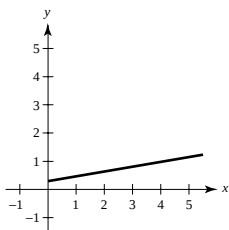
This is a minimum by the First Derivative Test. Hence, the closest point to the focus is the vertex, $(0, 0)$.



88. (a) $C = 0.0853t^2 + 0.2917t + 263.3559$



(c) $\frac{dC}{dt} = 0.1706t + 0.2971$



The consumption of fruits is increasing at a rate of 0.1706 pounds/year.

92. $x^2 = 20y$

$$y = \frac{x^2}{20}$$

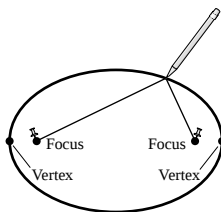
$$y' = \frac{x}{10}$$

$$S = 2\pi \int_0^r x \sqrt{1 + \left(\frac{x}{10}\right)^2} dx = 2\pi \int_0^r \frac{x\sqrt{100 + x^2}}{10} dx$$

$$= \left[\frac{\pi}{10} \cdot \frac{2}{3} (100 + x^2)^{3/2} \right]_0^r = \frac{\pi}{15} [(100 + r^2)^{3/2} - 1000]$$

96. (a) At the vertices we notice that the string is horizontal and has a length of $2a$.

- (b) The thumbtacks are located at the foci and the length of string is the constant sum of the distances from the foci.



100. $e = \frac{A - P}{A + P}$

$$= \frac{(122,000 + 4000) - (119 + 4000)}{(122,000 + 4000) + (119 + 4000)}$$

$$= \frac{121,881}{130,119} \approx 0.9367$$

90. $x = \frac{1}{4}y^2$

$$x' = \frac{1}{2}y$$

$$1 + (x')^2 = 1 + \frac{y^2}{4}$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{y^2}{4}\right)} dy = \frac{1}{2} \int_0^4 \sqrt{4 + y^2} dy$$

$$= \frac{1}{4} \left[y\sqrt{4 + y^2} + 4 \ln|y + \sqrt{4 + y^2}| \right]_0^4$$

$$= \frac{1}{4} [4\sqrt{20} + 4 \ln|4 + \sqrt{20}| - 4 \ln 2]$$

$$= 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916$$

94. $A = 2 \int_0^h \sqrt{4py} dy$

$$= 4\sqrt{p} \int_0^h y^{1/2} dy$$

$$= \left[4\sqrt{p} \left(\frac{2}{3} \right) y^{3/2} \right]_0^h$$

$$= \frac{8}{3} \sqrt{p} h^{3/2}$$

98. $e = \frac{c}{a}$

$$0.0167 = \frac{c}{149,570,000}$$

$$c \approx 2,497,819$$

$$\text{Least distance: } a - c = 147,072,181 \text{ km}$$

$$\text{Greatest distance: } a + c = 152,067,819 \text{ km}$$

102. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(b^2/a^2)} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(a^2 - c^2)/a^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

As $e \rightarrow 0$, $1 - e^2 \rightarrow 1$ and we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \text{ or the circle } x^2 + y^2 = a^2.$$

$$104. \frac{x^2}{(4.5)^2} + \frac{y^2}{(2.5)^2} = 1$$

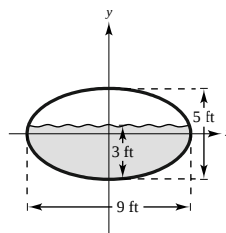
$$x^2 = (4.5)^2 \left[1 - \frac{y^2}{(2.5)^2} \right]$$

$$x = \pm \frac{9}{5} \sqrt{(2.5)^2 - y^2}$$

$$V = (\text{Area of bottom})(\text{Length}) + (\text{Area of top})(\text{Length})$$

$$V = \left[\frac{\pi(4.5)(2.5)}{2} \right](16) + 16 \int_0^{0.5} \frac{9}{5} \sqrt{(2.5)^2 - y^2} dy \quad (\text{Recall: Area of ellipse is } \pi ab.)$$

$$= 90\pi + \frac{144}{5} \cdot \frac{1}{2} \left[y \sqrt{(2.5)^2 - y^2} + (2.5)^2 \arcsin \frac{y}{2.5} \right]_0^{0.5} = 90\pi + \frac{72}{5} \left[0.5\sqrt{6} + (2.5)^2 \arcsin \frac{1}{5} \right] \approx 318.5 \text{ ft}^3$$



$$106. 9x^2 + 4y^2 + 36x - 24y + 36 = 0$$

$$18x + 8yy' + 36 - 24y' = 0$$

$$(8y - 24)y' = -(18x + 36)$$

$$y' = \frac{-(18x + 36)}{8y - 24}$$

$y' = 0$ when $x = -2$. y' undefined when $y = 3$.

At $x = -2$, $y = 0$ or 6 .

Endpoints of major axis: $(-2, 0)$, $(-2, 6)$

At $y = 3$, $x = 0$ or -4 .

Endpoints of minor axis: $(0, 3)$, $(-4, 3)$

Note: Equation of ellipse is $\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$

$$108. (a) A = 4 \int_0^4 \frac{3}{4} \sqrt{16 - x^2} dx = \frac{3}{2} \left[x \sqrt{16 - x^2} + 16 \arcsin \frac{x}{4} \right]_0^4 = 12\pi$$

$$(b) \text{ Disk: } V = 2\pi \int_0^4 \frac{9}{16} (16 - x^2) dx = \frac{9\pi}{8} \left[16x - \frac{1}{3}x^3 \right]_0^4 = 48\pi$$

$$y = \frac{3}{4} \sqrt{16 - x^2}$$

$$y' = \frac{-3x}{4\sqrt{16 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{9x^2}{16(16 - x^2)}}$$

$$S = 2(2\pi) \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \sqrt{\frac{16(16 - x^2) + 9x^2}{16(16 - x^2)}} dx = 4\pi \int_0^4 \frac{3}{4} \sqrt{16 - x^2} \frac{\sqrt{256 - 7x^2}}{4\sqrt{16 - x^2}} dx = \frac{3\pi}{4} \int_0^4 \sqrt{256 - 7x^2} dx$$

$$= \frac{3\pi}{8\sqrt{7}} \left[\sqrt{7}x \sqrt{256 - 7x^2} + 256 \arcsin \frac{\sqrt{7}x}{16} \right]_0^4 = \frac{3\pi}{8\sqrt{7}} \left(48\sqrt{7} + 256 \arcsin \frac{\sqrt{7}}{4} \right) \approx 138.93$$

—CONTINUED—

108. —CONTINUED—

$$(c) \text{ Shell: } V = 4\pi \int_0^4 x \left[\frac{3}{4} \sqrt{16 - x^2} \right] dx = 3\pi \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (16 - x^2)^{3/2} \right]_0^4 = 64\pi$$

$$x = \frac{4}{3} \sqrt{9 - y^2}$$

$$x' = \frac{-4y}{3\sqrt{9 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{16y^2}{9(9 - y^2)}}$$

$$S = 2(2\pi) \int_0^3 \frac{4}{3} \sqrt{9 - y^2} \sqrt{\frac{9(9 - y^2) + 16y^2}{9(9 - y^2)}} dy$$

$$= 4\pi \int_0^3 \frac{4}{9} \sqrt{81 + 7y^2} dy$$

$$= \frac{16}{9} \left(\frac{\pi}{2\sqrt{7}} \right) \left[\sqrt{7}y \sqrt{81 + 7y^2} + 81 \ln \left| \sqrt{7}y + \sqrt{81 + 7y^2} \right| \right]_0^3$$

$$= \frac{8\pi}{9\sqrt{7}} 3\sqrt{7}(12) + 81 \ln(3\sqrt{7} + 12) - 81 \ln 9 \approx 168.53$$

$$110. (a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$y' = -\frac{xb^2}{ya^2}$$

$$\text{At } P, y' = -\frac{b^2}{a^2} \cdot \frac{x_0}{y_0} = m.$$

$$(b) \text{ Slope of line through } (-c, 0) \text{ and } (x_0, y_0): m_1 = \frac{y_0}{x_0 + c}$$

$$\text{Slope of line through } (c, 0) \text{ and } (x_0, y_0): m_2 = \frac{y_0}{x_0 - c}$$

$$(c) \tan \alpha = \frac{m_2 - m}{1 + m_2 m} = \frac{\frac{y_0}{x_0 - c} - \left(-\frac{b^2 x_0}{a^2 y_0} \right)}{1 + \left(\frac{y_0}{x_0 - c} \right) \left(-\frac{b^2 x_0}{a^2 y_0} \right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 - c)}{a^2 y_0(x_0 - c) - b^2 x_0 y_0}$$

$$= \frac{a^2 y_0^2 + b^2 x_0^2 - b^2 x_0 c}{x_0 y_0(a^2 - b^2) - a^2 y_0 c} = \frac{a^2 b^2 - b^2 x_0 c}{x_0 y_0 c^2 - a^2 y_0 c} = \frac{b^2(a^2 - x_0 c)}{y_0 c(x_0 c - a^2)} = -\frac{b^2}{y_0 c}$$

$$\alpha = \arctan\left(-\frac{b^2}{y_0 c}\right) = -\arctan\left(\frac{b^2}{y_0 c}\right)$$

$$\tan \beta = \frac{m_1 - m}{1 + m_1 m} = \frac{\frac{y_0}{x_0 + c} - \left(-\frac{b^2 x_0}{a^2 y_0} \right)}{1 + \left(\frac{y_0}{x_0 + c} \right) \left(-\frac{b^2 x_0}{a^2 y_0} \right)} = \frac{a^2 y_0^2 + b^2 x_0(x_0 + c)}{a^2 y_0(x_0 + c) - b^2 x_0 y_0}$$

$$= \frac{a^2 y_0^2 + b^2 x_0^2 + b^2 x_0 c}{a^2 x_0 y_0 + a^2 c y_0 - b^2 x_0 y_0} = \frac{a^2 b^2 + b^2 x_0 c}{x_0 y_0(a^2 - b^2) + a^2 c y_0} = \frac{b^2(a^2 + x_0 c)}{y_0 c(x_0 c + a^2)} = \frac{b^2}{y_0 c}$$

$$\beta = \arctan\left(\frac{b^2}{y_0 c}\right)$$

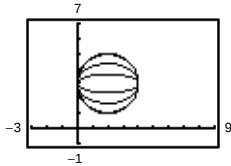
Since $|\alpha| = |\beta|$, the tangent line to an ellipse at a point P makes equal angles with the lines through P and the foci.

112. (a) $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \Rightarrow (ea)^2 - a^2 = b^2$. Hence,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1.$$

(b) $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{4(1-e^2)} = 1$



(c) As e approaches 0, the ellipse approaches a circle.

116. Center: $(0, 0)$

Horizontal transverse axis

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

The difference of the distances from any point on the hyperbola is constant. At a vertex, this constant difference is

$$(a + c) - (c - a) = 2a.$$

Now, for any point (x, y) on the hyperbola, the difference of the distances between (x, y) and the two foci must also be $2a$.

$$\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a + \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$-4xc - 4a^2 = 4a\sqrt{(x+c)^2 + y^2}$$

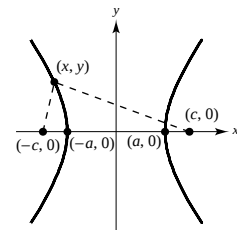
$$-(xc + a^2) = a\sqrt{(x+c)^2 + y^2}$$

$$x^2c^2 + 2a^2cx + a^4 = a^2[x^2 + 2cx + c^2 + y^2]$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Since $a^2 + b^2 = c^2$, we have $(x^2/a^2) - (y^2/b^2) = 1$.



118. $c = 150$, $2a = 0.001(186,000)$, $a = 93$,

$$b = \sqrt{150^2 - 93^2} = \sqrt{13,851}$$

$$\frac{x^2}{93^2} - \frac{y^2}{13,851} = 1$$

When $y = 75$, we have

$$x^2 = 93^2 \left(1 + \frac{75^2}{13,851} \right)$$

$$x \approx 110.3 \text{ miles.}$$

120. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \text{ or } y' = \frac{b^2x}{a^2y}$$

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$a^2y_0y - a^2y_0^2 = b^2x_0x - b^2x_0^2$$

$$b^2x_0^2 - a^2y_0^2 = b^2x_0x - a^2y_0y$$

$$a^2b^2 = b^2x_0x - a^2y_0y$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

- 122.
- $Ax^2 + Cy^2 + Dx + Ey + F = 0$
- (Assume
- $A \neq 0$
- and
- $C \neq 0$
- ; see (b) below)

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{C}y\right) = -F$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C} = R$$

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{C} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{A} = \frac{R}{AC}$$

- (a) If
- $A = C$
- , we have

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2C}\right)^2 = \frac{R}{A}$$

which is the standard equation of a circle.

- (c) If
- $AC > 0$
- , we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} + \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = 1$$

which is the equation of an ellipse.

- (b) If
- $C = 0$
- , we have

$$A\left(x + \frac{D}{2A}\right)^2 = -F - Ey + \frac{D^2}{4A}.$$

If $A = 0$, we have

$$C\left(y + \frac{E}{2C}\right)^2 = -F - Dx + \frac{E^2}{4C}.$$

These are the equations of parabolas.

- (d) If
- $AC < 0$
- , we have

$$\frac{\left[x + \left(\frac{D}{2A}\right)\right]^2}{\left|\frac{R}{A}\right|} - \frac{\left[y + \left(\frac{E}{2C}\right)\right]^2}{\left|\frac{R}{C}\right|} = \pm 1$$

which is the equation of a hyperbola.

124. True

126. False. The y^4 term should be y^2 .

128. True

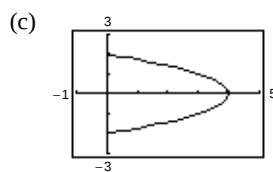
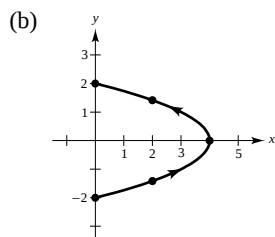
Section 9.2 Plane Curves and Parametric Equations

2. $x = 4 \cos^2 \theta$ $y = 2 \sin \theta$

$0 \leq x \leq 4$ $-2 \leq y \leq 2$

(a)

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	0	2	4	2	0
y	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2



(d) $\frac{x}{4} = \cos^2 \theta$

$\frac{y^2}{4} = \sin^2 \theta$

$\frac{x}{4} + \frac{y^2}{4} = 1$

$x = 4 - y^2, -2 \leq y \leq 2$

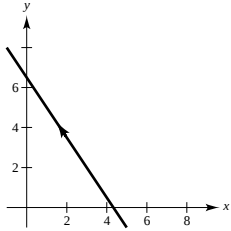
- (e) The graph would be oriented in the opposite direction.

4. $x = 3 - 2t$

$$y = 2 + 3t$$

$$y = 2 + 3\left(\frac{3-x}{2}\right)$$

$$2y + 3x - 13 = 0$$



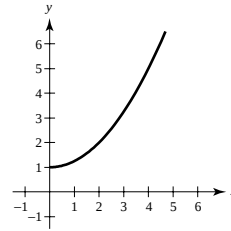
6. $x = 2t^2$

$$y = t^4 + 1$$

$$y = \left(\frac{x}{2}\right)^2 + 1 = \frac{x^2}{4} + 1, x \geq 0$$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.



8. $x = t^2 + t, y = t^2 - t$

Subtracting the second equation from the first, we have

$$x - y = 2t \quad \text{or} \quad t = \frac{x - y}{2}$$

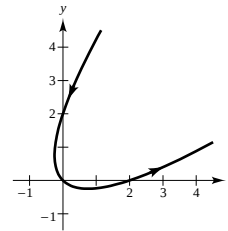
$$y = \frac{(x - y)^2}{4} - \frac{x - y}{2}$$

Since the discriminant is

$$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$$

the graph is a rotated parabola.

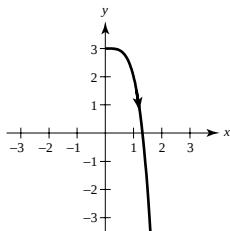
t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2



10. $x = \sqrt[4]{t}, t \geq 0$

$$y = 3 - t$$

$$y = 3 - x^4, x \geq 0$$

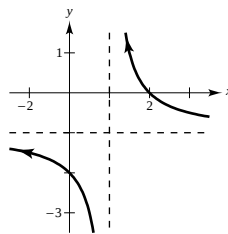


12. $x = 1 + \frac{1}{t}$

$$y = t - 1$$

$$x = 1 + \frac{1}{t} \text{ implies } t = \frac{1}{x - 1}$$

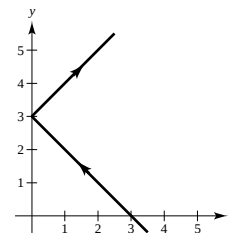
$$y = \frac{1}{x - 1} - 1$$



14. $x = |t - 1|$

$$y = t + 2$$

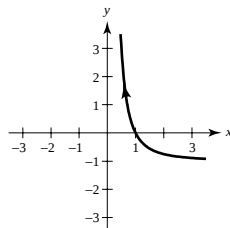
$$x = |(y - 2) - 1| = |y - 3|$$



16. $x = e^{-t}, x > 0$

$$y = e^{2t} - 1$$

$$y = x^{-2} - 1 = \frac{1}{x^2} - 1, x > 0$$



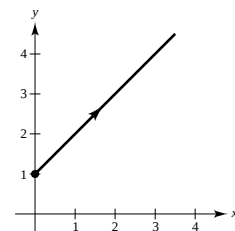
18. $x = \tan^2 \theta$

$$y = \sec^2 \theta$$

$$\sec^2 \theta = \tan^2 \theta + 1$$

$$y = x + 1$$

$$x \geq 0$$

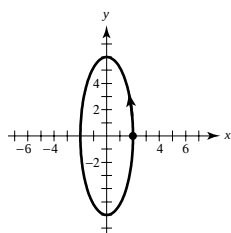


20. $x = 2 \cos \theta$

$y = 6 \sin \theta$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{6}\right)^2 = \cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{x^2}{4} + \frac{y^2}{36} = 1 \text{ ellipse}$$



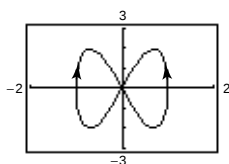
22. $x = \cos \theta$

$y = 2 \sin 2\theta$

$y = 4 \sin \theta \cos \theta$

$1 - x^2 = \sin^2 \theta$

$y = \pm 4x\sqrt{1 - x^2}$



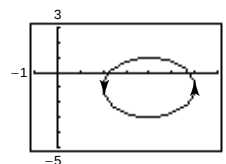
24. $x = 4 + 2 \cos \theta$

$y = -1 + 2 \sin \theta$

$(x - 4)^2 = 4 \cos^2 \theta$

$(y + 1)^2 = 4 \sin^2 \theta$

$(x - 4)^2 + (y + 1)^2 = 4$

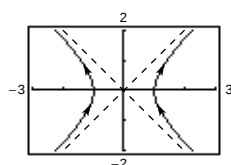


26. $x = \sec \theta$

$y = \tan \theta$

$x^2 = \sec^2 \theta$

$y^2 = \tan^2 \theta$

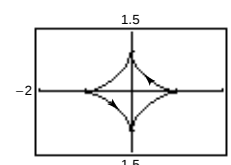


28. $x = \cos^3 \theta$

$y = \sin^3 \theta$

$x^{2/3} = \cos^2 \theta$

$y^{2/3} = \sin^2 \theta$

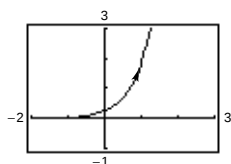


30. $x = \ln 2t$

$y = t^2$

$t = \frac{e^x}{2}$

$y = \frac{e^{2x}}{4} = \frac{1}{4}e^{2x}$



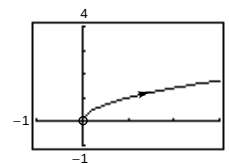
32. $x = e^{2t}$

$y = e^t$

$y^2 = x$

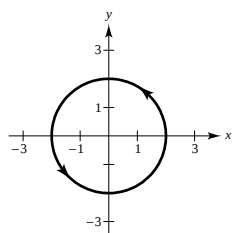
$y > 0$

$y = \sqrt{x}, x > 0$



34. By eliminating the parameters in (a) – (d), we get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

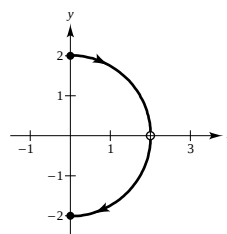
(a) $x = 2 \cos \theta, y = 2 \sin \theta$



(b) $x = \frac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \frac{1}{t^2}} \quad y = \frac{1}{t}$

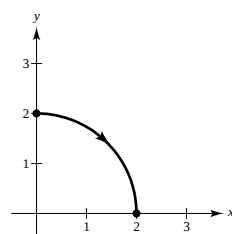
$x \geq 0, x \neq 2$

$y \neq 0$



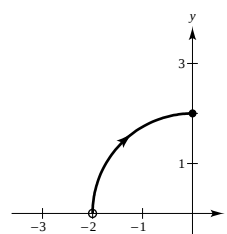
(c) $x = \sqrt{t} \quad y = \sqrt{4 - t}$

$x \geq 0 \quad y \geq 0$



(d) $x = -\sqrt{4 - e^{2t}} \quad y = e^t$

$-2 < x \leq 0 \quad y > 0$



36. The orientations are reversed. The graphs are the same. They are both smooth.

38. The set of points (x, y) corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

40. $x = h + r \cos \theta$

$$y = k + r \sin \theta$$

$$\cos \theta = \frac{x - h}{r}$$

$$\sin \theta = \frac{y - k}{r}$$

$$\cos^2 \theta + \sin^2 \theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$

$$(x - h)^2 + (y - k)^2 = r^2$$

42.

$$x = h + a \sec \theta$$

$$y = k + b \tan \theta$$

$$\frac{x - h}{a} = \sec \theta$$

$$\frac{y - k}{b} = \tan \theta$$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

44. From Exercise 39 we have

$$x = 1 + 4t$$

$$y = 4 - 6t.$$

Solution not unique

46. From Exercise 40 we have

$$x = -3 + 3 \cos \theta$$

$$y = 1 + 3 \sin \theta.$$

Solution not unique

48. From Exercise 41 we have

$$a = 5, c = 3 \Rightarrow b = 4$$

$$x = 4 + 5 \cos \theta$$

$$y = 2 + 4 \sin \theta.$$

Center: $(4, 2)$

Solution not unique

50. From Exercise 42 we have

$$a = 1, c = 2 \Rightarrow b = \sqrt{3}$$

$$x = \sqrt{3} \tan \theta$$

$$y = \sec \theta.$$

Center: $(0, 0)$

Solution not unique

The transverse axis is vertical, therefore, x and y are interchanged.

52. $y = \frac{2}{x - 1}$

Example

$$x = t, y = \frac{2}{t - 1}$$

$$x = -t, y = \frac{2}{-t - 1}$$

54. $y = x^2$

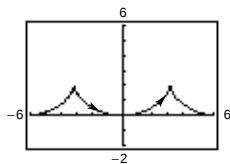
Example

$$x = t, y = t^2$$

$$x = t^3, y = t^6$$

56. $x = \theta + \sin \theta$

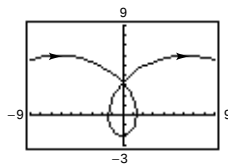
$$y = 1 - \cos \theta$$



Not smooth at $x = (2n - 1)\pi$

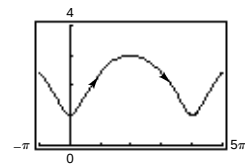
58. $x = 2\theta - 4 \sin \theta$

$$y = 2 - 4 \cos \theta$$



60. $x = 2\theta - \sin \theta$

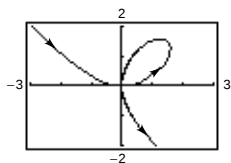
$$y = 2 - \cos \theta$$



Smooth everywhere

$$62. \quad x = \frac{3t}{1+t^3}$$

$$y = \frac{3t^2}{1+t^3}$$



Smooth everywhere

64. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t , plot (x, y) . As t increases, the curve is traced out in a specific direction called the orientation of the curve.

66. (a) Matches (ii) because $-1 \leq x \leq 0$ and $1 \leq y \leq 2$.

(b) Matches (i) because $x = (y - 2)^2 - 1$ for all y .

$$68. \quad x = \cos^3 \theta$$

$$y = 2 \sin^2 \theta$$

Matches (a)

$$70. \quad x = \cot \theta$$

$$y = 4 \sin \theta \cos \theta$$

Matches (c)

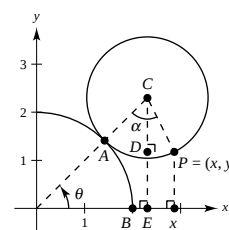
72. Let the circle of radius 1 be centered at C . A is the point of tangency on the line OC . $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle θ changes. $AB = AP$. $AB = 2\theta$ and $AP = \alpha \Rightarrow \alpha = 2\theta$. Form the right triangle $\triangle CDP$. The angle $OCE = (\pi/2) - \theta$ and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3 \sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3 \cos \theta - \cos 3\theta$$

$$y = EC - CD = 3 \sin \theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3 \sin \theta - \sin 3\theta$$

Hence, $x = 3 \cos \theta - \cos 3\theta$, $y = 3 \sin \theta - \sin 3\theta$.



74. False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and y is not a function of x .

$$76. \quad (a) \quad x = (v_0 \cos \theta)t$$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$t = \frac{x}{v_0 \cos \theta} \Rightarrow y = h + (v_0 \sin \theta) \frac{x}{v_0 \cos \theta} - 16 \left(\frac{x}{v_0 \cos \theta}\right)^2$$

$$y = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2} x^2$$

$$(b) \quad y = 5 + x - 0.005x^2 = h + (\tan \theta)x - \frac{16 \sec^2 \theta}{v_0^2} x^2$$

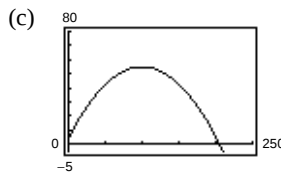
$$h = 5, \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \text{ and}$$

$$0.005 = \frac{16 \sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \Rightarrow v_0 = 80.$$

$$\text{Hence, } x = (80 \cos(45^\circ))t$$

$$y = 5 + (80 \sin(45^\circ))t - 16t^2.$$



(d) Maximum height: $y = 55$ (at $x = 100$)
Range: 204.88

Section 9.3 Parametric Equations and Calculus

$$2. \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$$

$$6. x = \sqrt{t}, y = 3t - 1$$

$$\frac{dy}{dx} = \frac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6 \text{ when } t = 1.$$

$$\frac{d^2y}{dx^2} = \frac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6 \text{ concave upwards}$$

$$10. x = \cos \theta, y = 3 \sin \theta$$

$$\frac{dy}{dx} = \frac{3 \cos \theta}{-\sin \theta} = -3 \cot \theta \cdot \frac{dy}{dx} \text{ is undefined when } \theta = 0.$$

$$\frac{d^2y}{dx^2} = \frac{3 \csc^2 \theta}{-\sin \theta} = \frac{-3}{\sin^3 \theta} \cdot \frac{d^2y}{dx^2} \text{ is undefined when } \theta = 0.$$

$$12. x = \sqrt{t}, y = \sqrt{t-1}$$

$$\frac{dy}{dx} = \frac{1/(2\sqrt{t-1})}{1/(2\sqrt{t})}$$

$$= \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{[\sqrt{t-1}/(2\sqrt{t}) - \sqrt{t}(1/2\sqrt{t-1})]/(t-1)}{1/(2\sqrt{t})}$$

$$= \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

concave downward

$$16. x = 2 - 3 \cos \theta, y = 3 + 2 \sin \theta$$

$$\frac{dy}{dx} = \frac{2 \cos \theta}{3 \sin \theta} = \frac{2}{3} \cot \theta$$

At $(-1, 3)$, $\theta = 0$, and $\frac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y = 5$

At $\left(\frac{4+3\sqrt{3}}{2}, 2\right)$, $\theta = \frac{7\pi}{6}$, and $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$.

Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3} \left(x - \frac{4+3\sqrt{3}}{2} \right)$$

$$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$$

$$4. \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(-1/2)e^{-\theta/2}}{2e^\theta} = -\frac{1}{4}e^{-3\theta/2} = \frac{-1}{4e^{3\theta/2}}$$

$$8. x = t^2 + 3t + 2, y = 2t$$

$$\frac{dy}{dx} = \frac{2}{2t+3} = \frac{2}{3} \text{ when } t = 0.$$

$$\frac{d^2y}{dx^2} = \frac{-2(2)/(2t+3)}{(2t+3)^2} = \frac{-4}{(2t+3)^2} = \frac{-4}{9} \text{ when } t = 0.$$

concave downward

$$14. x = \theta - \sin \theta, y = 1 - \cos \theta$$

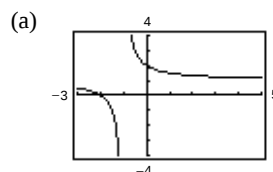
$$\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{[(1 - \cos \theta) \cos \theta - \sin^2 \theta]}{(1 - \cos \theta)^2}$$

$$= \frac{-1}{(1 - \cos \theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

concave downward

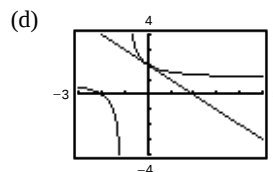
$$18. x = t - 1, y = \frac{1}{t} + 1, t = 1$$



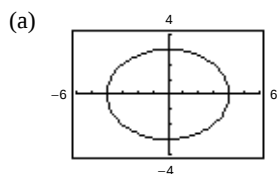
(b) At $t = 1$, $(x, y) = (0, 2)$, and

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -1, \frac{dy}{dx} = -1$$

(c) $\frac{dy}{dx} = -1$. At $(0, 2)$, $y - 2 = -1(x - 0)$
 $y = -x + 2$



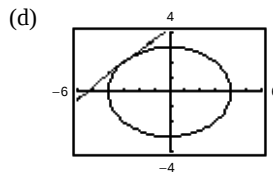
20. $x = 4 \cos \theta, y = 3 \sin \theta, \theta = \frac{3\pi}{4}$



(c) $\frac{dy}{dx} = \frac{3}{4}$. At $\left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), y - \frac{3}{\sqrt{2}} = \frac{3}{4}\left(x + \frac{4}{\sqrt{2}}\right)$
 $y = \frac{3}{4}x + 3\sqrt{2}$

(b) At $\theta = \frac{3\pi}{4}, (x, y) = \left(\frac{-4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, and

$$\frac{dx}{dt} = -2\sqrt{2}, \frac{dy}{dt} = -\frac{3\sqrt{2}}{2}, \frac{dy}{dx} = \frac{3}{4}$$



22. $x = t^2 - t, y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1, \frac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2, \frac{dy}{dx} = \frac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$. Tangent Line

24. $x = 2\theta, y = 2(1 - \cos \theta)$

Horizontal tangents: $\frac{dy}{d\theta} = 2 \sin \theta = 0$ when $\theta = 0, \pm\pi, \pm2\pi, \dots$

Points: $(4n\pi, 0), (2[2n - 1]\pi, 4)$ where n is an integer.

Points shown: $(0, 0), (2\pi, 4), (4\pi, 0)$

Vertical tangents: $\frac{dx}{d\theta} = 2 \neq 0$; none

26. $x = t + 1, y = t^2 + 3t$

Horizontal tangents: $\frac{dy}{dt} = 2t + 3 = 0$ when $t = -\frac{3}{2}$.

Point: $\left(-\frac{1}{2}, -\frac{9}{4}\right)$

Vertical tangents: $\frac{dx}{dt} = 1 \neq 0$; none

28. $x = t^2 - t + 2, y = t^3 - 3t$

Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\frac{dx}{dt} = 2t - 1 = 0$ when $t = \frac{1}{2}$.

Point: $\left(\frac{7}{4}, -\frac{11}{8}\right)$

30. $x = \cos \theta, y = 2 \sin 2\theta$

Horizontal tangents: $\frac{dy}{d\theta} = 4 \cos 2\theta = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Points: $\left(\frac{\sqrt{2}}{2}, 2\right), \left(-\frac{\sqrt{2}}{2}, -2\right), \left(-\frac{\sqrt{2}}{2}, 2\right), \left(\frac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\frac{dx}{d\theta} = -\sin \theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$

32. $x = 4 \cos^2 \theta$, $y = 2 \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = 2 \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Since $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\frac{dx}{d\theta} = -8 \cos \theta \sin \theta = 0$ when

$$\theta = 0, \pi.$$

Point: $(4, 0)$

34. $x = \cos^2 \theta$, $y = \cos \theta$

Horizontal tangents: $\frac{dy}{d\theta} = -\sin \theta = 0$ when $x = 0, \pi$.

Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\frac{dx}{d\theta} = -2 \cos \theta \sin \theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

(Exclude $0, \pi$.)

Point: $(0, 0)$

36. $x = t^2 + 1$, $y = 4t^3 + 3$, $-1 \leq t \leq 0$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 12t^2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 144t^4$$

$$s = \int_{-1}^0 \sqrt{4t^2 + 144t^4} dt = \int_{-1}^0 -2t\sqrt{1 + 36t^2} dt$$

$$= \left[-\frac{(1 + 36t^2)^{3/2}}{54} \right]_{-1}^0 = \frac{-1}{54}(1 - 37^{3/2}) \approx 4.149$$

38. $x = \arcsin t$, $y = \ln \sqrt{1 - t^2}$, $0 \leq t \leq \frac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}}, \frac{dy}{dt} = \frac{1}{2} \left(\frac{-2t}{1 - t^2} \right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{(1 - t^2)^2}} dt = \int_0^{1/2} \frac{1}{1 - t^2} dt$$

$$= \left[-\frac{1}{2} \ln \left| \frac{t - 1}{t + 1} \right| \right]_0^{1/2}$$

$$= -\frac{1}{2} \ln \left(\frac{1}{3} \right) = \frac{1}{2} \ln(3) \approx 0.549$$

40. $x = t$, $y = \frac{t^5}{10} + \frac{1}{6t^3}$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2} dt =$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2} dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right) dt$$

$$= \left[\frac{t^5}{10} - \frac{1}{6t^3} \right]_1^2 = \frac{779}{240}$$

42. $x = a \cos \theta$, $y = a \sin \theta$, $\frac{dx}{d\theta} = -a \sin \theta$, $\frac{dy}{d\theta} = a \cos \theta$

$$S = 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta} d\theta$$

$$= 4a \int_0^{\pi/2} d\theta = \left[4a\theta \right]_0^{\pi/2} = 2\pi a$$

44. $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$, $\frac{dx}{d\theta} = \theta \cos \theta$

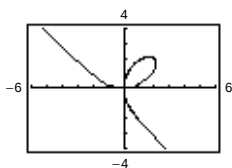
$$\frac{dy}{d\theta} = \theta \sin \theta$$

$$S = \int_0^{2\pi} \sqrt{\theta^2 \cos^2 \theta + \theta^2 \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} \theta d\theta = \left[\frac{\theta^2}{2} \right]_0^{2\pi} = 2\pi^2$$

46. $x = \frac{4t}{1+t^3}, y = \frac{4t^2}{1+t^3}$

(a) $x^3 + y^3 = 4xy$



(b) $\frac{dy}{dt} = \frac{(1+t^3)(8t) - 4t^2(3t^2)}{(1+t^3)^2}$
 $= \frac{4t(2-t^3)}{(1+t^3)^2} = 0$ when $t = 0$ or $t = \sqrt[3]{2}$.

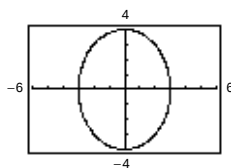
Points: $(0, 0), \left(\frac{4\sqrt[3]{2}}{3}, \frac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

(c) $s = 2 \int_0^1 \sqrt{\left[\frac{4(1-2t^3)}{(1+t^3)^2}\right]^2 + \left[\frac{4t(2-t^3)}{(1+t^3)^2}\right]^2} dt = 2 \int_0^1 \sqrt{\frac{16}{(1+t^3)^4} [t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1]} dt$
 $= 8 \int_0^1 \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1+t^3)^2} dt \approx 6.557$

48. $x = 3 \cos \theta, y = 4 \sin \theta$

$\frac{dx}{d\theta} = -3 \sin \theta, \frac{dy}{d\theta} = 4 \cos \theta$

$s = \int_0^{2\pi} \sqrt{9 \sin^2 \theta + 16 \cos^2 \theta} d\theta \approx 22.1$



50. $x = t, y = 4 - 2t, \frac{dx}{dt} = 1, \frac{dy}{dt} = -2$

(a) $S = 2\pi \int_0^2 (4-2t) \sqrt{1+4} dt$
 $= \left[2\sqrt{5}\pi(4t-t^2) \right]_0^2 = 8\pi\sqrt{5}$

(b) $S = 2\pi \int_0^2 t \sqrt{1+4} dt = \left[\sqrt{5}\pi t^2 \right]_0^2 = 4\pi\sqrt{5}$

52. $x = \frac{1}{3}t^3, y = t + 1, 1 \leq t \leq 2, y\text{-axis}$

$\frac{dx}{dt} = t^2, \frac{dy}{dt} = 1$

$S = 2\pi \int_1^2 \frac{1}{3}t^3 \sqrt{t^4+1} dt = \frac{\pi}{9} \left[(x^4+1)^{3/2} \right]_1^2$
 $= \frac{\pi}{9} (17^{3/2} - 2^{3/2}) \approx 23.48$

54. $x = a \cos \theta, y = b \sin \theta, \frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = b \cos \theta$

(a) $S = 4\pi \int_0^{\pi/2} b \sin \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$
 $= 4\pi \int_0^{\pi/2} ab \sin \theta \sqrt{1 - \left(\frac{a^2-b^2}{a^2}\right) \cos^2 \theta} d\theta = \frac{-4ab\pi}{e} \int_0^{\pi/2} (-e \sin \theta) \sqrt{1 - e^2 \cos^2 \theta} d\theta$
 $= \frac{-2ab\pi}{e} \left[e \cos \theta \sqrt{1 - e^2 \cos^2 \theta} + \arcsin(e \cos \theta) \right]_0^{\pi/2} = \frac{-ab\pi}{e} [e \sqrt{1 - e^2} + \arcsin(e)]$
 $= 2\pi b^2 + \left(\frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \right) \arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) \arcsin(e)$
 $\left(e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}; \text{eccentricity} \right)$

—CONTINUED—

54. —CONTINUED—

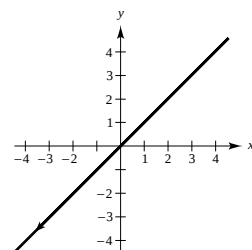
$$\begin{aligned}
 \text{(b) } S &= 4\pi \int_0^{\pi/2} a \cos \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta \\
 &= 4\pi \int_0^{\pi/2} a \cos \theta \sqrt{b^2 + c^2 \sin^2 \theta} d\theta = \frac{4a\pi}{c} \int_0^{\pi/2} c \cos \theta \sqrt{b^2 + c^2 \sin^2 \theta} d\theta \\
 &= \frac{2a\pi}{c} \left[c \sin \theta \sqrt{b^2 + c^2 \sin^2 \theta} + b^2 \ln |c \sin \theta + \sqrt{b^2 + c^2 \sin^2 \theta}| \right]_0^{\pi/2} \\
 &= \frac{2a\pi}{c} \left[c \sqrt{b^2 + c^2} + b^2 \ln |c + \sqrt{b^2 + c^2}| - b^2 \ln b \right] \\
 &= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}} \ln \left| \frac{a + \sqrt{a^2 - b^2}}{b} \right| = 2\pi a^2 + \left(\frac{\pi b^2}{e} \right) \ln \left| \frac{1+e}{1-e} \right|
 \end{aligned}$$

56. (a) 0

(b) 4

58. One possible answer is the graph given by

$$x = -t, y = -t.$$



$$\begin{aligned}
 \text{60. (a) } S &= 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 \text{(b) } S &= 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt
 \end{aligned}$$

62. Let y be a continuous function of x on $a \leq x \leq b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t) dt$ and

$$\int_a^b y dx = \int_{t_1}^{t_2} g(t) f'(t) dt.$$

$$\text{64. } x = \sqrt{4-t}, y = \sqrt{t}, \frac{dx}{dt} = -\frac{1}{2\sqrt{4-t}}, 0 \leq t \leq 4$$

$$A = \int_4^0 \sqrt{t} \left(-\frac{1}{2\sqrt{4-t}} \right) dt = \int_0^2 \sqrt{4-u^2} du = \frac{1}{2} \left[u \sqrt{4-u^2} + 4 \arcsin \frac{u}{2} \right]_0^2 = \pi$$

Let $u = \sqrt{4-t}$, then $du = -1/(2\sqrt{4-t}) dt$ and $\sqrt{t} = \sqrt{4-u^2}$.

$$\bar{x} = \frac{1}{\pi} \int_4^0 \sqrt{4-t} \sqrt{t} \left(-\frac{1}{2\sqrt{4-t}} \right) dt = -\frac{1}{2\pi} \int_4^0 \sqrt{t} dt = \left[-\frac{1}{2\pi} \frac{2}{3} t^{3/2} \right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi} \int_4^0 (\sqrt{t})^2 \left(-\frac{1}{2\sqrt{4-t}} \right) dt = -\frac{1}{4\pi} \int_4^0 \frac{t}{\sqrt{4-t}} dt = -\frac{1}{4\pi} \left[\frac{-2(8+t)}{3} \sqrt{4-t} \right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi} \right)$$

$$\text{66. } x = \cos \theta, y = 3 \sin \theta, \frac{dx}{d\theta} = -\sin \theta$$

$$\begin{aligned}
 V &= 2\pi \int_{\pi/2}^0 (3 \sin \theta)^2 (-\sin \theta) d\theta \\
 &= -18\pi \int_{\pi/2}^0 \sin^3 \theta d\theta = -18\pi \left[-\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 = 12\pi
 \end{aligned}$$

68. $x = 2 \cot \theta$, $y = 2 \sin^2 \theta$, $\frac{dx}{d\theta} = -2 \csc^2 \theta$

$$A = 2 \int_{\pi/2}^0 (2 \sin^2 \theta)(-2 \csc^2 \theta) d\theta = -8 \int_{\pi/2}^0 d\theta = \left[-8\theta \right]_{\pi/2}^0 = 4\pi$$

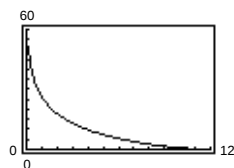
70. $\frac{3}{8}\pi a^2$ is area of asteroid (b).

72. $2\pi a^2$ is area of deltoid (c).

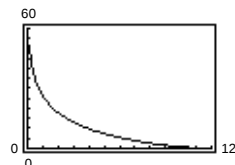
74. $2\pi ab$ is area of teardrop (e).

76. (a) $y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$

$$0 < x \leq 12$$



(b) $x = 12 \operatorname{sech} \frac{t}{12}$, $y = t - 12 \tanh \frac{t}{12}$, $0 \leq t$



Same as the graph in (a), but has the advantage of showing the position of the object and any given time t .

(c) $\frac{dy}{dx} = \frac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12) \tanh(t/12)} = -\sinh \frac{t}{12}$

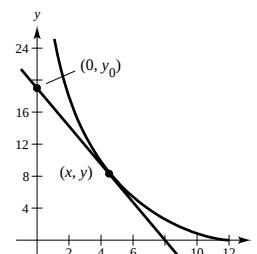
Tangent line: $y - \left(t_0 - 12 \tanh \frac{t_0}{12}\right) = -\sinh \frac{t_0}{12} \left(x - 12 \operatorname{sech} \frac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh \frac{t_0}{12}\right)x$$

y-intercept: $(0, t_0)$

Distance between $(0, t_0)$ and (x, y) : $d = \sqrt{\left(12 \operatorname{sech} \frac{t_0}{12}\right)^2 + \left(-12 \tanh \frac{t_0}{12}\right)^2} = 12$

$$d = 12 \text{ for any } t \geq 0.$$



78. False. Both dx/dt and dy/dt are zero when $t = 0$. By eliminating the parameter, we have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

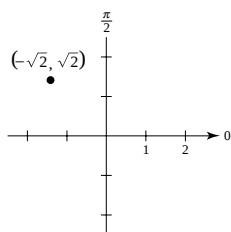
Section 9.4 Polar Coordinates and Polar Graphs

2. $\left(-2, \frac{7\pi}{4}\right)$

$$x = -2 \cos\left(\frac{7\pi}{4}\right) = -\sqrt{2}$$

$$y = -2 \sin\left(\frac{7\pi}{4}\right) = \sqrt{2}$$

$$(x, y) = (-\sqrt{2}, \sqrt{2})$$

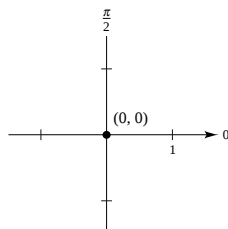


4. $\left(0, -\frac{7\pi}{6}\right)$

$$x = 0 \cos\left(-\frac{7\pi}{6}\right) = 0$$

$$y = 0 \sin\left(-\frac{7\pi}{6}\right) = 0$$

$$(x, y) = (0, 0)$$

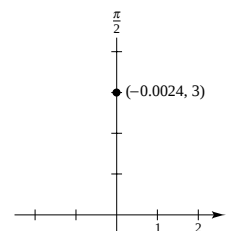


6. $(-3, -1.57)$

$$x = -3 \cos(-1.57) \approx -0.0024$$

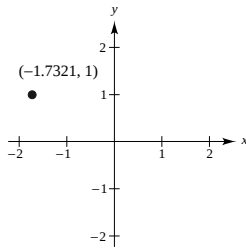
$$y = -3 \sin(-1.57) \approx 3$$

$$(x, y) = (-0.0024, 3)$$



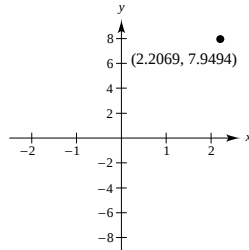
$$8. (r, \theta) = \left(-2, \frac{11\pi}{6}\right)$$

$$(x, y) = (-1.7321, 1)$$



$$10. (r, \theta) = (8.25, 1.3)$$

$$(x, y) = (2.2069, 7.9494)$$

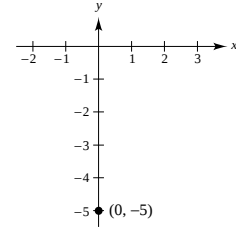


$$12. (x, y) = (0, -5)$$

$$r = \pm 5$$

$$\tan \theta \text{ undefined}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \left(5, \frac{3\pi}{2}\right), \left(-5, \frac{\pi}{2}\right)$$



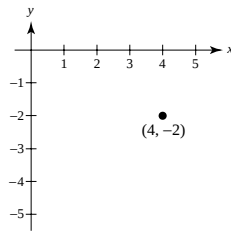
$$14. (x, y) = (4, -2)$$

$$r = \pm \sqrt{16 + 4} = \pm 2\sqrt{5}$$

$$\tan \theta = -\frac{2}{4} = -\frac{1}{2}$$

$$\theta \approx -0.464$$

$$(2\sqrt{5}, -0.464), (-2\sqrt{5}, 2.678)$$



$$16. (x, y) = (3\sqrt{2}, 3\sqrt{2})$$

$$(r, \theta) = (6, 0.785)$$

$$18. (x, y) = (0, -5)$$

$$(r, \theta) = (5, -1.571)$$

20. (a) Moving horizontally, the x -coordinate changes. Moving vertically, the y -coordinate changes.

(b) Both r and θ values change.

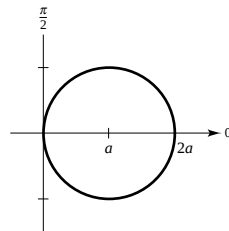
(c) In polar mode, horizontal (or vertical) changes result in changes in both r and θ .

$$22. x^2 + y^2 - 2ax = 0$$

$$r^2 - 2ar \cos \theta = 0$$

$$r(r - 2a \cos \theta) = 0$$

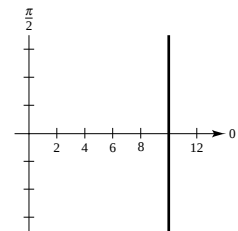
$$r = 2a \cos \theta$$



$$24. x = 10$$

$$r \cos \theta = 10$$

$$r = 10 \sec \theta$$

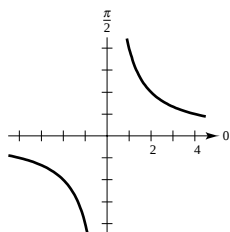


$$26. xy = 4$$

$$(r \cos \theta)(r \sin \theta) = 4$$

$$r^2 = 4 \sec \theta \csc \theta$$

$$= 8 \csc 2\theta$$

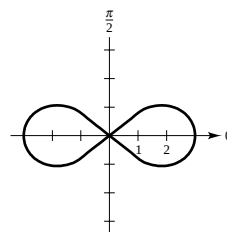


$$28. (x^2 + y^2)^2 - 9(x^2 - y^2) = 0$$

$$(r^2)^2 - 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta) = 0$$

$$r^2[r^2 - 9(\cos 2\theta)] = 0$$

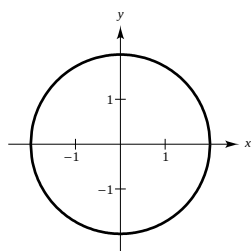
$$r^2 = 9 \cos 2\theta$$



30. $r = -2$

$$r^2 = 4$$

$$x^2 + y^2 = 4$$



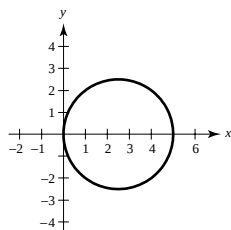
32. $r = 5 \cos \theta$

$$r^2 = 5r \cos \theta$$

$$x^2 + y^2 = 5x$$

$$x^2 - 5x + \frac{25}{4} + y^2 = \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

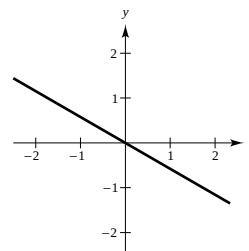


34. $\theta = \frac{5\pi}{6}$

$$\tan \theta = \tan \frac{5\pi}{6}$$

$$\frac{y}{x} = -\frac{\sqrt{3}}{3}$$

$$y = -\frac{\sqrt{3}}{3}x$$

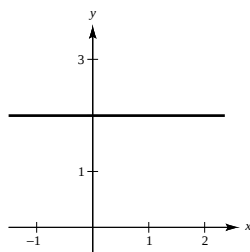


36. $r = 2 \csc \theta$

$$r \sin \theta = 2$$

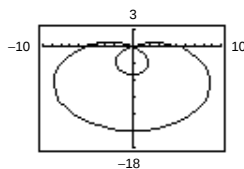
$$y = 2$$

$$y - 2 = 0$$



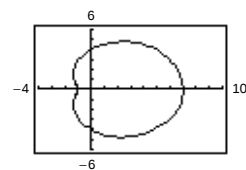
38. $r = 5(1 - 2 \sin \theta)$

$$0 \leq \theta < 2\pi$$



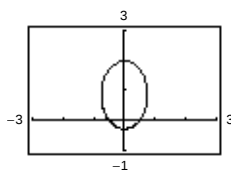
40. $r = 4 + 3 \cos \theta$

$$0 \leq \theta < 2\pi$$



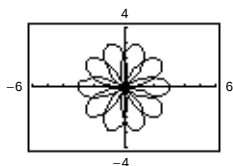
42. $r = \frac{2}{4 - 3 \sin \theta}$

Traced out once on $0 \leq \theta \leq 2\pi$



44. $r = 3 \sin\left(\frac{5\theta}{2}\right)$

$$0 \leq \theta < 4\pi$$

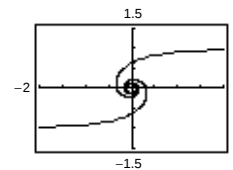


46. $r^2 = \frac{1}{\theta}$

Graph as

$$r_1 = \frac{1}{\sqrt{\theta}}, r_2 = -\frac{1}{\sqrt{\theta}}$$

It is traced out once on $[0, \infty)$.



48. (a) The rectangular coordinates of (r_1, θ_1) are $(r_1 \cos \theta_1, r_1 \sin \theta_1)$. The rectangular coordinates of (r_2, θ_2) are $(r_2 \cos \theta_2, r_2 \sin \theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\ &= r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1 \\ &= r_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2) + r_1^2 (\cos^2 \theta_1 + \sin^2 \theta_1) - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)} \end{aligned}$$

- (b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2| \end{aligned}$$

- (c) If $\theta_1 - \theta_2 = 90^\circ$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

- (d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = (2, \pi/2)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2) \cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

$$\text{Using } (r_1, \theta_1) = (-1, \pi) \text{ and } (r_2, \theta_2) = [2, (5\pi/2)], d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2) \cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}.$$

You always obtain the same distance.

50. $\left(10, \frac{7\pi}{6}\right), (3, \pi)$

$$\begin{aligned} d &= \sqrt{10^2 + 3^2 - 2(10)(3) \cos\left(\frac{7\pi}{6} - \pi\right)} \\ &= \sqrt{109 - 60 \cos \frac{\pi}{6}} = \sqrt{109 - 30\sqrt{3}} \approx 7.6 \end{aligned}$$

52. $(4, 2.5), (12, 1)$

$$\begin{aligned} d &= \sqrt{4^2 + 12^2 - 2(4)(12) \cos(2.5 - 1)} \\ &= \sqrt{160 - 96 \cos 1.5} \approx 12.3 \end{aligned}$$

54. $r = 2(1 - \sin \theta)$

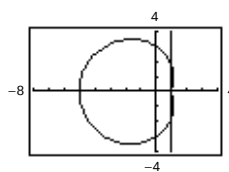
$$\frac{dy}{dx} = \frac{-2 \cos \theta \sin \theta + 2 \cos \theta (1 - \sin \theta)}{-2 \cos \theta \cos \theta - 2 \sin \theta (1 - \sin \theta)}$$

At $(2, 0)$, $\frac{dy}{dx} = -1$.

At $\left(3, \frac{7\pi}{6}\right)$, $\frac{dy}{dx}$ is undefined.

At $\left(4, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

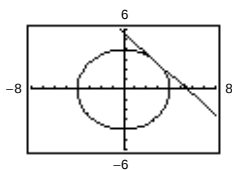
56. (a), (b) $r = 3 - 2 \cos \theta$



$$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$$

Tangent line: $x = 1$

- (c) At $\theta = 0$, $\frac{dy}{dx}$ does not exist (vertical tangent).

58. (a), (b) $r = 4$


$$\text{at } (r, \theta) = \left(4, \frac{\pi}{4}\right) \Rightarrow (x, y) = (2\sqrt{2}, 2\sqrt{2})$$

$$\begin{aligned} \text{Tangent line: } y - 2\sqrt{2} &= -1(x - 2\sqrt{2}) \\ y &= -x + 4\sqrt{2} \end{aligned}$$

 (c) At $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = -1$.

 62. $r = a \sin \theta \cos^2 \theta$

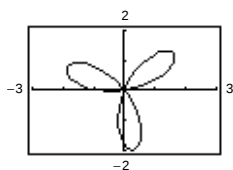
$$\frac{dy}{d\theta} = a \sin \theta \cos^3 \theta + [-2a \sin^2 \theta \cos \theta + a \cos^3 \theta] \sin \theta$$

$$= 2a[\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta]$$

$$= 2a \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) = 0$$

$$\theta = 0, \tan^2 \theta = 1, \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\text{Horizontal: } \left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$$

 66. $r = 2 \cos(3\theta - 2)$


Horizontal tangents:

$$(1.894, 0.776), (1.755, 2.594), (1.998, -1.442)$$

 70. $r = 3(1 - \cos \theta)$

Cardioid

 Symmetric to polar axis since r is a function of $\cos \theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6

 60. $r = a \sin \theta$

$$\frac{dy}{d\theta} = a \sin \theta \cos \theta + a \cos \theta \sin \theta$$

$$= 2a \sin \theta \cos \theta = 0$$

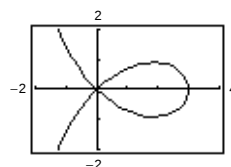
$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a \sin^2 \theta + a \cos^2 \theta = a(1 - 2 \sin^2 \theta) = 0$$

$$\sin \theta = \pm \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\text{Horizontal: } (0, 0), \left(a, \frac{\pi}{2}\right)$$

$$\text{Vertical: } \left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$$

 64. $r = 3 \cos 2\theta \sec \theta$

 Horizontal tangents: $(2.133, \pm 0.4352)$

 68. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

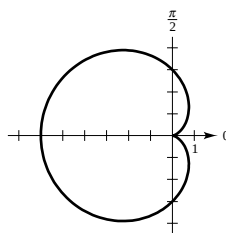
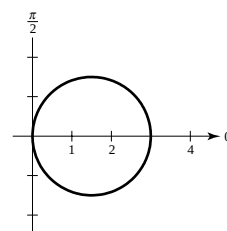
$$x^2 + y^2 = 3x$$

$$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$$

$$\text{Circle: } r = \frac{3}{2}$$

$$\text{Center: } \left(\frac{3}{2}, 0\right)$$

$$\text{Tangent at pole: } \theta = \frac{\pi}{2}$$



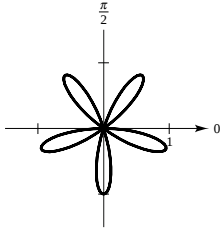
72. $r = -\sin(5\theta)$

Rose curve with five petals

 Symmetric to $\theta = \frac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5 \cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

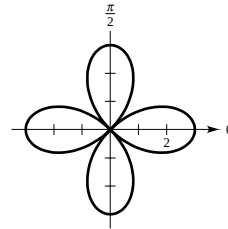
 Tangents at the pole: $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$


74. $r = 3 \cos 2\theta$

Rose curve with four petals

 Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

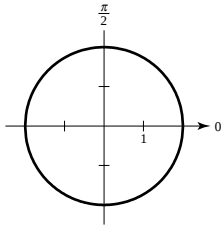
 Relative extrema: $(3, 0), (-3, \frac{\pi}{2}), (3, \pi), (-3, \frac{3\pi}{2})$

 Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$
 $\theta = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ given the same tangents.


76. $r = 2$

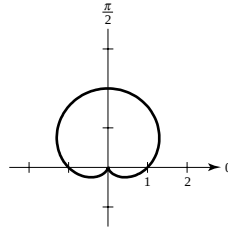
Circle radius: 2

$$x^2 + y^2 = 4$$



78. $r = 1 + \sin \theta$

Cardioid

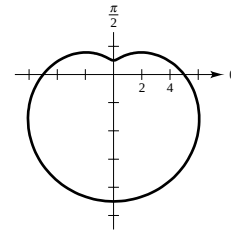


80. $r = 5 - 4 \sin \theta$

Limaçon

 Symmetric to $\theta = \frac{\pi}{2}$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	9	7	5	3	1

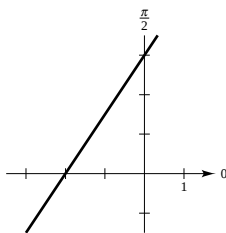


82. $r = \frac{6}{2 \sin \theta - 3 \cos \theta}$

$$2r \sin \theta - 3r \cos \theta = 6$$

$$2y - 3x = 6$$

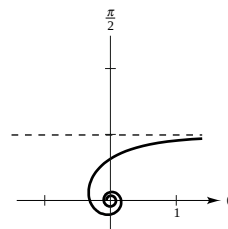
Line



84. $r = \frac{1}{\theta}$

Hyperbolic spiral

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	$\frac{4}{\pi}$	$\frac{2}{\pi}$	$\frac{4}{3\pi}$	$\frac{1}{\pi}$	$\frac{4}{5\pi}$	$\frac{2}{3\pi}$

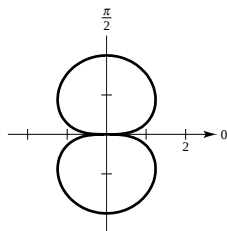


86. $r^2 = 4 \sin \theta$

Lemniscate

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, \frac{\pi}{2})$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
r	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0

Tangent at the pole: $\theta = 0$ 

88. Since

$$r = 2 + \csc \theta = 2 + \frac{1}{\sin \theta},$$

the graph has symmetry with respect to $\theta = \pi/2$.

Furthermore,

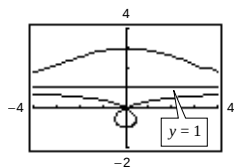
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

$$\text{Also, } r = 2 + \frac{1}{\sin \theta} = 2 + \frac{r}{\sin \theta} = 2 + \frac{r}{y}$$

$$ry = 2y + r$$

$$r = \frac{2y}{y-1}.$$

Thus, $r \Rightarrow \pm\infty$ as $y \Rightarrow 1$.

92. $x = r \cos \theta, y = r \sin \theta$

$$x^2 + y^2 = r^2, \tan \theta = \frac{y}{x}$$

96. $r = 4 \cos 2\theta$

Rose curve

Matches (b)

90. $r = 2 \cos 2\theta \sec \theta$

Strophoid

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

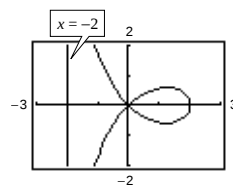
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$

$$r = 2 \cos 2\theta \sec \theta = 2(2 \cos^2 \theta - 1) \sec \theta$$

$$r \cos \theta = 4 \cos^2 \theta - 2$$

$$x = 4 \cos^2 \theta - 2$$

$$\lim_{\theta \rightarrow \pm\pi/2} (4 \cos^2 \theta - 2) = -2$$



94. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos \theta + f'(\theta)\sin \theta}{-f(\theta)\sin \theta + f'(\theta)\cos \theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

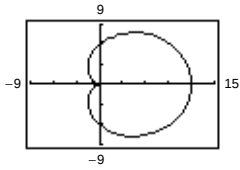
98. $r = 2 \sec \theta$

Line

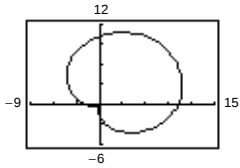
Matches (d)

100. $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos \theta]$



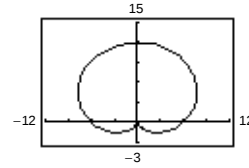
(b) $\theta = \frac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \frac{\pi}{4}\right)\right]$



The graph of $r = 6[1 + \cos \theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \frac{\pi}{2}$

$$\begin{aligned} r &= 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right] \\ &= 6\left[1 + \cos \theta \cos \frac{\pi}{2} + \sin \theta \sin \frac{\pi}{2}\right] \\ &= 6[1 + \sin \theta] \end{aligned}$$



The graph of $r = 6[1 + \cos \theta]$ is rotated through the angle $\pi/2$.

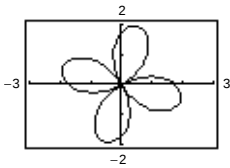
102. (a) $\sin\left(\theta - \frac{\pi}{2}\right) = \sin \theta \cos\left(\frac{\pi}{2}\right) - \cos \theta \sin\left(\frac{\pi}{2}\right)$
 $= -\cos \theta$
 $r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$
 $= f(-\cos \theta)$

(c) $\sin\left(\theta - \frac{3\pi}{2}\right) = \sin \theta \cos\left(\frac{3\pi}{2}\right) - \cos \theta \sin\left(\frac{3\pi}{2}\right)$
 $= \cos \theta$
 $r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos \theta)$

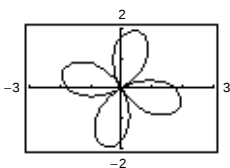
(b) $\sin(\theta - \pi) = \sin \theta \cos \pi - \cos \theta \sin \pi$
 $= -\sin \theta$
 $r = f[\sin(\theta - \pi)]$
 $= f(-\sin \theta)$

104. $r = 2 \sin 2\theta = 4 \sin \theta \cos \theta$

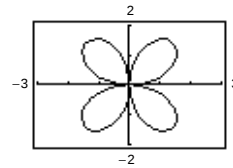
(a) $r = 4 \sin\left(\theta - \frac{\pi}{6}\right) \cos\left(\theta - \frac{\pi}{6}\right)$



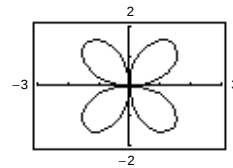
(c) $r = 4 \sin\left(\theta - \frac{2\pi}{3}\right) \cos\left(\theta - \frac{2\pi}{3}\right)$



(b) $r = 4 \sin\left(\theta - \frac{\pi}{2}\right) \cos\left(\theta - \frac{\pi}{2}\right) = -4 \sin \theta \cos \theta$



(d) $r = 4 \sin(\theta - \pi) \cos(\theta - \pi) = 4 \sin \theta \cos \theta$

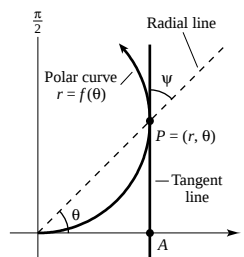


106. By Theorem 9.11, the slope of the tangent line through A and P is

$$\frac{f \cos \theta + f' \sin \theta}{-f \sin \theta + f' \cos \theta}$$

This is equal to

$$\tan(\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\sin \theta + \cos \theta \tan \psi}{\cos \theta - \sin \theta \tan \psi}.$$



Equating the expressions and cross-multiplying, you obtain

$$(f \cos \theta + f' \sin \theta)(\cos \theta - \sin \theta \tan \psi) = (\sin \theta + \cos \theta \tan \psi)(-f \sin \theta + f' \cos \theta)$$

$$f \cos^2 \theta - f \cos \theta \sin \theta \tan \psi + f' \sin \theta \cos \theta - f' \sin^2 \theta \tan \psi = -f \sin^2 \theta - f \sin \theta \cos \theta \tan \psi + f' \sin \theta \cos \theta + f' \cos^2 \theta \tan \psi$$

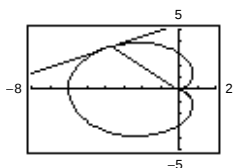
$$f(\cos^2 \theta + \sin^2 \theta) = f' \tan \psi (\cos^2 \theta + \sin^2 \theta)$$

$$\tan \psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$

108. $\tan \psi = \frac{r}{dr/d\theta} = \frac{3(1 - \cos \theta)}{3 \sin \theta}$

At $\theta = \frac{3\pi}{4}$, $\tan \psi = \frac{1 + (\sqrt{2}/2)}{\sqrt{2}} = \frac{2 + \sqrt{2}}{\sqrt{2}}.$

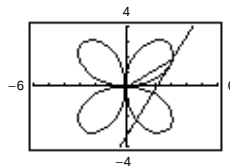
$$\psi = \arctan\left(\frac{2 + \sqrt{2}}{\sqrt{2}}\right) \approx 1.041 (\approx 59.64^\circ)$$



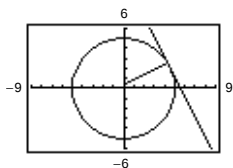
110. $\tan \psi = \frac{r}{dr/d\theta} = \frac{4 \sin 2\theta}{8 \cos 2\theta}$

At $\theta = \frac{\pi}{6}$, $\tan \psi = \frac{\sin(\pi/3)}{2 \cos(\pi/3)} = \frac{\sqrt{3}}{2}.$

$$\psi = \arctan\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 (\approx 40.89^\circ)$$



112. $\tan \psi = \frac{r}{dr/d\theta} = \frac{5}{0} \text{ undefined} \Rightarrow \psi = \frac{\pi}{2}.$

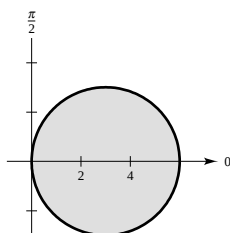


114. True

116. True

Section 9.5 Area and Arc Length in Polar Coordinates

2. (a) $r = 3 \cos \theta$



$$A = \pi \left(\frac{3}{2}\right)^2 = \frac{9\pi}{4}$$

(b) $A = 2 \left(\frac{1}{2}\right) \int_0^{\pi/2} [3 \cos \theta]^2 d\theta$

$$= 9 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= \frac{9}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

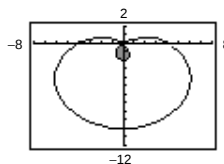
$$= \frac{9}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{9\pi}{4}$$

$$\begin{aligned}
 4. \quad A &= 2 \left[\frac{1}{2} \int_0^{\pi/4} (6 \sin 2\theta)^2 d\theta \right] = 36 \int_0^{\pi/4} \sin^2 2\theta d\theta \\
 &= 36 \int_0^{\pi/4} \frac{1 - \cos 4\theta}{2} d\theta \\
 &= 18 \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/4} \\
 &= 18 \left[\frac{\pi}{4} \right] = \frac{9\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad A &= 2 \left[\frac{1}{2} \int_0^{\pi/2} (1 - \sin \theta)^2 d\theta \right] \\
 &= \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = \frac{3\pi - 8}{4}
 \end{aligned}$$

$$\begin{aligned}
 6. \quad A &= 2 \left[\frac{1}{2} \int_0^{\pi/10} (\cos 5\theta)^2 d\theta \right] \\
 &= \frac{1}{2} \left[\theta + \frac{1}{10} \sin(10\theta) \right]_0^{\pi/10} = \frac{\pi}{20}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad A &= 2 \left[\frac{1}{2} \int_{\arcsin(2/3)}^{\pi/2} (4 - 6 \sin \theta)^2 d\theta \right] \\
 &= \int_{\arcsin(2/3)}^{\pi/2} [16 - 48 \sin \theta + 36 \sin^2 \theta] d\theta \\
 &= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48 \sin \theta + 36 \left(\frac{1 - \cos 2\theta}{2} \right) \right] d\theta \\
 &= \left[34\theta + 48 \cos \theta - 9 \sin 2\theta \right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635
 \end{aligned}$$



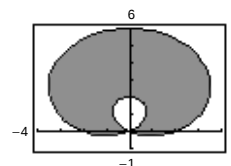
12. Four times the area in Exercise 11, $A = 4(\pi + 3\sqrt{3})$. More specifically, we see that the area inside the outer loop is

$$2 \left[\frac{1}{2} \int_{-\pi/6}^{\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = \int_{-\pi/6}^{\pi/2} (4 + 16 \sin \theta + 16 \sin^2 \theta) d\theta = 8\pi + 6\sqrt{3}.$$

The area inside the inner loop is

$$2 \frac{1}{2} \left[\int_{7\pi/6}^{3\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = 4\pi - 6\sqrt{3}.$$

Thus, the area between the loops is $(8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.



$$14. \quad r = 3(1 + \sin \theta)$$

$$r = 3(1 - \sin \theta)$$

Solving simultaneously,

$$3(1 + \sin \theta) = 3(1 - \sin \theta)$$

$$2 \sin \theta = 0$$

$$\theta = 0, \pi.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin \theta) = 3(1 - \sin \theta)$, $\sin \theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0)$, $(3, \pi)$, $(0, 0)$

$$16. \quad r = 2 - 3 \cos \theta$$

$$r = \cos \theta$$

Solving simultaneously,

$$2 - 3 \cos \theta = \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$, $(0, 0)$

18. $r = 1 + \cos \theta$

$r = 3 \cos \theta$

Solving simultaneously,

$1 + \cos \theta = 3 \cos \theta$

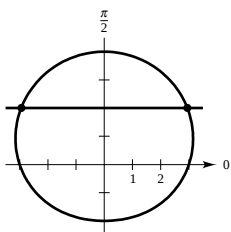
$\cos \theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, 0)$

22. $r = 3 + \sin \theta$

$r = 2 \csc \theta$



The graph of $r = 3 + \sin \theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2 \csc \theta$ is the horizontal line $y = 2$. Therefore, there are two points of intersection. Solving simultaneously,

$3 + \sin \theta = 2 \csc \theta$

$\sin^2 \theta + 3 \sin \theta - 2 = 0$

$\sin \theta = \frac{-3 \pm \sqrt{17}}{2}$

$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596$

24. $r = 3(1 - \cos \theta)$

$r = \frac{6}{1 - \cos \theta}$

The graph of $r = 3(1 - \cos \theta)$ is a cardioid with polar axis symmetry. The graph of

$r = 6/(1 - \cos \theta)$

is a parabola with focus at the pole, vertex $(3, \pi)$, and polar axis symmetry. Therefore, there are two points of intersection. Solving simultaneously,

$3(1 - \cos \theta) = \frac{6}{1 - \cos \theta}$

$(1 - \cos \theta)^2 = 2$

$\cos \theta = 1 \pm \sqrt{2}$

$\theta = \arccos(1 - \sqrt{2})$

Points of intersection: $(3\sqrt{2}, \arccos(1 - \sqrt{2})) \approx (4.243, 1.998), (3\sqrt{2}, 2\pi - \arccos(1 - \sqrt{2})) \approx (4.243, 4.285)$

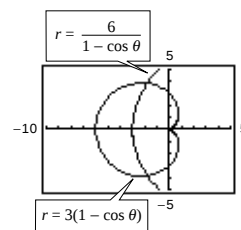
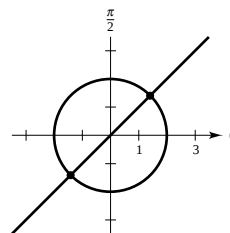
20. $\theta = \frac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection:

$\left(2, \frac{\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$

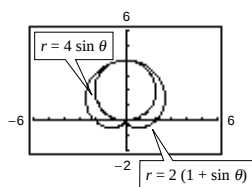


26. $r = 4 \sin \theta$

$$r = 2(1 + \sin \theta)$$

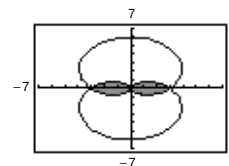
Points of intersection: $(0, 0), \left(4, \frac{\pi}{2}\right)$

The graphs reach the pole at different times (θ values).



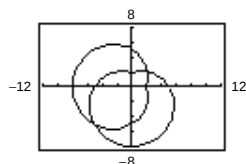
$$\begin{aligned} 28. A &= 4 \left[\frac{1}{2} \int_0^{\pi/2} 9(1 - \sin \theta)^2 d\theta \right] \\ &= 18 \int_0^{\pi/2} (1 - \sin \theta)^2 d\theta = \frac{9}{2} (3\pi - 8) \end{aligned}$$

(from Exercise 14)

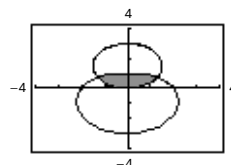


30. $r = 5 - 3 \sin \theta$ and $r = 5 - 3 \cos \theta$ intersect at $\theta = \pi/4$ and $\theta = 5\pi/4$.

$$\begin{aligned} A &= 2 \left[\frac{1}{2} \int_{\pi/4}^{5\pi/4} (5 - 3 \sin \theta)^2 d\theta \right] \\ &= \left[\frac{59}{2} \theta + 30 \cos \theta - \frac{9}{4} \sin 2\theta \right]_{\pi/4}^{5\pi/4} \\ &= \left(\frac{59}{2} \left(\frac{5\pi}{4} \right) - 30 \frac{\sqrt{2}}{2} - \frac{9}{4} \right) - \left(\frac{59}{2} \left(\frac{\pi}{4} \right) + 30 \frac{\sqrt{2}}{2} - \frac{9}{4} \right) \\ &= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251 \end{aligned}$$



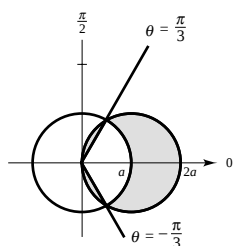
$$\begin{aligned} 32. A &= 2 \left[\frac{1}{2} \int_{\pi/6}^{\pi/2} (3 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/2} (2 - \sin \theta)^2 d\theta \right] \\ &= \int_{\pi/6}^{\pi/2} (-4 \cos 2\theta + 4 \sin \theta) d\theta \\ &= \left[-2 \sin(2\theta) - 4 \cos \theta \right]_{\pi/6}^{\pi/2} = 3\sqrt{3} \end{aligned}$$



34. Area = Area of $r = 2a \cos \theta$ - Area of sector - twice area between $r = 2a \cos \theta$ and the lines

$$\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}.$$

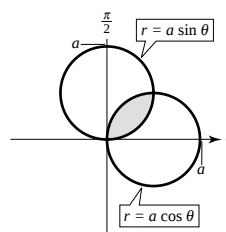
$$\begin{aligned} A &= \pi a^2 - \left(\frac{\pi}{3} \right) a^2 - 2 \left[\frac{1}{2} \int_{\pi/3}^{\pi/2} (2a \cos \theta)^2 d\theta \right] \\ &= \frac{2\pi a^2}{3} - 2a^2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{2\pi a^2}{3} - 2a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/3}^{\pi/2} \\ &= \frac{2\pi a^2}{3} - 2a^2 \left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6} \end{aligned}$$



36. $r = a \cos \theta, r = a \sin \theta$

$$\tan \theta = 1, \theta = \pi/4$$

$$\begin{aligned} A &= 2 \left[\frac{1}{2} \int_0^{\pi/2} (a \cos \theta)^2 d\theta \right] \\ &= a^2 \int_0^{\pi/4} \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{1}{2} a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} \\ &= \frac{1}{2} a^2 \left[\frac{\pi}{4} + \frac{1}{2} \right] \\ &= \frac{1}{4} a^2 + \frac{1}{8} a^2 \pi \end{aligned}$$



38. By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

$$\begin{aligned} A_1 &= A_2 = \frac{1}{2} \int_{-\pi/3}^{\pi/6} [(2a \cos \theta)^2 - (a)^2] d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/4} [(2a \cos \theta)^2 - (2a \sin \theta)^2] d\theta \\ &= \frac{a^2}{2} \int_{-\pi/3}^{\pi/6} (4 \cos^2 \theta - 1) d\theta + 2a^2 \int_{\pi/6}^{\pi/4} \cos 2\theta d\theta \\ &= \frac{a^2}{2} \left[\theta + \sin 2\theta \right]_{-\pi/3}^{\pi/6} + a^2 \left[\sin 2\theta \right]_{\pi/6}^{\pi/4} = \frac{a^2}{2} \left(\frac{\pi}{2} + \sqrt{3} \right) + a^2 \left(1 - \frac{\sqrt{3}}{2} \right) = a^2 \left(\frac{\pi}{4} + 1 \right) \end{aligned}$$

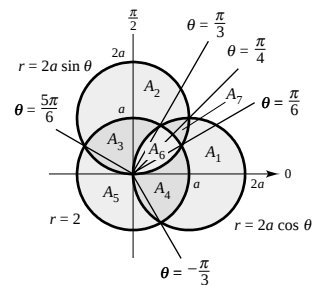
$$A_3 = A_4 = \frac{1}{2} \left(\frac{\pi}{2} \right) a^2 = \frac{\pi a^2}{4}$$

$$\begin{aligned} A_5 &= \frac{1}{2} \left(\frac{5\pi}{6} \right) a^2 - 2 \left(\frac{1}{2} \right) \int_{5\pi/6}^{\pi} (2a \sin \theta)^2 d\theta \\ &= \frac{5\pi a^2}{12} - 2a^2 \int_{5\pi/6}^{\pi} (1 - \cos 2\theta) d\theta \\ &= \frac{5\pi a^2}{12} - a^2 \left[2\theta - \sin 2\theta \right]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = a^2 \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\begin{aligned} A_6 &= 2 \left(\frac{1}{2} \right) \int_0^{\pi/6} (2a \sin \theta)^2 d\theta + 2 \left(\frac{1}{2} \right) \int_{\pi/6}^{\pi/4} a^2 d\theta \\ &= 2a^2 \int_0^{\pi/6} (1 - \cos 2\theta) d\theta + \left[a^2 \theta \right]_{\pi/6}^{\pi/4} \\ &= a^2 \left[2\theta - \sin 2\theta \right]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) + \frac{\pi a^2}{12} = a^2 \left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\begin{aligned} A_7 &= 2 \left(\frac{1}{2} \right) \int_{\pi/6}^{\pi/4} [(2a \sin \theta)^2 - (a)^2] d\theta \\ &= a^2 \int_{\pi/6}^{\pi/4} (4 \sin^2 \theta - 1) d\theta = a^2 \left[\theta - \sin 2\theta \right]_{\pi/6}^{\pi/4} = a^2 \left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2} \right) \end{aligned}$$

[Note: $A_1 + A_6 + A_7 + A_4 = \pi a^2 = \text{area of circle of radius } a]$



40. $r = \sec \theta - 2 \cos \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$r \cos \theta = 1 - 2 \cos^2 \theta$$

$$x = 1 - 2 \frac{r^2 \cos^2 \theta}{r^2} = 1 - 2 \left(\frac{x^2}{x^2 + y^2} \right)$$

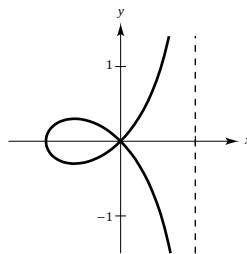
$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

$$y^2 = \frac{x^2(1+x)}{1-x}$$

$$A = 2 \left(\frac{1}{2} \right) \int_0^{\pi/4} (\sec \theta - 2 \cos \theta)^2 d\theta$$

$$= \int_0^{\pi/4} (\sec^2 \theta - 4 + 4 \cos^2 \theta) d\theta = \int_0^{\pi/4} (\sec^2 \theta - 4 + 2(1 + \cos 2\theta)) d\theta = \left[\tan \theta - 2\theta + \sin 2\theta \right]_0^{\pi/4} = 2 - \frac{\pi}{2}$$



42. $r = 2a \cos \theta$

$$r' = -2a \sin \theta$$

$$s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a \cos \theta)^2 + (-2a \sin \theta)^2} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2a d\theta = \left[2\theta \right]_{-\pi/2}^{\pi/2} = 2\pi a$$

44. $r = 8(1 + \cos \theta), 0 \leq \theta \leq 2\pi$

$$r' = -8 \sin \theta$$

$$s = 2 \int_0^\pi \sqrt{[8(1 + \cos \theta)]^2 + (-8 \sin \theta)^2} d\theta$$

$$= 16 \int_0^\pi \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta$$

$$= 16 \sqrt{2} \int_0^\pi \sqrt{1 + \cos \theta} d\theta$$

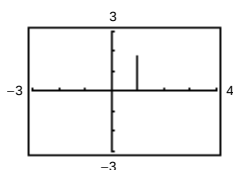
$$= 16 \sqrt{2} \int_0^\pi \sqrt{1 + \cos \theta} \cdot \left(\frac{\sqrt{1 - \cos \theta}}{\sqrt{1 - \cos \theta}} \right) d\theta$$

$$= 16 \sqrt{2} \int_0^\pi \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta$$

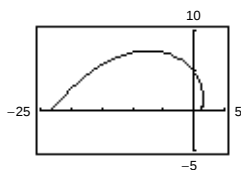
$$= \left[32 \sqrt{2} \sqrt{1 - \cos \theta} \right]_0^\pi$$

$$= 64$$

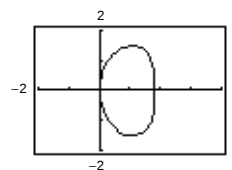
46. $r = \sec \theta, 0 \leq \theta \leq \frac{\pi}{3}$

Length ≈ 1.73 (exact $\sqrt{3}$)

48. $r = e^\theta, 0 \leq \theta \leq \pi$

Length ≈ 31.31

50. $r = 2 \sin(2 \cos \theta), 0 \leq \theta \leq \pi$

Length ≈ 7.78

52. $r = a \cos \theta$

$$r' = -a \sin \theta$$

$$S = 2\pi \int_0^{\pi/2} a \cos \theta (\cos \theta) \sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta} d\theta$$

$$= 2\pi a^2 \int_0^{\pi/2} \cos^2 \theta d\theta = \pi a^2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta$$

$$= \left[\pi a^2 \left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

54. $r = a(1 + \cos \theta)$

$$r' = -a \sin \theta$$

$$S = 2\pi \int_0^\pi a(1 + \cos \theta) \sin \theta \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta = 2\pi a^2 \int_0^\pi \sin \theta (1 + \cos \theta) \sqrt{2 + 2 \cos \theta} d\theta$$

$$= -2\sqrt{2} \pi a^2 \int_0^\pi (1 + \cos \theta)^{3/2} (-\sin \theta) d\theta = -\frac{4\sqrt{2} \pi a^2}{5} \left[(1 + \cos \theta)^{5/2} \right]_0^\pi = \frac{32 \pi a^2}{5}$$

56. $r = \theta$

$$r' = 1$$

$$S = 2\pi \int_0^\pi \theta \sin \theta \sqrt{\theta^2 + 1} d\theta \approx 42.32$$

58. The curves might intersect for different values of θ :

See page 696.

60. (a) $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

(b) $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$

62. $r = 8 \cos \theta, 0 \leq \theta \leq \pi$

(a) $A = \frac{1}{2} \int_0^{\pi} r^2 d\theta = \frac{1}{2} \int_0^{\pi} 64 \cos^2 \theta d\theta = 32 \int_0^{\pi} \frac{1 + \cos 2\theta}{2} d\theta = 16 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi} = 16\pi$

(Area circle = $\pi r^2 = \pi 4^2 = 16\pi$)

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c), (d) For $\frac{1}{4}$ of area ($4\pi \approx 12.57$): 0.42
 For $\frac{1}{2}$ of area ($8\pi \approx 25.13$): 1.57 ($\pi/2$)
 For $\frac{3}{4}$ of area ($12\pi \approx 37.70$): 2.73

(e) No, it does not depend on the radius.

64. False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

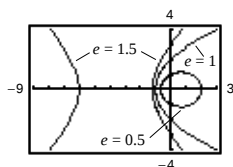
Section 9.6 Polar Equations of Conics and Kepler's Laws

2. $r = \frac{2e}{1 - e \cos \theta}$

(a) $e = 1, r = \frac{2}{1 - \cos \theta}$, parabola

(b) $e = 0.5, r = \frac{1}{1 - 0.5 \cos \theta} = \frac{2}{2 - \cos \theta}$, ellipse

(c) $e = 1.5, r = \frac{3}{1 - 1.5 \cos \theta} = \frac{6}{2 - 3 \cos \theta}$, hyperbola

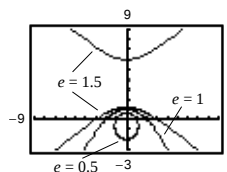


4. $r = \frac{2e}{1 + e \sin \theta}$

(a) $e = 1, r = \frac{2}{1 + \sin \theta}$, parabola

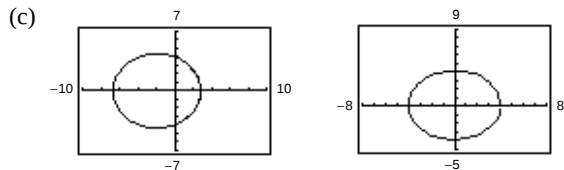
(b) $e = 0.5, r = \frac{1}{1 + 0.5 \sin \theta} = \frac{2}{2 + \sin \theta}$, ellipse

(c) $e = 1.5, r = \frac{3}{1 + 1.5 \sin \theta} = \frac{6}{2 + 3 \sin \theta}$, hyperbola



6. $r = \frac{4}{1 - 0.4 \cos \theta}$

(a) Because $e = 0.4 < 1$, the conic is an ellipse with vertical directrix to the left of the pole.



(b) $r = \frac{4}{1 + 0.4 \cos \theta}$

The ellipse is shifted to the left. The vertical directrix is to the right of the pole

$$r = \frac{4}{1 - 0.4 \sin \theta}$$

The ellipse has a horizontal directrix below the pole.

8. Ellipse; Matches (f)

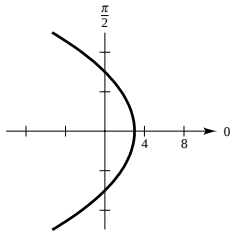
10. Parabola; Matches (e)

12. Hyperbola; Matches (d)

$$14. r = \frac{6}{1 + \cos \theta}$$

Parabola since $e = 1$

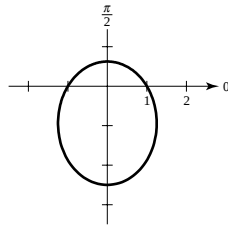
Vertex: (3, 0)



$$16. r = \frac{5}{5 + 3 \sin \theta} = \frac{1}{1 + (3/5) \sin \theta}$$

Ellipse since $e = \frac{3}{5} < 1$

Vertices: $(\frac{5}{8}, \frac{\pi}{2}), (\frac{5}{2}, \frac{3\pi}{2})$



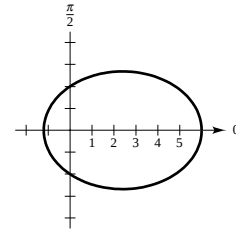
$$18. r(3 - 2 \cos \theta) = 6$$

$$r = \frac{6}{3 - 2 \cos \theta}$$

$$= \frac{2}{1 - (2/3) \cos \theta}$$

Ellipse since $e = \frac{2}{3} < 1$

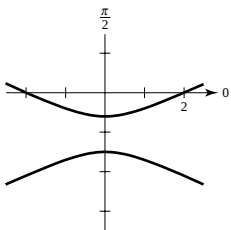
Vertices: (6, 0), $(\frac{6}{5}, \pi)$



$$20. r = \frac{-6}{3 + 7 \sin \theta} = \frac{-2}{1 + (7/3) \sin \theta}$$

Hyperbola since $e = \frac{7}{3} > 1$.

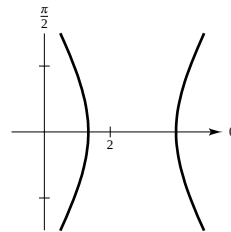
Vertices: $(-\frac{3}{5}, \frac{\pi}{2}), (\frac{3}{2}, \frac{3\pi}{2})$



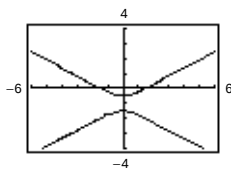
$$22. r = \frac{4}{1 + 2 \cos \theta}$$

Hyperbola since $e = 2 > 1$

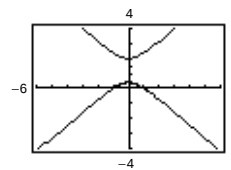
Vertices: $(\frac{4}{3}, 0), (-4, \pi)$



24. Hyperbola



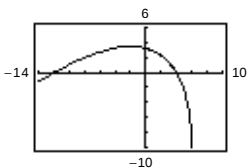
26. Hyperbola



$$28. r = \frac{6}{1 + \cos(\theta - \frac{\pi}{3})}$$

Rotate the graph of $r = \frac{6}{1 + \cos \theta}$

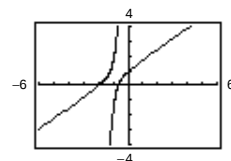
counterclockwise through the angle $\frac{\pi}{3}$.



$$30. r = \frac{-6}{3 + 7 \sin(\theta + (2\pi/3))}$$

Rotate graph of $r = \frac{-6}{3 + 7 \sin \theta}$

Clockwise through angle of $2\pi/3$.



32. Change θ to $\theta - \frac{\pi}{6}$: $r = \frac{2}{1 + \sin\left(\theta - \frac{\pi}{6}\right)}$

34. Parabola

$$e = 1, y = 1, d = 1$$

$$r = \frac{ed}{1 + e \sin \theta} = \frac{1}{1 + \sin \theta}$$

36. Ellipse

$$e = \frac{3}{4}, y = -2, d = 2$$

$$\begin{aligned} r &= \frac{ed}{1 - e \sin \theta} \\ &= \frac{2(3/4)}{1 - (3/4) \sin \theta} \\ &= \frac{6}{4 - 3 \sin \theta} \end{aligned}$$

38. Hyperbola

$$e = \frac{3}{2}, x = -1, d = 1$$

$$\begin{aligned} r &= \frac{ed}{1 - e \cos \theta} \\ &= \frac{3/2}{1 - (3/2) \cos \theta} \\ &= \frac{3}{2 - 3 \cos \theta} \end{aligned}$$

40. Parabola

Vertex: $(5, \pi)$

$$e = 1, d = 10$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{10}{1 - \cos \theta}$$

42. Ellipse

$$\text{Vertices: } \left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$$

$$e = \frac{1}{3}, d = 8$$

$$\begin{aligned} r &= \frac{ed}{1 + e \sin \theta} \\ &= \frac{8/3}{1 + (1/3) \sin \theta} \\ &= \frac{8}{3 + \sin \theta} \end{aligned}$$

44. Hyperbola

Vertices: $(2, 0), (10, 0)$

$$e = \frac{3}{2}, d = \frac{10}{3}$$

$$\begin{aligned} r &= \frac{ed}{1 + e \cos \theta} \\ &= \frac{5}{1 + (3/2) \cos \theta} \\ &= \frac{10}{2 + 3 \cos \theta} \end{aligned}$$

46. $r = \frac{4}{1 + \sin \theta}$ is a parabola with horizontal directrix above the pole.

(a) Parabola with vertical directrix to left pole.

(c) Parabola with vertical directrix to right of pole.

(b) Parabola with horizontal directrix below pole.

(d) Parabola (b) rotated counterclockwise $\pi/4$.

48. (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x^2 b^2 + y^2 a^2 = a^2 b^2$$

$$b^2 r^2 \cos^2 \theta + a^2 r^2 \sin^2 \theta = a^2 b^2$$

$$r^2 [b^2 \cos^2 \theta + a^2 (1 - \cos^2 \theta)] = a^2 b^2$$

$$r^2 [a^2 + \cos^2 \theta (b^2 - a^2)] = a^2 b^2$$

$$r^2 = \frac{a^2 b^2}{a^2 + (b^2 - a^2) \cos^2 \theta} = \frac{a^2 b^2}{a^2 - c^2 \cos^2 \theta}$$

$$= \frac{b^2}{1 - (c/a)^2 \cos^2 \theta} = \frac{b^2}{1 - e^2 \cos^2 \theta}$$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$x^2 b^2 - y^2 a^2 = a^2 b^2$$

$$b^2 r^2 \cos^2 \theta - a^2 r^2 \sin^2 \theta = a^2 b^2$$

$$r^2 [b^2 \cos^2 \theta - a^2 (1 - \cos^2 \theta)] = a^2 b^2$$

$$r^2 [-a^2 + \cos^2 \theta (a^2 + b^2)] = a^2 b^2$$

$$r^2 = \frac{a^2 b^2}{-a^2 + c^2 \cos^2 \theta} = \frac{b^2}{-1 + (c^2/a^2) \cos^2 \theta}$$

$$= \frac{-b^2}{1 - e^2 \cos^2 \theta}$$

50. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16) \cos^2 \theta}$$

52. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4) \cos^2 \theta}$$

54. $A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{2}{3 - 2 \sin \theta} \right)^2 d\theta \right] = 4 \int_{-\pi/2}^{\pi/2} \frac{1}{(3 - 2 \sin \theta)^2} d\theta \approx 3.37$

56. (a) $r = \frac{ed}{1 - e \cos \theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \frac{(1 - e^2)a}{1 - e \cos \theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi, r = \frac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0, r = \frac{(1 - e^2)a}{1 - e} = a(1 + e)$.

58. $a = 1.427 \times 10^9 \text{ km}$

$$e = 0.0543$$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543 \cos \theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9 \text{ km}$

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9 \text{ km}$

60. $a = 36.0 \times 10^6 \text{ mi}, e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{34.472 \times 10^6}{1 - 0.206 \cos \theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6 \text{ mi}$

Aphelion distance: $a(1 + e) = 43.416 \times 10^6 \text{ mi}$

62. $r = a \sin \theta + b \cos \theta$

$$r^2 = ar \sin \theta + br \cos \theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

Review Exercises for Chapter 9

2. Matches (b) - hyperbola

4. Matches (c) - hyperbola

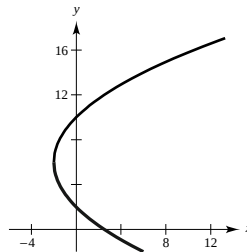
6. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



8. $4x^2 + y^2 - 16x + 15 = 0$

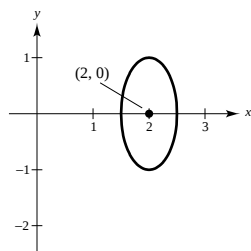
$$4(x^2 - 4x + 4) + y^2 = -15 + 16$$

$$\frac{(x - 2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: (2, 0)

Vertices: (2, ± 1)



10. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$$

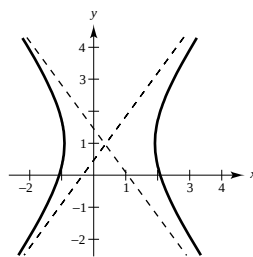
$$\frac{[x - (1/2)]^2}{2} - \frac{(y - 1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$

Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$



12. Vertex: (4, 2)

Focus: (4, 0)

Parabola opens downward

$$p = -2$$

$$(x - 4)^2 = 4(-2)(y - 2)$$

$$x^2 - 8x + 8y = 0$$

14. Center: (0, 0)

Solution points: (1, 2), (2, 0)

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, \quad 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \quad \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

18. $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20 \int_0^{\pi/2} \sqrt{1 - \frac{21}{25} \sin^2 \theta} d\theta \approx 23.01.$$

16. Foci: (0, ± 8)

Asymptotes: $y = \pm 4x$

Center: (0, 0)

Vertical transverse axis

$$c = 8$$

$$y = \frac{a}{b}x = 4x \text{ asymptote} \rightarrow a = 4b$$

$$b^2 = c^2 - a^2 = 64 - (4b)^2 \Rightarrow 17b^2 = 64$$

$$\Rightarrow b^2 = \frac{64}{17} \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

20. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: (0, 50)

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10,000}}$

$S = 2\pi \int_0^{100} x \sqrt{1 + \frac{x^2}{10,000}} dx \approx 38,294.49$

22. (a) $A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx = \frac{4b}{a} \left(\frac{1}{2} \right) \left[x \sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right) \right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi \int_0^b \frac{a^2}{b^2} (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{1}{3} y^3 \right]_0^b = \frac{4}{3} \pi a^2 b$

$$S = 4\pi \int_0^b \frac{a}{b} \sqrt{b^2 - y^2} \left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b \sqrt{b^2 - y^2}} \right) dy$$

$$= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + c^2 y^2} dy = \frac{2\pi a}{b^2 c} \left[cy \sqrt{b^4 + c^2 y^2} + b^4 \ln|cy + \sqrt{b^4 + c^2 y^2}| \right]_0^b$$

$$= \frac{2\pi a}{b^2 c} [b^2 c \sqrt{b^2 + c^2} + b^4 \ln|cb + b \sqrt{b^2 + c^2}| - b^4 \ln(b^2)]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c} \ln\left(\frac{c+a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right) \ln\left(\frac{1+e}{1-e}\right)$$

(c) **Disk:** $V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} \pi ab^2$

$$S = 2(2\pi) \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a \sqrt{a^2 - x^2}} \right) dx$$

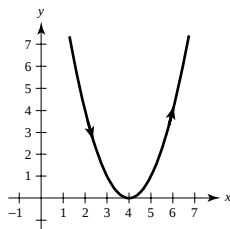
$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - c^2 x^2} dx = \frac{2\pi b}{a^2 c} \left[cx \sqrt{a^4 - c^2 x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right) \right]_0^a$$

$$= \frac{a\pi b}{a^2 c} \left[a^2 c \sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right) \right] = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) \arcsin(e)$$

24. $x = t + 4, y = t^2$

$t = x - 4 \Rightarrow y = (x - 4)^2$

Parabola

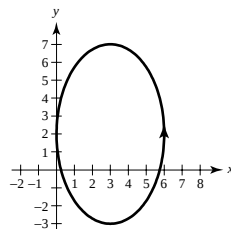


26. $x = 3 + 3 \cos \theta, y = 2 + 5 \sin \theta$

$\left(\frac{x-3}{3}\right)^2 + \left(\frac{y-2}{5}\right)^2 = 1$

$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

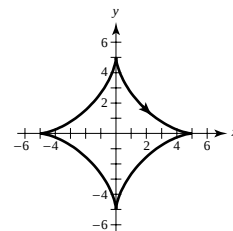
Ellipse



28. $x = 5 \sin^3 \theta, y = 5 \cos^3 \theta$

$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$

$x^{2/3} + y^{2/3} = 5^{2/3}$



30. $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$$

32. $a = 4, c = 5, b^2 = c^2 - a^2 = 9, \frac{y^2}{16} - \frac{x^2}{9} = 1$

Let $\frac{y^2}{16} = \sec^2 \theta$ and $\frac{x^2}{9} = \tan^2 \theta$.

Then $x = 3 \tan \theta$ and $y = 4 \sec \theta$.

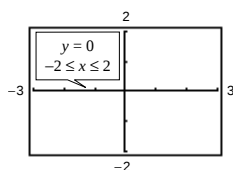
34. $x = (a - b) \cos t + b \cos \left(\frac{a - b}{b} t \right)$

$$y = (a - b) \sin t - b \sin \left(\frac{a - b}{b} t \right)$$

(a) $a = 2, b = 1$

$$x = \cos t + \cos t = 2 \cos t$$

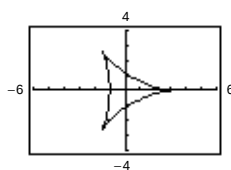
$$y = \sin t - \sin t = 0$$



(b) $a = 3, b = 1$

$$x = 2 \cos t + \cos 2t$$

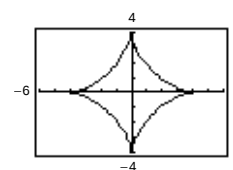
$$y = 2 \sin t - \sin 2t$$



(c) $a = 4, b = 1$

$$x = 3 \cos t + \cos 3t$$

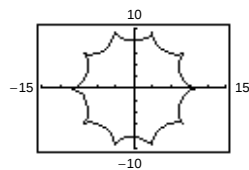
$$y = 3 \sin t - \sin 3t$$



(d) $a = 10, b = 1$

$$x = 9 \cos t + \cos 9t$$

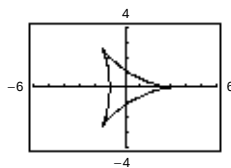
$$y = 9 \sin t - \sin 9t$$



(e) $a = 3, b = 2$

$$x = \cos t + 2 \cos \frac{t}{2}$$

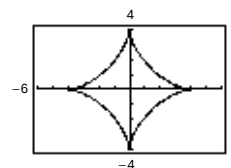
$$y = \sin t - 2 \sin \frac{t}{2}$$



(f) $a = 4, b = 3$

$$x = \cos t + 3 \cos \frac{t}{3}$$

$$y = \sin t - 3 \sin \frac{t}{3}$$

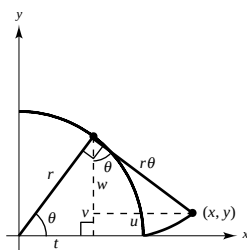


36. $x = t + u = r \cos \theta + r \theta \sin \theta$

$$= r(\cos \theta + \theta \sin \theta)$$

$$y = v - w = r \sin \theta - r \theta \cos \theta$$

$$= r(\sin \theta - \theta \cos \theta)$$



38. $x = t + 4$

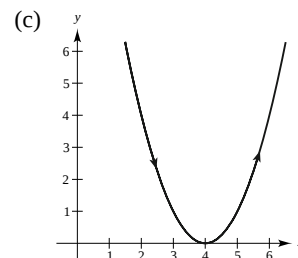
$$y = t^2$$

(a) $\frac{dy}{dx} = \frac{2t}{1} = 2t = 0$ when $t = 0$.

 Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$$y = (x - 4)^2$$



40. $x = \frac{1}{t}$

$y = t^2$

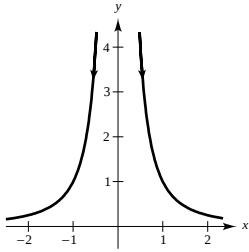
(a) $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents ($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{1}{x^2}$

(c)



44. $x = 6 \cos \theta$

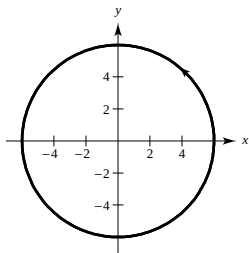
$y = 6 \sin \theta$

(a) $\frac{dy}{dx} = \frac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

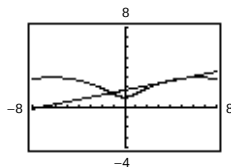
(c)



48. $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)



(b) At $\theta = \frac{\pi}{6}, \frac{dx}{d\theta} \approx 1.134, \left(2 - \frac{\sqrt{3}}{2}\right),$

$\frac{dy}{dt} = 0.5,$ and $\frac{dy}{dx} \approx 0.441$

42. $x = 2t - 1$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

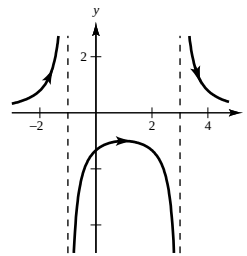
$= \frac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \frac{x + 1}{2}$

$y = \frac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \frac{4}{(x - 3)(x + 1)}$

(c)



46. $x = e^t$

$y = e^{-t}$

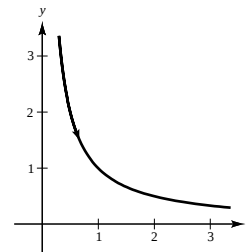
(a) $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$

(c)



50. $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\frac{dx}{d\theta} = -6 \sin \theta$

$\frac{dy}{d\theta} = 6 \cos \theta$

$s = \int_0^\pi \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta} d\theta = \left[6\theta\right]_0^\pi = 6\pi$

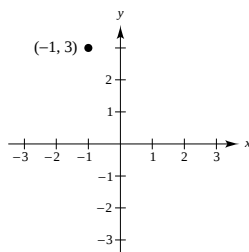
(one-half circumference of circle)

52. $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89 \text{ (108.43}^\circ\text{)}$$

$$(r, \theta) = (\sqrt{10}, 1.89), (-\sqrt{10}, 5.03)$$



54. $r = 10$

$$r^2 = 100$$

$$x^2 + y^2 = 100$$

56.
$$r = \frac{1}{2 - \cos \theta}$$

$$2r - r \cos \theta = 1$$

$$2(\pm \sqrt{x^2 + y^2}) - x = 1$$

$$4(x^2 + y^2) = (x + 1)^2$$

$$3x^2 + 4y^2 - 2x - 1 = 0$$

58.
$$r = 4 \sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$

$$= \frac{4}{(1/2) \cos \theta + (\sqrt{3}/2) \sin \theta}$$

$$r(\cos \theta + \sqrt{3} \sin \theta) = 8$$

$$x + \sqrt{3}y = 8$$

60.
$$\theta = \frac{3\pi}{4}$$

$$\tan \theta = -1$$

$$\frac{y}{x} = -1$$

$$y = -x$$

62. $x^2 + y^2 - 4x = 0$

$$r^2 - 4r \cos \theta = 0$$

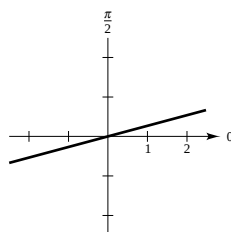
$$r = 4 \cos \theta$$

64.
$$(x^2 + y^2) \left(\arctan \frac{y}{x} \right)^2 = a^2$$

$$r^2 \theta^2 = a^2$$

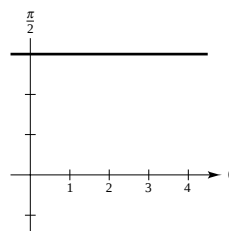
66. $\theta = \frac{\pi}{12}$

Line



68. $r = 3 \csc \theta, r \sin \theta = 3, y = 3$

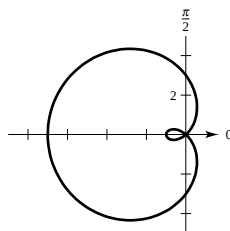
Horizontal line



70. $r = 3 - 4 \cos \theta$

Limaçon

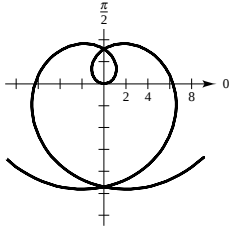
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

72. $r = 2\theta$

Spiral

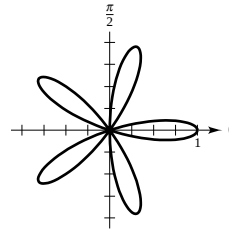
Symmetric to $\theta = \pi/2$ 

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{5}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

74. $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), (-1, \frac{\pi}{5}), (1, \frac{2\pi}{5}), (-1, \frac{3\pi}{5}), (1, \frac{4\pi}{5})$ Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ 

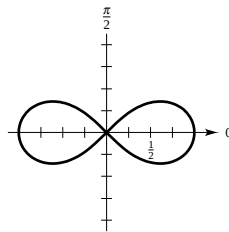
76. $r^2 = \cos(2\theta)$

Lemniscate

Symmetric to the polar axis

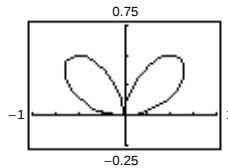
Relative extrema: $(\pm 1, 0)$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm \frac{\sqrt{2}}{2}$	0



78. $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to $\theta = \pi/2$ 

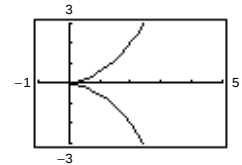
80. $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

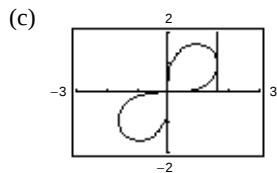
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



82. $r^2 = 4 \sin(2\theta)$

(a) $2r \left(\frac{dr}{d\theta} \right) = 8 \cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4 \cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

$$(b) \frac{dy}{dx} = \frac{r \cos \theta + [(4 \cos 2\theta \sin \theta)/r]}{-r \sin \theta + [(4 \cos 2\theta \cos \theta)/r]}$$

$$= \frac{\cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta}{\cos(2\theta) \cos \theta - \sin(2\theta) \sin \theta}$$

Horizontal tangents:

$$\frac{dy}{dx} = 0 \text{ when } \cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta = 0,$$

$$\tan \theta = -\tan(2\theta), \theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{3} \right)$$

Vertical tangents when $\cos 2\theta \cos \theta - \sin 2\theta \sin \theta = 0$:

$$\tan 2\theta \tan \theta = 1, \theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{6} \right)$$

84. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

86. $r = a \sin \theta, r = a \cos \theta$

The points of intersection are $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$. For $r = a \sin \theta$,

$$m_1 = \frac{dy}{dx} = \frac{a \cos \theta \sin \theta + a \sin \theta \cos \theta}{a \cos^2 \theta - a \sin^2 \theta} = \frac{2 \sin \theta \cos \theta}{\cos 2\theta}.$$

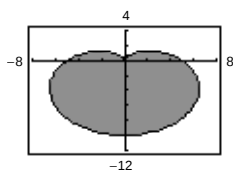
At $(a/\sqrt{2}, \pi/4)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a \cos \theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a \sin^2 \theta + a \cos^2 \theta}{-a \sin \theta \cos \theta - a \cos \theta \sin \theta} = \frac{\cos 2\theta}{-2 \sin \theta \cos \theta}.$$

At $(a/\sqrt{2}, \pi/4)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$.

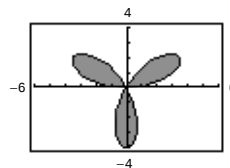
88. $r = 5(1 - \sin \theta)$

$$A = 2 \left[\frac{1}{2} \int_{\pi/2}^{3\pi/2} [5(1 - \sin \theta)]^2 d\theta \right] \approx 117.81 \left(75 \frac{\pi}{2} \right)$$



90. $r = 4 \sin 3\theta$

$$A = 3 \left[\frac{1}{2} \int_0^{\pi/3} (4 \sin 3\theta)^2 d\theta \right] \approx 12.57 (4\pi)$$



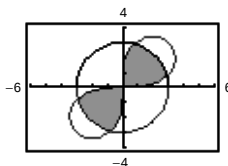
92. $r = 3, r^2 = 18 \sin 2\theta$

$$9 = r^2 = 18 \sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

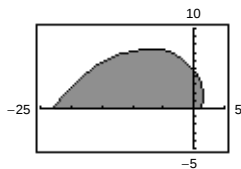
$$\theta = \frac{\pi}{12}$$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/12} 18 \sin 2\theta d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} 9 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} 18 \sin 2\theta d\theta \right] \approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$



94. $r = e^\theta, 0 \leq \theta \leq \pi$

$$A = \frac{1}{2} \int_0^\pi (e^\theta)^2 d\theta \approx 133.62$$



96. $r = a \cos 2\theta, \frac{dr}{d\theta} = -2a \sin 2\theta$

$$\begin{aligned} s &= 8 \int_0^{\pi/4} \sqrt{a^2 \cos^2 2\theta + 4a^2 \sin^2 2\theta} d\theta \\ &= 8a \int_0^{\pi/4} \sqrt{1 + 3 \sin^2 2\theta} d\theta \quad (\text{Simpson's Rule: } n = 4) \\ &\approx \frac{\pi a}{6} [1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2] \\ &\approx 9.69a \end{aligned}$$

50. $a = 4, c = 5, b = 3, e = \frac{5}{4}$

$$r^2 = \frac{-9}{1 - (25/16) \cos^2 \theta}$$

52. $a = 2, b = 1, c = \sqrt{3}, e = \frac{\sqrt{3}}{2}$

$$r^2 = \frac{1}{1 - (3/4) \cos^2 \theta}$$

54. $A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{2}{3 - 2 \sin \theta} \right)^2 d\theta \right] = 4 \int_{-\pi/2}^{\pi/2} \frac{1}{(3 - 2 \sin \theta)^2} d\theta \approx 3.37$

56. (a) $r = \frac{ed}{1 - e \cos \theta}$

When $\theta = 0, r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ed}{1 - e}$$

$$a(1 + e)(1 - e) = ed$$

$$a(1 - e^2) = ed.$$

Thus, $r = \frac{(1 - e^2)a}{1 - e \cos \theta}$.

(b) The perihelion distance is $a - c = a - ea = a(1 - e)$.

When $\theta = \pi, r = \frac{(1 - e^2)a}{1 + e} = a(1 - e)$.

The aphelion distance is $a + c = a + ea = a(1 + e)$.

When $\theta = 0, r = \frac{(1 - e^2)a}{1 - e} = a(1 + e)$.

58. $a = 1.427 \times 10^9 \text{ km}$

$$e = 0.0543$$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{1.422792505 \times 10^9}{1 - 0.0543 \cos \theta}$$

Perihelion distance: $a(1 - e) = 1.3495139 \times 10^9 \text{ km}$

Aphelion distance: $a(1 + e) = 1.5044861 \times 10^9 \text{ km}$

60. $a = 36.0 \times 10^6 \text{ mi}, e = 0.206$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{34.472 \times 10^6}{1 - 0.206 \cos \theta}$$

Perihelion distance: $a(1 - e) = 28.582 \times 10^6 \text{ mi}$

Aphelion distance: $a(1 + e) = 43.416 \times 10^6 \text{ mi}$

62. $r = a \sin \theta + b \cos \theta$

$$r^2 = ar \sin \theta + br \cos \theta$$

$$x^2 + y^2 = ay + bx$$

$x^2 + y^2 - bx - ay = 0$ represents a circle.

Review Exercises for Chapter 9

2. Matches (b) - hyperbola

4. Matches (c) - hyperbola

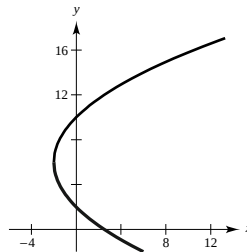
6. $y^2 - 12y - 8x + 20 = 0$

$$y^2 - 12y + 36 = 8x - 20 + 36$$

$$(y - 6)^2 = 4(2)(x + 2)$$

Parabola

Vertex: $(-2, 6)$



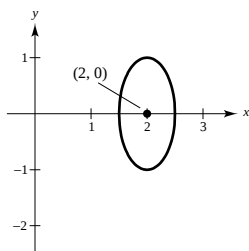
8. $4x^2 + y^2 - 16x + 15 = 0$

$$4(x^2 - 4x + 4) + y^2 = -15 + 16$$

$$\frac{(x - 2)^2}{1/4} + \frac{y^2}{1} = 1$$

Ellipse

Center: (2, 0)

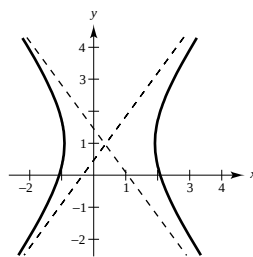
Vertices: (2, ± 1)

10. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

$$4\left(x^2 - x + \frac{1}{4}\right) - 4(y^2 - 2y + 1) = 11 + 1 - 4$$

$$\frac{[x - (1/2)]^2}{2} - \frac{(y - 1)^2}{2} = 1$$

Hyperbola

Center: $\left(\frac{1}{2}, 1\right)$ Vertices: $\left(\frac{1}{2} \pm \sqrt{2}, 1\right)$ Asymptotes: $y = 1 \pm \left(x - \frac{1}{2}\right)$ 

12. Vertex: (4, 2)

Focus: (4, 0)

Parabola opens downward

$$p = -2$$

$$(x - 4)^2 = 4(-2)(y - 2)$$

$$x^2 - 8x + 8y = 0$$

14. Center: (0, 0)

Solution points: (1, 2), (2, 0)

Substituting the values of the coordinates of the given points into

$$\left(\frac{x^2}{b^2}\right) + \left(\frac{y^2}{a^2}\right) = 1,$$

we obtain the system

$$\left(\frac{1}{b^2}\right) + \left(\frac{4}{a^2}\right) = 1, 4/b^2 = 1.$$

Solving the system, we have

$$a^2 = \frac{16}{3} \text{ and } b^2 = 4, \left(\frac{x^2}{4}\right) + \left(\frac{3y^2}{16}\right) = 1.$$

18. $\frac{x^2}{4} + \frac{y^2}{25} = 1, a = 5, b = 2, c = \sqrt{21}, e = \frac{\sqrt{21}}{5}$

By Example 5 of Section 9.1,

$$C = 20 \int_0^{\pi/2} \sqrt{1 - \frac{21}{25} \sin^2 \theta} d\theta \approx 23.01.$$

16. Foci: (0, ± 8)

Asymptotes: $y = \pm 4x$

Center: (0, 0)

Vertical transverse axis

$$c = 8$$

$$y = \frac{a}{b}x = 4x \text{ asymptote} \rightarrow a = 4b$$

$$b^2 = c^2 - a^2 = 64 - (4b)^2 \Rightarrow 17b^2 = 64$$

$$\Rightarrow b^2 = \frac{64}{17} \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

20. $y = \frac{1}{200}x^2$

(a) $x^2 = 200y$

$x^2 = 4(50)y$

Focus: (0, 50)

(b) $y = \frac{1}{200}x^2$

$y' = \frac{1}{100}x$

$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{10,000}}$

$S = 2\pi \int_0^{100} x \sqrt{1 + \frac{x^2}{10,000}} dx \approx 38,294.49$

22. (a) $A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx = \frac{4b}{a} \left(\frac{1}{2} \right) \left[x \sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right) \right]_0^a = \pi ab$

(b) **Disk:** $V = 2\pi \int_0^b \frac{a^2}{b^2} (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{1}{3} y^3 \right]_0^b = \frac{4}{3} \pi a^2 b$

$$S = 4\pi \int_0^b \frac{a}{b} \sqrt{b^2 - y^2} \left(\frac{\sqrt{b^4 + (a^2 - b^2)y^2}}{b \sqrt{b^2 - y^2}} \right) dy$$

$$= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + c^2 y^2} dy = \frac{2\pi a}{b^2 c} \left[cy \sqrt{b^4 + c^2 y^2} + b^4 \ln|cy + \sqrt{b^4 + c^2 y^2}| \right]_0^b$$

$$= \frac{2\pi a}{b^2 c} [b^2 c \sqrt{b^2 + c^2} + b^4 \ln|cb + b \sqrt{b^2 + c^2}| - b^4 \ln(b^2)]$$

$$= 2\pi a^2 + \frac{\pi ab^2}{c} \ln\left(\frac{c+a}{e}\right)^2 = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right) \ln\left(\frac{1+e}{1-e}\right)$$

(c) **Disk:** $V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} \pi ab^2$

$$S = 2(2\pi) \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \left(\frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a \sqrt{a^2 - x^2}} \right) dx$$

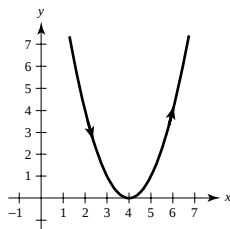
$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - c^2 x^2} dx = \frac{2\pi b}{a^2 c} \left[cx \sqrt{a^4 - c^2 x^2} + a^4 \arcsin\left(\frac{cx}{a^2}\right) \right]_0^a$$

$$= \frac{a\pi b}{a^2 c} \left[a^2 c \sqrt{a^2 - c^2} + a^4 \arcsin\left(\frac{c}{a}\right) \right] = 2\pi b^2 + 2\pi \left(\frac{ab}{e}\right) \arcsin(e)$$

24. $x = t + 4, y = t^2$

$t = x - 4 \Rightarrow y = (x - 4)^2$

Parabola

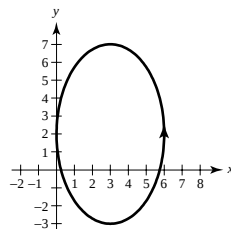


26. $x = 3 + 3 \cos \theta, y = 2 + 5 \sin \theta$

$\left(\frac{x-3}{3}\right)^2 + \left(\frac{y-2}{5}\right)^2 = 1$

$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

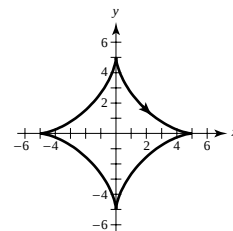
Ellipse



28. $x = 5 \sin^3 \theta, y = 5 \cos^3 \theta$

$\left(\frac{x}{5}\right)^{2/3} + \left(\frac{y}{5}\right)^{2/3} = 1$

$x^{2/3} + y^{2/3} = 5^{2/3}$



30. $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 5)^2 + (y - 3)^2 = 2^2 = 4$$

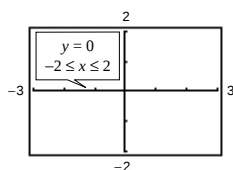
34. $x = (a - b) \cos t + b \cos \left(\frac{a - b}{b} t \right)$

$$y = (a - b) \sin t - b \sin \left(\frac{a - b}{b} t \right)$$

(a) $a = 2, b = 1$

$$x = \cos t + \cos t = 2 \cos t$$

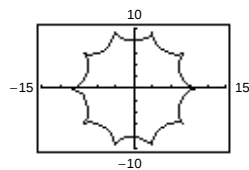
$$y = \sin t - \sin t = 0$$



(d) $a = 10, b = 1$

$$x = 9 \cos t + \cos 9t$$

$$y = 9 \sin t - \sin 9t$$

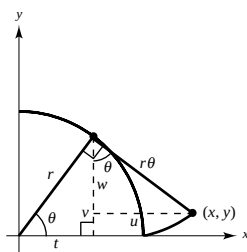


36. $x = t + u = r \cos \theta + r \theta \sin \theta$

$$= r(\cos \theta + \theta \sin \theta)$$

$$y = v - w = r \sin \theta - r \theta \cos \theta$$

$$= r(\sin \theta - \theta \cos \theta)$$



32. $a = 4, c = 5, b^2 = c^2 - a^2 = 9, \frac{y^2}{16} - \frac{x^2}{9} = 1$

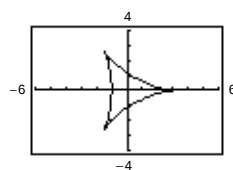
Let $\frac{y^2}{16} = \sec^2 \theta$ and $\frac{x^2}{9} = \tan^2 \theta$.

Then $x = 3 \tan \theta$ and $y = 4 \sec \theta$.

(b) $a = 3, b = 1$

$$x = 2 \cos t + \cos 2t$$

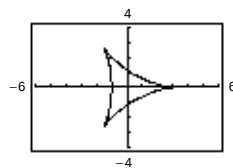
$$y = 2 \sin t - \sin 2t$$



(e) $a = 3, b = 2$

$$x = \cos t + 2 \cos \frac{t}{2}$$

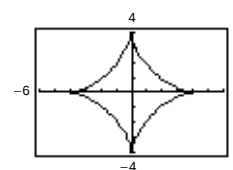
$$y = \sin t - 2 \sin \frac{t}{2}$$



(c) $a = 4, b = 1$

$$x = 3 \cos t + \cos 3t$$

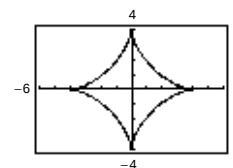
$$y = 3 \sin t - \sin 3t$$



(f) $a = 4, b = 3$

$$x = \cos t + 3 \cos \frac{t}{3}$$

$$y = \sin t - 3 \sin \frac{t}{3}$$



38. $x = t + 4$

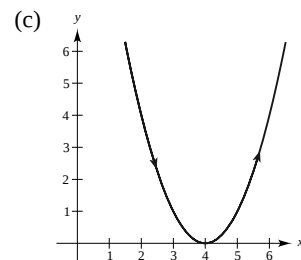
$$y = t^2$$

(a) $\frac{dy}{dx} = \frac{2t}{1} = 2t = 0$ when $t = 0$.

 Point of horizontal tangency: $(4, 0)$

(b) $t = x - 4$

$$y = (x - 4)^2$$



40. $x = \frac{1}{t}$

$y = t^2$

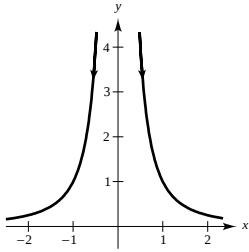
(a) $\frac{dy}{dx} = \frac{2t}{-1/t^2} = -2t^3$

No horizontal tangents ($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{1}{x^2}$

(c)



44. $x = 6 \cos \theta$

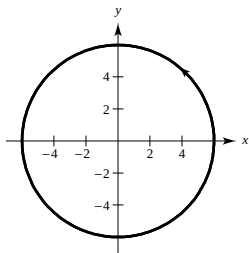
$y = 6 \sin \theta$

(a) $\frac{dy}{dx} = \frac{6 \cos \theta}{-6 \sin \theta} = -\cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(0, 6), (0, -6)$

(b) $\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$

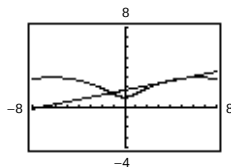
(c)



48. $x = 2\theta - \sin \theta$

$y = 2 - \cos \theta$

(a), (c)



(b) At $\theta = \frac{\pi}{6}, \frac{dx}{d\theta} \approx 1.134, \left(2 - \frac{\sqrt{3}}{2}\right),$

$\frac{dy}{dt} = 0.5,$ and $\frac{dy}{dx} \approx 0.441$

42. $x = 2t - 1$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{-(t^2 - 2t)^{-2}(2t - 2)}{2}$

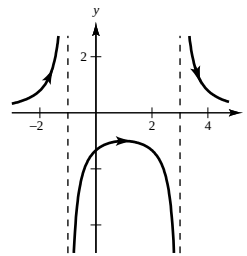
$= \frac{1 - t}{t^2(t - 2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(1, -1)$

(b) $t = \frac{x + 1}{2}$

$y = \frac{1}{[(x + 1)/2]^2 - 2[(x + 1)/2]} = \frac{4}{(x - 3)(x + 1)}$

(c)



46. $x = e^t$

$y = e^{-t}$

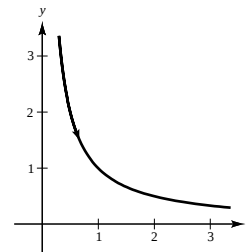
(a) $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -\frac{1}{e^{2t}} = -\frac{1}{x^2}$

No horizontal tangents

(b) $t = \ln x$

$y = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}, x > 0$

(c)



50. $x = 6 \cos \theta$

$y = 6 \sin \theta$

$\frac{dx}{d\theta} = -6 \sin \theta$

$\frac{dy}{d\theta} = 6 \cos \theta$

$s = \int_0^\pi \sqrt{36 \sin^2 \theta + 36 \cos^2 \theta} d\theta = \left[6\theta\right]_0^\pi = 6\pi$

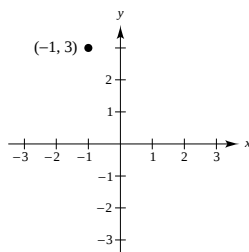
(one-half circumference of circle)

52. $(x, y) = (-1, 3)$

$$r = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\theta = \arctan(-3) \approx 1.89 \text{ (108.43}^\circ\text{)}$$

$$(r, \theta) = (\sqrt{10}, 1.89), (-\sqrt{10}, 5.03)$$



54. $r = 10$

$$r^2 = 100$$

$$x^2 + y^2 = 100$$

56.
$$r = \frac{1}{2 - \cos \theta}$$

$$2r - r \cos \theta = 1$$

$$2(\pm \sqrt{x^2 + y^2}) - x = 1$$

$$4(x^2 + y^2) = (x + 1)^2$$

$$3x^2 + 4y^2 - 2x - 1 = 0$$

58.
$$r = 4 \sec\left(\theta - \frac{\pi}{3}\right) = \frac{4}{\cos[\theta - (\pi/3)]}$$

$$= \frac{4}{(1/2) \cos \theta + (\sqrt{3}/2) \sin \theta}$$

$$r(\cos \theta + \sqrt{3} \sin \theta) = 8$$

$$x + \sqrt{3}y = 8$$

60. $\theta = \frac{3\pi}{4}$

$$\tan \theta = -1$$

$$\frac{y}{x} = -1$$

$$y = -x$$

62. $x^2 + y^2 - 4x = 0$

$$r^2 - 4r \cos \theta = 0$$

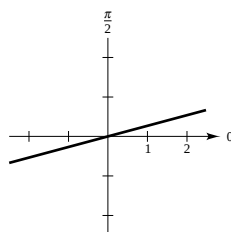
$$r = 4 \cos \theta$$

64. $(x^2 + y^2)\left(\arctan \frac{y}{x}\right)^2 = a^2$

$$r^2 \theta^2 = a^2$$

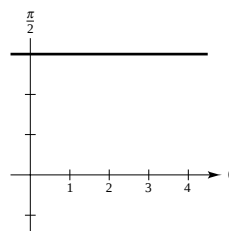
66. $\theta = \frac{\pi}{12}$

Line



68. $r = 3 \csc \theta, r \sin \theta = 3, y = 3$

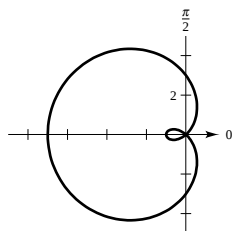
Horizontal line



70. $r = 3 - 4 \cos \theta$

Limaçon

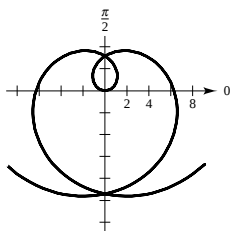
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-1	1	3	5	7

72. $r = 2\theta$

Spiral

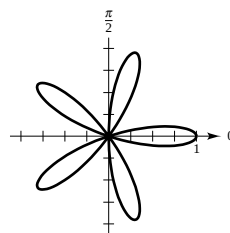
Symmetric to $\theta = \pi/2$ 

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{5}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

74. $r = \cos(5\theta)$

Rose curve with five petals

Symmetric to polar axis

Relative extrema: $(1, 0), (-1, \frac{\pi}{5}), (1, \frac{2\pi}{5}), (-1, \frac{3\pi}{5}), (1, \frac{4\pi}{5})$ Tangents at the pole: $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$ 

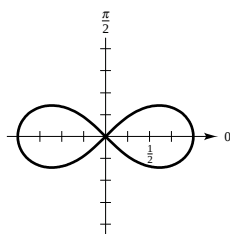
76. $r^2 = \cos(2\theta)$

Lemniscate

Symmetric to the polar axis

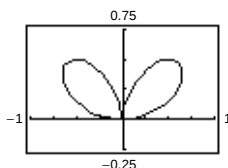
Relative extrema: $(\pm 1, 0)$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 1	$\pm \frac{\sqrt{2}}{2}$	0



78. $r = 2 \sin \theta \cos^2 \theta$

Bifolium

Symmetric to $\theta = \pi/2$ 

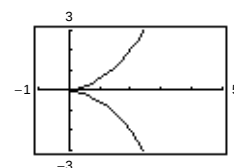
80. $r = 4(\sec \theta - \cos \theta)$

Semicubical parabola

Symmetric to the polar axis

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi^-}{2}$$

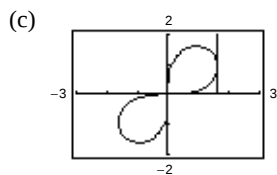
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{-\pi^+}{2}$$



82. $r^2 = 4 \sin(2\theta)$

(a) $2r \left(\frac{dr}{d\theta} \right) = 8 \cos(2\theta)$

$$\frac{dr}{d\theta} = \frac{4 \cos(2\theta)}{r}$$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

$$(b) \frac{dy}{dx} = \frac{r \cos \theta + [(4 \cos 2\theta \sin \theta)/r]}{-r \sin \theta + [(4 \cos 2\theta \cos \theta)/r]}$$

$$= \frac{\cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta}{\cos(2\theta) \cos \theta - \sin(2\theta) \sin \theta}$$

Horizontal tangents:

$$\frac{dy}{dx} = 0 \text{ when } \cos(2\theta) \sin \theta + \sin(2\theta) \cos \theta = 0,$$

$$\tan \theta = -\tan(2\theta), \theta = 0, \frac{\pi}{3}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{3} \right)$$

Vertical tangents when $\cos 2\theta \cos \theta - \sin 2\theta \sin \theta = 0$:

$$\tan 2\theta \tan \theta = 1, \theta = 0, \frac{\pi}{6}, (0, 0), \left(\pm \sqrt{2\sqrt{3}}, \frac{\pi}{6} \right)$$

84. False. There are an infinite number of polar coordinate representations of a point. For example, the point $(x, y) = (1, 0)$ has polar representations $(r, \theta) = (1, 0), (1, 2\pi), (-1, \pi)$, etc.

86. $r = a \sin \theta, r = a \cos \theta$

The points of intersection are $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$. For $r = a \sin \theta$,

$$m_1 = \frac{dy}{dx} = \frac{a \cos \theta \sin \theta + a \sin \theta \cos \theta}{a \cos^2 \theta - a \sin^2 \theta} = \frac{2 \sin \theta \cos \theta}{\cos 2\theta}.$$

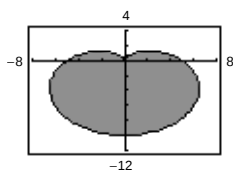
At $(a/\sqrt{2}, \pi/4)$, m_1 is undefined and at $(0, 0)$, $m_1 = 0$. For $r = a \cos \theta$,

$$m_2 = \frac{dy}{dx} = \frac{-a \sin^2 \theta + a \cos^2 \theta}{-a \sin \theta \cos \theta - a \cos \theta \sin \theta} = \frac{\cos 2\theta}{-2 \sin \theta \cos \theta}.$$

At $(a/\sqrt{2}, \pi/4)$, $m_2 = 0$ and at $(0, 0)$, m_2 is undefined. Therefore, the graphs are orthogonal at $(a/\sqrt{2}, \pi/4)$ and $(0, 0)$.

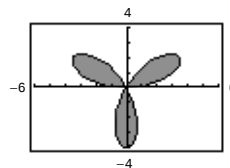
88. $r = 5(1 - \sin \theta)$

$$A = 2 \left[\frac{1}{2} \int_{\pi/2}^{3\pi/2} [5(1 - \sin \theta)]^2 d\theta \right] \approx 117.81 \left(75 \frac{\pi}{2} \right)$$



90. $r = 4 \sin 3\theta$

$$A = 3 \left[\frac{1}{2} \int_0^{\pi/3} (4 \sin 3\theta)^2 d\theta \right] \approx 12.57 (4\pi)$$



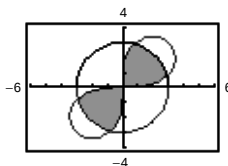
92. $r = 3, r^2 = 18 \sin 2\theta$

$$9 = r^2 = 18 \sin 2\theta$$

$$\sin 2\theta = \frac{1}{2}$$

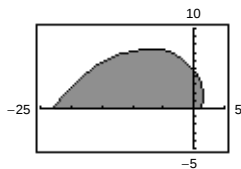
$$\theta = \frac{\pi}{12}$$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/12} 18 \sin 2\theta d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} 9 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} 18 \sin 2\theta d\theta \right] \approx 1.2058 + 9.4248 + 1.2058 \approx 11.84$$



94. $r = e^\theta, 0 \leq \theta \leq \pi$

$$A = \frac{1}{2} \int_0^\pi (e^\theta)^2 d\theta \approx 133.62$$

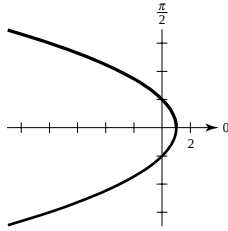


96. $r = a \cos 2\theta, \frac{dr}{d\theta} = -2a \sin 2\theta$

$$\begin{aligned} s &= 8 \int_0^{\pi/4} \sqrt{a^2 \cos^2 2\theta + 4a^2 \sin^2 2\theta} d\theta \\ &= 8a \int_0^{\pi/4} \sqrt{1 + 3 \sin^2 2\theta} d\theta \quad (\text{Simpson's Rule: } n = 4) \\ &\approx \frac{\pi a}{6} [1 + 4(1.1997) + 2(1.5811) + 4(1.8870) + 2] \\ &\approx 9.69a \end{aligned}$$

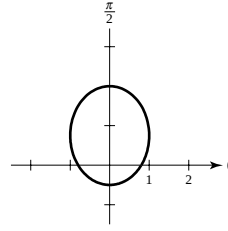
$$98. r = \frac{2}{1 + \cos \theta}, e = 1$$

Parabola



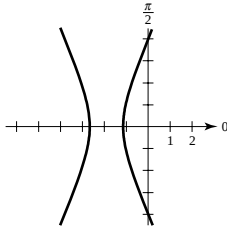
$$100. r = \frac{4}{5 - 3 \sin \theta} = \frac{4/5}{1 - (3/5) \sin \theta}, e = \frac{3}{5}$$

Ellipse



$$102. r = \frac{8}{2 - 5 \cos \theta} = \frac{4}{1 - (5/2) \cos \theta}, e = \frac{5}{2}$$

Hyperbola



104. Line

Slope: $\sqrt{3}$

Solution point: $(0, 0)$

$$y = \sqrt{3}x, r \sin \theta = \sqrt{3}r \cos \theta,$$

$$\tan \theta = \sqrt{3}, \theta = \frac{\pi}{3}$$

106. Parabola

Vertex: $\left(2, \frac{\pi}{2}\right)$

Focus: $(0, 0)$

$e = 1, d = 4$

$$r = \frac{4}{1 + \sin \theta}$$

108. Hyperbola

Vertices: $(1, 0), (7, 0)$

Focus: $(0, 0)$

$$a = 3, c = 4, e = \frac{4}{3}, d = \frac{7}{4}$$

$$r = \frac{\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)}{1 + \left(\frac{4}{3}\right) \cos \theta} = \frac{7}{3 + 4 \cos \theta}$$

Problem Solving for Chapter 9

2. Assume $p > 0$.

Let $y = mx + p$ be the equation of the focal chord.

First find x -coordinates of focal chord endpoints:

$$x^2 = 4py = 4p(mx + p)$$

$$x^2 - 4pmx - 4p^2 = 0$$

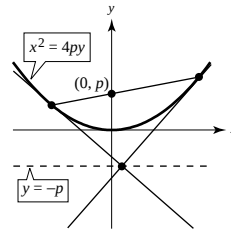
$$x = \frac{4pm \pm \sqrt{16p^2m^2 + 16p^2}}{2} = 2pm \pm 2p\sqrt{m^2 + 1}$$

$$x^2 = 4py, 2x = 4py' \Rightarrow y' = \frac{x}{2p}.$$

(a) The slopes of the tangent lines at the endpoints are perpendicular because

$$\frac{1}{2p}[2pm + 2p\sqrt{m^2 + 1}] \cdot \frac{1}{2p}[2pm - 2p\sqrt{m^2 + 1}] = \frac{1}{4p^2}[4p^2m^2 - 4p^2(m^2 + 1)] = \frac{1}{4p^2}[-4p^2] = -1$$

—CONTINUED—



2. —CONTINUED—

(b) Finally, we show that the tangent lines intersect at a point on the directrix $y = -p$.

$$\text{Let } b = 2pm + 2p\sqrt{m^2 + 1} \text{ and } c = 2pm - 2p\sqrt{m^2 + 1}.$$

$$b^2 = 8p^2m^2 + 4p^2 + 8p^2m\sqrt{m^2 + 1}$$

$$c^2 = 8p^2m^2 + 4p^2 - 8p^2m\sqrt{m^2 + 1}$$

$$\frac{b^2}{4p} = 2pm^2 + p + 2pm\sqrt{m^2 + 1}$$

$$\frac{c^2}{4p} = 2pm^2 + p - 2pm\sqrt{m^2 + 1}$$

$$\text{Tangent line at } x = b: y - \frac{b^2}{4p} = \frac{b}{2p}(x - b) \Rightarrow y = \frac{bx}{2p} - \frac{b^2}{4p}$$

$$\text{Tangent line at } x = c: y - \frac{c^2}{4p} = \frac{c}{2p}(x - c) \Rightarrow y = \frac{cx}{2p} - \frac{c^2}{4p}$$

$$\begin{aligned} \text{Intersection of tangent lines: } \quad \frac{bx}{2p} - \frac{b^2}{4p} &= \frac{cx}{2p} - \frac{c^2}{4p} \\ 2bx - b^2 &= 2cx - c^2 \\ 2x(b - c) &= b^2 - c^2 \\ 2x(4p\sqrt{m^2 + 1}) &= 16p^2m\sqrt{m^2 + 1} \\ x &= 2pm \end{aligned}$$

Finally, the corresponding y -value is $y = -p$, which shows that the intersection point lies on the directrix.

$$4. \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a^2 + b^2 = c^2, MF_2 - MF_1 = 2a$$

$$y' = \frac{b^2x}{a^2y}$$

$$\text{Tangent line at } M(x_0, y_0): y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{x_0x - x_0^2}{a^2}$$

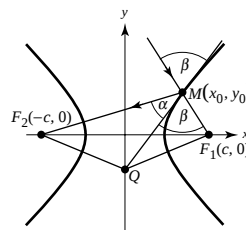
$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

$$\text{At } x = 0, y = -\frac{b^2}{y_0} \Rightarrow Q = \left(0, -\frac{b^2}{y_0}\right).$$

$$QF_2 = QF_1 = \sqrt{c^2 + \frac{b^4}{y_0^2}} = d$$

$$MQ = \sqrt{x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2} = f$$



By the Law of Cosines,

$$(F_2Q)^2 = (MF_2)^2 + (MQ)^2 - 2(MF_2)(MQ)\cos \alpha$$

$$d^2 = (MF_2)^2 + f^2 - 2f(MF_2)\cos \alpha$$

$$(F_1Q)^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos \beta$$

$$d^2 = (MF_1)^2 + f^2 - 2f(MF_1)\cos \beta.$$

$$\cos \alpha = \frac{(MF_2)^2 f^2 - d^2}{2f(MF_2)}, \cos \beta = \frac{(MF_1)^2 f^2 - d^2}{2f(MF_1)}$$

$$MF_2 = MF_1 + 2a. \text{ Let } z = MF_1.$$

$$\text{Slopes: } MF_1: \frac{y_0}{x_0 - c}; QF_1: \frac{-b^2}{y_0 c}; QF_2: \frac{b^2}{y_0 c}$$

—CONTINUED—

4. —CONTINUED—

To show $\alpha = \beta$, consider

$$\begin{aligned}
 & [(MF_2)^2 + f^2 - d^2][2f(MF_1)] = [(MF_1)^2 + f^2 - d^2][2f(MF_2)] \\
 \Leftrightarrow & [(z + 2a)^2 + f^2 - d^2][z] = [z^2 + f^2 - d^2][z + 2a] \\
 \Leftrightarrow & z^2 + 2az = f^2 - d^2 \\
 \Leftrightarrow & (x_0 - c)^2 + y_0^2 + 2az = \left(x_0^2 + \left(y_0 + \frac{b^2}{y_0}\right)^2\right) - \left(c^2 + \frac{b^4}{y_0^2}\right) \\
 \Leftrightarrow & az - x_0c + a^2 = 0 \\
 \Leftrightarrow & a\sqrt{(x_0 - c)^2 + y_0^2} = x_0c - a^2 \\
 \Leftrightarrow & x_0^2b^2 - a^2y_0^2 = a^2b^2 \\
 \Leftrightarrow & \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.
 \end{aligned}$$

Thus, $\alpha = \beta$ and the reflective property is verified.

$$6. (a) \quad y^2 = \frac{t^2(1 - t^2)^2}{(1 + t^2)^2}, \quad x^2 = \frac{(1 - t^2)^2}{(1 + t^2)^2}$$

$$\frac{1 - x}{1 + x} = \frac{1 - \left(\frac{1 - t^2}{1 + t^2}\right)}{1 + \left(\frac{1 - t^2}{1 + t^2}\right)} = \frac{2t^2}{2} = t^2$$

$$\text{Thus, } y^2 = x^2 \left(\frac{1 - x}{1 + x}\right).$$

$$(b) \quad r^2 \sin^2 \theta = r^2 \cos^2 \theta \left(\frac{1 - r \cos \theta}{1 + r \cos \theta}\right)$$

$$\sin^2 \theta (1 + r \cos \theta) = \cos^2 \theta (1 - r \cos \theta)$$

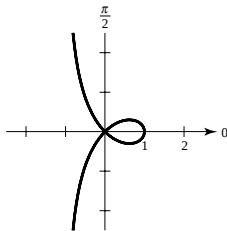
$$r \cos \theta \sin^2 \theta + \sin^2 \theta = \cos^2 \theta - r \cos^3 \theta$$

$$r \cos \theta (\sin^2 \theta + \cos^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$r \cos \theta = \cos 2\theta$$

$$r = \cos 2\theta \cdot \sec \theta$$

(c)



$$(d) \quad r(\theta) = 0 \text{ for } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

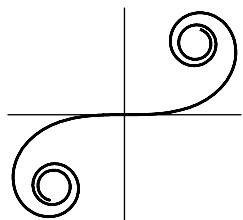
Thus, $y = x$ and $y = -x$ are tangent lines to curve at the origin.

$$(e) \quad y'(t) = \frac{(1 + t^2)(1 - 3t^2) - (t - t^3)(2t)}{(1 + t^2)^2} = \frac{1 - 4t^2 - t^4}{(1 + t^2)^2} = 0$$

$$\begin{aligned}
 t^4 + 4t^2 - 1 = 0 & \Rightarrow t^2 = -2 \pm \sqrt{5} \Rightarrow x = \frac{1 - (-2 \pm \sqrt{5})}{1 + (-2 \pm \sqrt{5})} = \frac{3 \mp \sqrt{5}}{-1 \pm \sqrt{5}} \\
 & = \frac{3 - \sqrt{5}}{-1 + \sqrt{5}} = \frac{\sqrt{5} - 1}{2}
 \end{aligned}$$

$$\left(\frac{\sqrt{5} - 1}{2}, \pm \frac{\sqrt{5} - 1}{2} \sqrt{-2 + \sqrt{5}}\right)$$

8. (a)



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$$(b) \quad (-x, -y) = \left(-\int_0^t \cos \frac{\pi u^2}{2} du, -\int_0^t \sin \frac{\pi u^2}{2} du \right) \text{ is on}$$

the curve whenever (x, y) is on the curve.

$$(c) \quad x'(t) = \cos \frac{\pi t^2}{2}, y'(t) = \sin \frac{\pi t^2}{2}, x'(t)^2 + y'(t)^2 = 1$$

$$\text{Thus, } s = \int_0^a dt = a.$$

$$\text{On } [-\pi, \pi], s = 2\pi.$$

12. Let (r, θ) be on the graph.

$$\sqrt{r^2 + 1 + 2r \cos \theta} \sqrt{r^2 + 1 - 2r \cos \theta} = 1$$

$$(r^2 + 1)^2 - 4r^2 \cos^2 \theta = 1$$

$$r^4 + 2r^2 + 1 - 4r^2 \cos^2 \theta = 1$$

$$r^2(r^2 - 4 \cos^2 \theta + 2) = 0$$

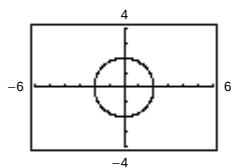
$$r^2 = 4 \cos^2 \theta - 2$$

$$r^2 = 2(2 \cos^2 \theta - 1)$$

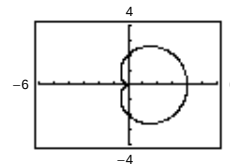
$$r^2 = 2 \cos 2\theta$$

14. (a) $r = 2$

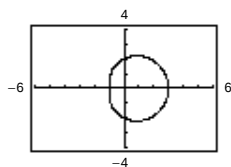
Circle radius 2


(c) $r = 2 + 2 \cos \theta$

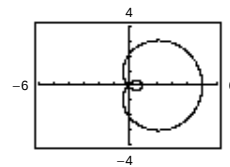
Cardioid


(b) $r = 2 + \cos \theta$

Convex limaçon


(d) $r = 2 + 3 \cos \theta$

Limaçon with inner loop



16. The curve is produced over the interval

$$0 \leq \theta \leq 9\pi.$$

$$10. \quad r = \frac{ab}{a \sin \theta + b \cos \theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$r(a \sin \theta + b \cos \theta) = ab$$

$$ay + bx = ab$$

$$\frac{y}{b} + \frac{x}{a} = 1$$

Line segment

$$\text{Area} = \frac{1}{2}ab$$

CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1	Conics and Calculus	177
Section 9.2	Plane Curves and Parametric Equations	188
Section 9.3	Parametric Equations and Calculus	192
Section 9.4	Polar Coordinates and Polar Graphs	198
Section 9.5	Area and Arc Length in Polar Coordinates	205
Section 9.6	Polar Equations of Conics and Kepler's Laws	210
	Review Exercises	214
	Problem Solving	222

CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1 Conics and Calculus

Solutions to Odd-Numbered Exercises

1. $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right

Matches graph (h).

5. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Center: $(0, 0)$

Ellipse

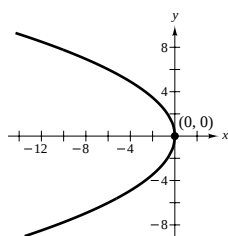
Matches (f)

9. $y^2 = -6x = 4\left(-\frac{3}{2}\right)x$

Vertex: $(0, 0)$

Focus: $\left(-\frac{3}{2}, 0\right)$

Directrix: $x = \frac{3}{2}$



3. $(x + 3)^2 = -2(y - 2)$

Vertex: $(-3, 2)$

$p = -\frac{1}{2} < 0$

Opens downward

Matches graph (e).

7. $\frac{y^2}{16} - \frac{x^2}{1} = 1$

Hyperbola

Center: $(0, 0)$

Vertical transverse axis.

Matches (c)

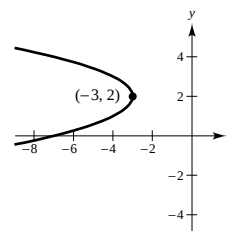
11. $(x + 3) + (y - 2)^2 = 0$

$(y - 2)^2 = 4\left(-\frac{1}{4}\right)(x + 3)$

Vertex: $(-3, 2)$

Focus: $(-3.25, 2)$

Directrix: $x = -2.75$



13. $y^2 - 4y - 4x = 0$

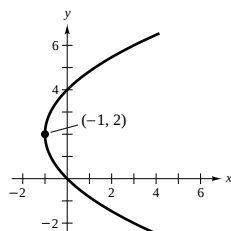
$y^2 - 4y + 4 = 4x + 4$

$(y - 2)^2 = 4(1)(x + 1)$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



15. $x^2 + 4x + 4y - 4 = 0$

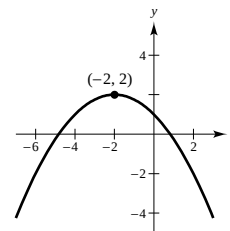
$x^2 + 4x + 4 = -4y + 4 + 4$

$(x + 2)^2 = 4(-1)(y - 2)$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



17. $y^2 + x + y = 0$

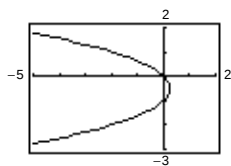
$$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$$

$$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$



19. $y^2 - 4x - 4 = 0$

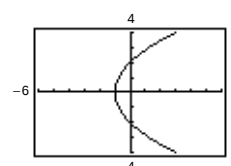
$$y^2 = 4x + 4$$

$$= 4(1)(x + 1)$$

Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$



21. $(y - 2)^2 = 4(-2)(x - 3)$

$$y^2 - 4y + 8x - 20 = 0$$

23. $(x - h)^2 = 4p(y - k)$

$$x^2 = 4(6)(y - 4)$$

$$x^2 - 24y + 96 = 0$$

25. $y = 4 - x^2$

$$x^2 + y - 4 = 0$$

27. Since the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, we obtain

$$3 = c, 4 = 9a + 3b + c, 11 = 16a + 4b + c.$$

Solving this system, we have $a = \frac{5}{3}$, $b = -\frac{14}{3}$, $c = 3$. Therefore,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

29. $x^2 + 4y^2 = 4$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

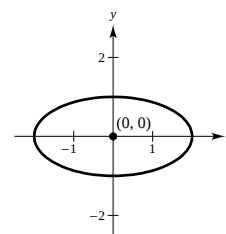
$$a^2 = 4, b^2 = 1, c^2 = 3$$

Center: $(0, 0)$

Foci: $(\pm\sqrt{3}, 0)$

Vertices: $(\pm 2, 0)$

$$e = \frac{\sqrt{3}}{2}$$



31. $\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1$

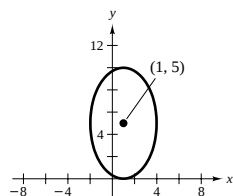
$$a^2 = 25, b^2 = 9, c^2 = 16$$

Center: $(1, 5)$

Foci: $(1, 9), (1, 1)$

Vertices: $(1, 10), (1, 0)$

$$e = \frac{4}{5}$$



33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36 = 36$$

$$\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$$

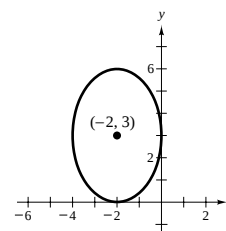
$$a^2 = 9, b^2 = 4, c^2 = 5$$

Center: $(-2, 3)$

Foci: $(-2, 3 \pm \sqrt{5})$

Vertices: $(-2, 6), (-2, 0)$

$$e = \frac{\sqrt{5}}{3}$$



35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$

$$= 60$$

$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$$a^2 = 5, b^2 = 3, c^2 = 2$$

$$\text{Center: } \left(\frac{1}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{1}{2} \pm \sqrt{2}, -1\right)$$

$$\text{Vertices: } \left(\frac{1}{2} \pm \sqrt{5}, -1\right)$$

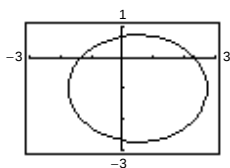
Solve for y:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$

$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



39. Center: (0, 0)

Focus: (2, 0)

Vertex: (3, 0)

Horizontal major axis

$$a = 3, c = 2 \Rightarrow b = \sqrt{5}$$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

43. Center: (0, 0)

Horizontal major axis

Points on ellipse: (3, 1), (4, 0)

Since the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, we have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16$, $b^2 = 16/7$.

Therefore,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$$a^2 = 4, b^2 = 2, c^2 = 2$$

$$\text{Center: } \left(\frac{3}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{3}{2} \pm \sqrt{2}, -1\right)$$

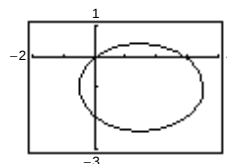
$$\text{Vertices: } \left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$$

$$\text{Solve for y: } 2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



41. Vertices: (3, 1), (3, 9)

Minor axis length: 6

Vertical major axis

Center: (3, 5)

$$a = 4, b = 3$$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

45. $\frac{y^2}{1} - \frac{x^2}{4} = 1$

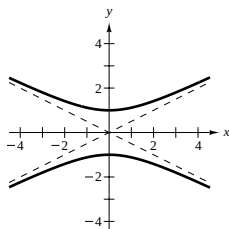
$a = 1, b = 2, c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $(0, \pm 1)$

Foci: $(0, \pm \sqrt{5})$

Asymptotes: $y = \pm \frac{1}{2}x$



47. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$

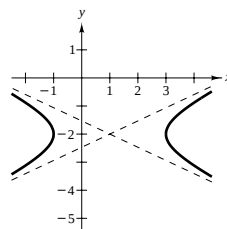
$a = 2, b = 1, c = \sqrt{5}$

Center: $(1, -2)$

Vertices: $(-1, -2), (3, -2)$

Foci: $(1 \pm \sqrt{5}, -2)$

Asymptotes: $y = -2 \pm \frac{1}{2}(x-1)$



49. $9x^2 - y^2 - 36x - 6y + 18 = 0$

$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$

$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

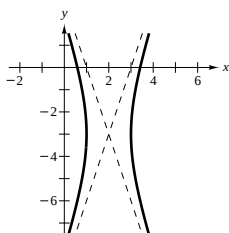
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $(2 \pm \sqrt{10}, -3)$

Asymptotes: $y = -3 \pm 3(x-2)$



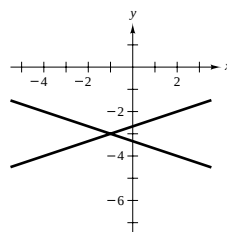
51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$

$(x+1)^2 - 9(y+3)^2 = 0$

$y+3 = \pm \frac{1}{3}(x+1)$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.



53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$

$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $(1, -3 \pm \sqrt{2})$

Foci: $(1, -3 \pm 2\sqrt{5})$

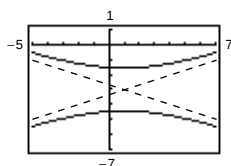
Solve for y :

$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$

$(y+3)^2 = \frac{x^2 - 2x + 19}{9}$

$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$

(Graph each curve separately.)



55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$

$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$

$a = 2, b = \sqrt{6}, c = \sqrt{10}$

Center: $(1, -3)$

Vertices: $(-1, -3), (3, -3)$

Foci: $(1 \pm \sqrt{10}, -3)$

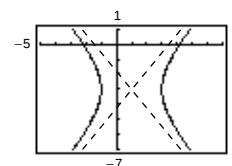
Solve for y :

$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$

$(y+3)^2 = \frac{3x^2 - 6x - 9}{2}$

$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$

(Graph each curve separately.)



57. Vertices:
- $(\pm 1, 0)$

Asymptotes: $y = \pm 3x$

Horizontal transverse axis

Center: $(0, 0)$

$$a = 1, \pm \frac{b}{a} = \pm \frac{b}{1} = \pm 3 \Rightarrow b = 3$$

$$\text{Therefore, } \frac{x^2}{1} - \frac{y^2}{9} = 1.$$

61. Center:
- $(0, 0)$

Vertex: $(0, 2)$ Focus: $(0, 4)$

Vertical transverse axis

$$a = 2, c = 4, b^2 = c^2 - a^2 = 12$$

$$\text{Therefore, } \frac{y^2}{4} - \frac{x^2}{12} = 1.$$

65. (a)
- $\frac{x^2}{9} - y^2 = 1, \frac{2x}{9} - 2yy' = 0, \frac{x}{9y} = y'$

$$\text{At } x = 6: y = \pm \sqrt{3}, y' = \frac{\pm 6}{9\sqrt{3}} = \frac{\pm 2\sqrt{3}}{9}$$

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x - 3\sqrt{3}y - 3 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{-2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x + 3\sqrt{3}y - 3 = 0$$

- 67.
- $x^2 + 4y^2 - 6x + 16y + 21 = 0$

$$A = 1, C = 4$$

$$AC = 4 > 0$$

Ellipse

- 69.
- $y^2 - 4y - 4x = 0$

$$A = 0, C = 1$$

Parabola

- 71.
- $4x^2 + 4y^2 - 16y + 15 = 0$

$$A = C = 4$$

Circle

- 73.
- $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

$$A = C = 9$$

Circle

59. Vertices:
- $(2, \pm 3)$

Point on graph: $(0, 5)$

Vertical transverse axis

Center: $(2, 0)$

$$a = 3$$

Therefore, the equation is of the form

$$\frac{y^2}{9} - \frac{(x - 2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, we have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

$$\text{Therefore, the equation is } \frac{y^2}{9} - \frac{(x - 2)^2}{9/4} = 1.$$

63. Vertices:
- $(0, 2), (6, 2)$

$$\text{Asymptotes: } y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$$

Horizontal transverse axis

Center: $(3, 2)$

$$a = 3$$

$$\text{Slopes of asymptotes: } \pm \frac{b}{a} = \pm \frac{2}{3}$$

Thus, $b = 2$. Therefore,

$$\frac{(x - 3)^2}{9} - \frac{(y - 2)^2}{4} = 1.$$

- (b) From part (a) we know that the slopes of the normal lines must be
- $\mp 9/(2\sqrt{3})$
- .

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = -\frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x + 2\sqrt{3}y - 60 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x - 2\sqrt{3}y - 60 = 0$$

- 75.
- $3x^2 - 6x + 3 = 6 + 2y^2 + 4y + 2$

$$3x^2 - 2y^2 - 6x - 4y - 5 = 0$$

$$A = 3, C = -2, AC < 0$$

Hyperbola

77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$

(c) See Theorem 9.2.

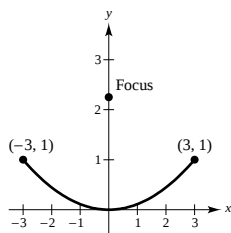
81. Assume that the vertex is at the origin.

$$x^2 = 4py$$

$$(3)^2 = 4p(1)$$

$$\frac{9}{4} = p$$

The pipe is located $\frac{9}{4}$ meters from the vertex.



79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distance fixed points (foci) is constant.

(b) $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ or $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$

(c) $y = k \pm \frac{b}{a}(x - h)$ or $y = k \pm \frac{a}{b}(x - h)$

83. $y = ax^2$

$$y' = 2ax$$

The equation of the tangent line is

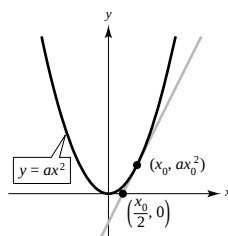
$$y - ax_0^2 = 2ax_0(x - x_0) \text{ or } y = 2ax_0x - ax_0^2.$$

Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$

$$ax_0^2 = 2ax_0x$$

Therefore, $\frac{x_0}{2} = x$ is the x -intercept.



85. (a) Consider the parabola $x^2 = 4py$. Let m_0 be the slope of the one tangent line at (x_1, y_1) and therefore, $-1/m_0$ is the slope of the second at (x_2, y_2) . From the derivative given in Exercise 32 we have:

$$m_0 = \frac{1}{2p}x_1 \text{ or } x_1 = 2pm_0$$

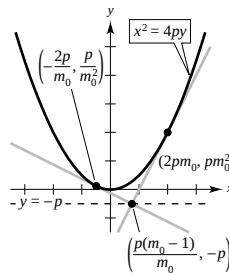
$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \text{ or } x_2 = \frac{-2p}{m_0}$$

Substituting these values of x into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \text{ and } \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is

$$\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right) \text{ and is on the directrix, } y = -p.$$



—CONTINUED—

85. —CONTINUED—

$$(b) \quad x^2 - 4x - 4y + 8 = 0$$

$$(x - 2)^2 = 4(y - 1). \text{ Vertex } (2, 1)$$

$$2x - 4 - 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

At $(-2, 5)$, $dy/dx = -2$. At $(3, \frac{5}{4})$, $dy/dx = \frac{1}{2}$.

Tangent line at $(-2, 5)$: $y - 5 = -2(x + 2) \Rightarrow 2x + y - 1 = 0$.

Tangent line at $(3, \frac{5}{4})$: $y - \frac{5}{4} = \frac{1}{2}(x - 3) \Rightarrow 2x - 4y - 1 = 0$.

Since $m_1 m_2 = (-2)(\frac{1}{2}) = -1$, the lines are perpendicular.

$$\text{Point of intersection: } -2x + 1 = \frac{1}{2}x - \frac{1}{4}$$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $(\frac{1}{2}, 0)$ lies on this line.

87. $y = x - x^2$

$$\frac{dy}{dx} = 1 - 2x$$

At (x_1, y_1) on the mountain, $m = 1 - 2x_1$. Also, $m = \frac{y_1 - 1}{x_1 + 1}$.

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for x_1 , we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2(-1 + \sqrt{3}) = 3 - 2\sqrt{3}$$

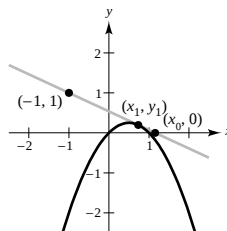
$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$

$$\text{Thus, } -\frac{1}{x_0 + 1} = 3 - 2\sqrt{3}$$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

$$\frac{2\sqrt{3}}{3} = x_0$$



The closest the receiver can be to the hill is $(2\sqrt{3}/3) - 1 \approx 0.155$.

89. Parabola

Vertex: (0, 4)

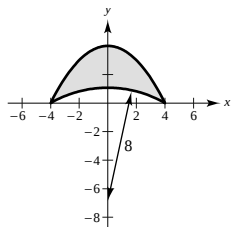
$$x^2 = 4p(y - 4)$$

$$4^2 = 4p(0 - 4)$$

$$p = -1$$

$$x^2 = -4(y - 4)$$

$$y = 4 - \frac{x^2}{4}$$



Circle

Center: (0, k)

Radius: 8

$$x^2 + (y - k)^2 = 64$$

$$4^2 + (0 - k)^2 = 64$$

$$k^2 = 48$$

$$k = -4\sqrt{3} \text{ (Center is on the negative y-axis.)}$$

$$x^2 + (y + 4\sqrt{3})^2 = 64$$

$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

Since the y-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

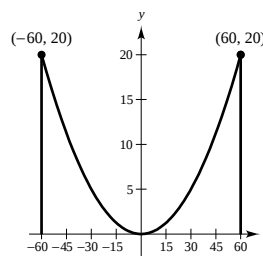
$$\begin{aligned} A &= 2 \int_0^4 \left[\left(4 - \frac{x^2}{4} \right) - \left(-4\sqrt{3} + \sqrt{64 - x^2} \right) \right] dx \\ &= 2 \left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2} \left(x\sqrt{64 - x^2} + 64 \arcsin \frac{x}{8} \right) \right]_0^4 \\ &= 2 \left[16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin \frac{1}{2} \right] \\ &= \frac{16(4 + 3\sqrt{3} - 2\pi)}{3} \approx 15.536 \text{ square feet} \end{aligned}$$

91. (a) Assume that $y = ax^2$.

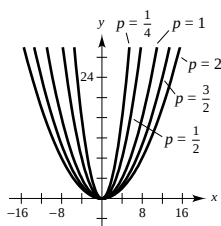
$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$

$$(b) f(x) = \frac{1}{180}x^2, f'(x) = \frac{1}{90}x$$

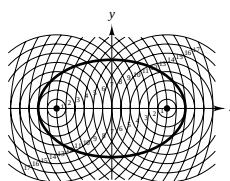
$$\begin{aligned} S &= 2 \int_0^{60} \sqrt{1 + \left(\frac{1}{90}x \right)^2} dx = \frac{2}{90} \int_0^{60} \sqrt{90^2 + x^2} dx \\ &= \frac{2}{90} \cdot \frac{1}{2} \left[x\sqrt{90^2 + x^2} + 90^2 \ln|x + \sqrt{90^2 + x^2}| \right]_0^{60} \quad (\text{formula 26}) \\ &= \frac{1}{90} [60\sqrt{11,700} + 90^2 \ln(60 + \sqrt{11,700}) - 90^2 \ln 90] \\ &= \frac{1}{90} [1800\sqrt{13} + 90^2 \ln(60 + 30\sqrt{13}) - 90^2 \ln 90] \\ &= 20\sqrt{13} + 90 \ln \left(\frac{60 + 30\sqrt{13}}{90} \right) \\ &= 10 \left[2\sqrt{13} + 9 \ln \left(\frac{2 + \sqrt{13}}{3} \right) \right] \approx 128.4 \text{ m} \end{aligned}$$

93. $x^2 = 4py$, $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

As p increases, the graph becomes wider.



95.



$$97. a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

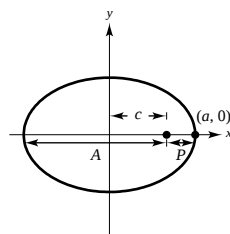
$$99. e = \frac{c}{a}$$

$$A + P = 2a$$

$$a = \frac{A + P}{2}$$

$$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$$

$$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$$



$$101. e = \frac{A - P}{A + P} = \frac{35.34au - 0.59au}{35.34au + 0.59au} \approx 0.9672$$

$$103. \frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2x}{10^2y} = \frac{-x}{4y}$$

$$\text{At } (-8, 3): y' = \frac{8}{12} = \frac{2}{3}$$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the y -axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

$$105. 16x^2 + 9y^2 + 96x + 36y + 36 = 0$$

$$32x + 18yy' + 96 + 36y' = 0$$

$$y'(18y + 36) = -(32x + 96)$$

$$y' = \frac{-(32x + 96)}{18y + 36}$$

$y' = 0$ when $x = -3$. y' is undefined when $y = -2$.

At $x = -3$, $y = 2$ or -6 .

Endpoints of major axis: $(-3, 2)$, $(-3, -6)$

At $y = -2$, $x = 0$ or -6 .

Endpoints of minor axis: $(0, -2)$, $(-6, -2)$

Note: Equation of ellipse is $\frac{(x + 3)^2}{9} + \frac{(y + 2)^2}{16} = 1$

$$107. (a) A = 4 \int_0^2 \frac{1}{2} \sqrt{4 - x^2} dx = \left[x\sqrt{4 - x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]_0^2 = 2\pi \quad [\text{or, } A = \pi ab = \pi(2)(1) = 2\pi]$$

$$(b) \text{ Disk: } V = 2\pi \int_0^2 \frac{1}{4}(4 - x^2) dx = \frac{1}{2}\pi \left[4x - \frac{1}{3}x^3 \right]_0^2 = \frac{8\pi}{3}$$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi) \int_0^2 y \left(\frac{\sqrt{16 - 3x^2}}{4y} \right) dx = \frac{\pi}{2\sqrt{3}} \left[\sqrt{3}x\sqrt{16 - 3x^2} + 16 \arcsin\left(\frac{\sqrt{3}x}{4}\right) \right]_0^2 = \frac{2\pi}{9}(9 + 4\sqrt{3}\pi) \approx 21.48$$

—CONTINUED—

107. —CONTINUED—

(c) **Shell:** $V = 2\pi \int_0^2 x \sqrt{4 - x^2} dx = -\pi \int_0^2 2x(4 - x^2)^{1/2} dx = -\frac{2\pi}{3} \left[(4 - x^2)^{3/2} \right]_0^2 = \frac{16\pi}{3}$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi) \int_0^1 2\sqrt{1 - y^2} \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}} dy = 8\pi \int_0^1 \sqrt{1 + 3y^2} dy$$

$$= \frac{8\pi}{2\sqrt{3}} \left[\sqrt{3}y\sqrt{1 + 3y^2} + \ln \left| \sqrt{3}y + \sqrt{1 + 3y^2} \right| \right]_0^1 = \frac{4\pi}{3} \left| 6 + \sqrt{3} \ln(2 + \sqrt{3}) \right| \approx 34.69$$

109. From Example 5,

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, we have

$$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}.$$

$$C = 4(7) \int_0^{\pi/2} \sqrt{1 - \frac{24}{49} \sin^2 \theta} d\theta$$

$$\approx 28(1.3558) \approx 37.9614$$

113. The transverse axis is horizontal since (2, 2) and (10, 2) are the foci (see definition of hyperbola).

Center: (6, 2)

$$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$$

Therefore, the equation is

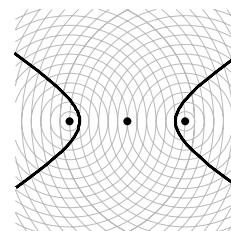
$$\frac{(x - 6)^2}{9} - \frac{(y - 2)^2}{7} = 1.$$

111. Area circle = $\pi r^2 = 100\pi$ Area ellipse = $\pi ab = \pi a(10)$

$$2(100\pi) = 10\pi a \Rightarrow a = 20$$

Hence, the length of the major axis is $2a = 40$.115. $2a = 10 \Rightarrow a = 5$

$$c = 6 \Rightarrow b = \sqrt{11}$$

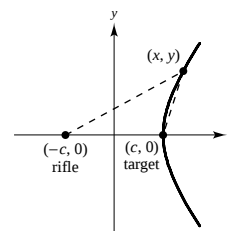
117. Time for sound of bullet hitting target to reach (x, y): $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s}$ Time for sound of rifle to reach (x, y): $\frac{\sqrt{(x + c)^2 + y^2}}{v_s}$ Since the times are the same, we have: $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s} = \frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{v_m^2} + \frac{4c}{v_m v_s} \sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{v_s^2} = \frac{(x + c)^2 + y^2}{v_s^2}$$

$$\sqrt{(x - c)^2 + y^2} = \frac{v_m^2 x - v_s^2 c}{v_s v_m}$$

$$\left(1 - \frac{v_m^2}{v_s^2}\right)x^2 + y^2 = \left(\frac{v_s^2}{v_m^2} - 1\right)c^2$$

$$\frac{x^2}{c^2 v_s^2 / v_m^2} - \frac{y^2}{c^2 (v_m^2 - v_s^2) / v_m^2} = 1$$



119. The point (x, y) lies on the line between $(0, 10)$ and $(10, 0)$. Thus, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, we have:

$$\frac{x^2}{36} - \frac{(10-x)^2}{64} = 1$$

$$16x^2 - 9(10-x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} = \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for x we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and } y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

121. $\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2}, c^2 = a^2 - b^2$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \Rightarrow 2 = x^2 \left(\frac{1}{a^2} + \frac{1}{a^2 - b^2} \right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \Rightarrow x = \pm \frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2} \left(\frac{2a^2c^2}{2a^2 - b^2} \right) \Rightarrow \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \Rightarrow y = \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right), \left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \Rightarrow y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \Rightarrow y'_h = \frac{b^2x}{2c^2y}$$

At $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$, the slopes of the tangent lines are:

$$y'_e = \frac{-b^2 \left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2a^2 \left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2 \left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2c^2 \left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = \frac{a}{c}$$

Since the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

123. False. See the definition of a parabola.

125. True

127. False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines.

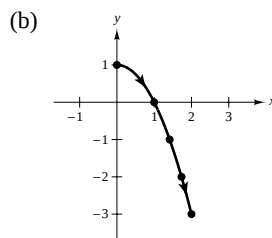
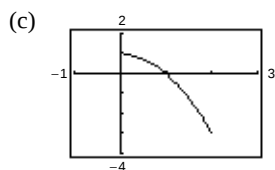
129. True

Section 9.2 Plane Curves and Parametric Equations

1. $x = \sqrt{t}, y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3



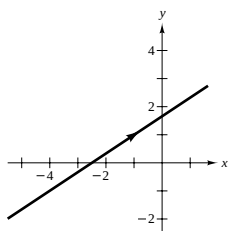
(d) $x^2 = t$
 $y = 1 - x^2, x \geq 0$

3. $x = 3t - 1$

$y = 2t + 1$

$y = 2\left(\frac{x+1}{3}\right) + 1$

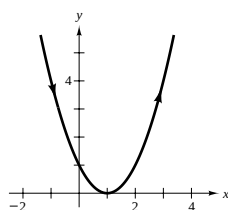
$2x - 3y + 5 = 0$



5. $x = t + 1$

$y = t^2$

$y = (x - 1)^2$

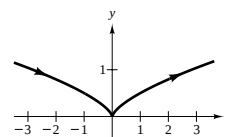


7. $x = t^3$

$y = \frac{1}{2}t^2$

$x = t^3$ implies $t = x^{1/3}$

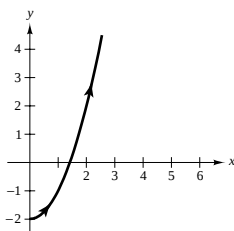
$y = \frac{1}{2}x^{2/3}$



9. $x = \sqrt{t}, t \geq 0$

$y = t - 2$

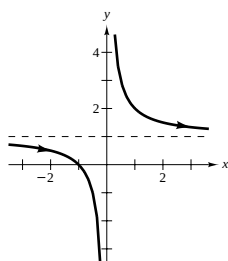
$y = x^2 - 2, x \geq 0$



11. $x = t - 1$

$y = \frac{t}{t-1}$

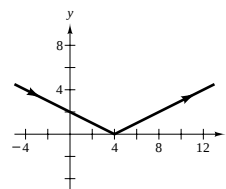
$y = \frac{x+1}{x}$



13. $x = 2t$

$y = |t - 2|$

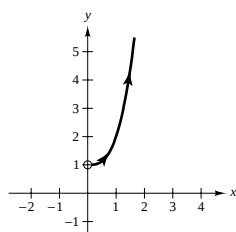
$y = \left|\frac{x}{2} - 2\right| = \frac{|x - 4|}{2}$



15. $x = e^t, x > 0$

$y = e^{3t} + 1$

$y = x^3 + 1, x > 0$



17. $x = \sec \theta$

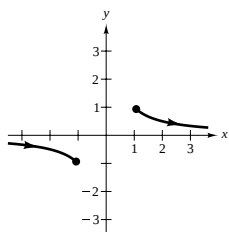
$$y = \cos \theta$$

$$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$$

$$xy = 1$$

$$y = \frac{1}{x}$$

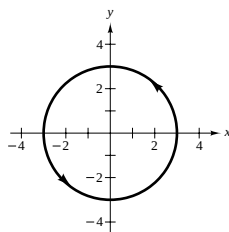
$$|x| \geq 1, |y| \leq 1$$



19. $x = 3 \cos \theta, y = 3 \sin \theta$

Squaring both equations and adding, we have

$$x^2 + y^2 = 9.$$



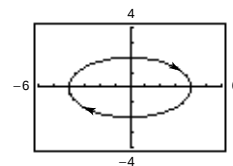
21. $x = 4 \sin 2\theta$

$$y = 2 \cos 2\theta$$

$$\frac{x^2}{16} = \sin^2 2\theta$$

$$\frac{y^2}{4} = \cos^2 2\theta$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$



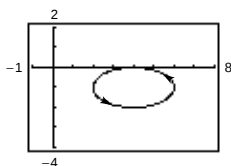
23. $x = 4 + 2 \cos \theta$

$$y = -1 + \sin \theta$$

$$\frac{(x-4)^2}{4} = \cos^2 \theta$$

$$\frac{(y+1)^2}{1} = \sin^2 \theta$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$$



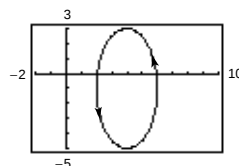
25. $x = 4 + 2 \cos \theta$

$$y = -1 + 4 \sin \theta$$

$$\frac{(x-4)^2}{4} = \cos^2 \theta$$

$$\frac{(y+1)^2}{16} = \sin^2 \theta$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1$$



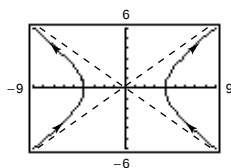
27. $x = 4 \sec \theta$

$$y = 3 \tan \theta$$

$$\frac{x^2}{16} = \sec^2 \theta$$

$$\frac{y^2}{9} = \tan^2 \theta$$

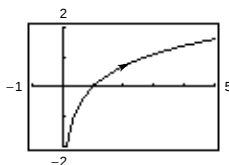
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



29. $x = t^3$

$$y = 3 \ln t$$

$$y = 3 \ln \sqrt[3]{x} = \ln x$$



31. $x = e^{-t}$

$$y = e^{3t}$$

$$e^t = \frac{1}{x}$$

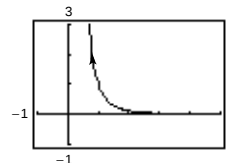
$$e^t = \sqrt[3]{y}$$

$$\sqrt[3]{y} = \frac{1}{x}$$

$$y = \frac{1}{x^3}$$

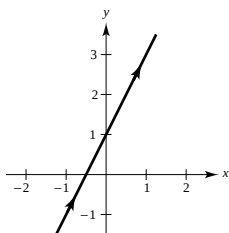
$$x > 0$$

$$y > 0$$



33. By eliminating the parameters in (a) – (d), we get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

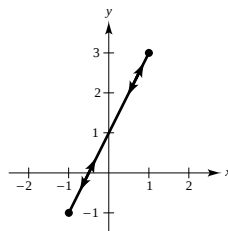
(a) $x = t, y = 2t + 1$



(b) $x = \cos \theta, y = 2 \cos \theta + 1$

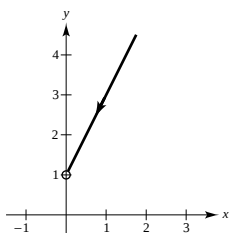
$-1 \leq x \leq 1, -1 \leq y \leq 3$

$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm2\pi, \dots$



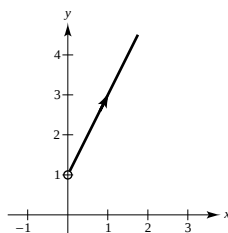
(c) $x = e^{-t}, y = 2e^{-t} + 1$

$x > 0, y > 1$



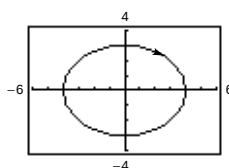
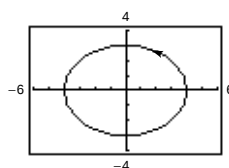
(d) $x = e^t, y = 2e^t + 1$

$x > 0, y > 1$



35. The curves are identical on $0 < \theta < \pi$. They are both smooth. Represent $y = 2(1 - x^2)$

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Many answers possible. For example, $x = 1 + t, y = 1 + 2t$, and $x = 1 - t, x = 1 - 2t$.

39. $x = x_1 + t(x_2 - x_1)$

$y = y_1 + t(y_2 - y_1)$

$\frac{x - x_1}{x_2 - x_1} = t$

$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - y_1 = m(x - x_1)$

41. $x = h + a \cos \theta$

$y = k + b \sin \theta$

$\frac{x - h}{a} = \cos \theta$

$\frac{y - k}{b} = \sin \theta$

$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

43. From Exercise 39 we have

$x = 5t$

$y = -2t$

Solution not unique

45. From Exercise 40 we have

$x = 2 + 4 \cos \theta$

$y = 1 + 4 \sin \theta$

Solution not unique

47. From Exercise 41 we have

$a = 5, c = 4 \Rightarrow b = 3$

$x = 5 \cos \theta$

$y = 3 \sin \theta$

Center: $(0, 0)$

Solution not unique

49. From Exercise 42 we have

$$a = 4, c = 5 \Rightarrow b = 3$$

$$x = 4 \sec \theta$$

$$y = 3 \tan \theta.$$

Center: $(0, 0)$

Solution not unique

51. $y = 3x - 2$

Example

$$x = t, \quad y = 3t - 2$$

$$x = t - 3, \quad y = 3t - 11$$

53. $y = x^3$

Example

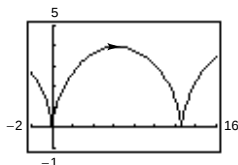
$$x = t, \quad y = t^3$$

$$x = \sqrt[3]{t}, \quad y = t$$

$$x = \tan t, \quad y = \tan^3 t$$

55. $x = 2(\theta - \sin \theta)$

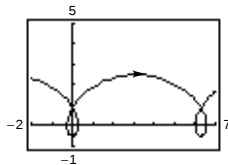
$$y = 2(1 - \cos \theta)$$



Not smooth at $\theta = 2n\pi$

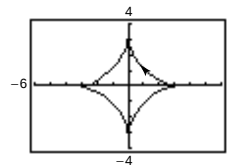
57. $x = \theta - \frac{3}{2} \sin \theta$

$$y = 1 - \frac{3}{2} \cos \theta$$



59. $x = 3 \cos^3 \theta$

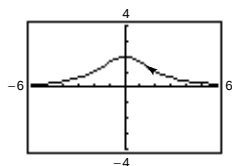
$$y = 3 \sin^3 \theta$$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

61. $x = 2 \cot \theta$

$$y = 2 \sin^2 \theta$$



Smooth everywhere

63. See definition on page 665.

65. A plane curve C , represented by $x = f(t)$, $y = g(t)$, is smooth if f' and g' are continuous and not simultaneously 0. See page 670.

67. $x = 4 \cos \theta$

$$y = 2 \sin 2\theta$$

Matches (d)

69. $x = \cos \theta + \theta \sin \theta$

$$y = \sin \theta - \theta \cos \theta$$

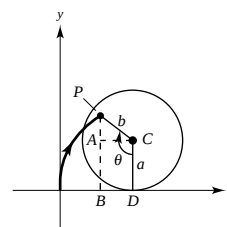
Matches (b)

71. When the circle has rolled θ radians, we know that the center is at $(a\theta, a)$.

$$\sin \theta = \sin(180^\circ - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \quad \text{or} \quad |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180^\circ - \theta) = \frac{|AP|}{-b} \quad \text{or} \quad |AP| = -b \cos \theta$$

Therefore, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.



73. False

$$x = t^2 \Rightarrow x \geq 0$$

$$x = t^2 \Rightarrow y \geq 0$$

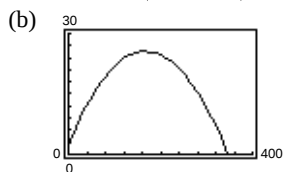
The graph of the parametric equations is only a portion of the line $y = x$.

75. (a) $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$

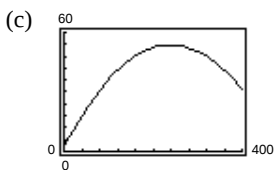
$$x = (v_0 \cos \theta)t = \left(\frac{440}{3} \cos \theta\right)t$$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$= 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2$$



It is not a home run—when $x = 400$, $y \leq 20$.



Yes, it's a home run when $x = 400$, $y > 10$.

(d) We need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3} \cos \theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440 \cos \theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{1200}{440 \cos \theta}\right) - 16\left(\frac{1200}{440 \cos \theta}\right)^2$$

$$7 = 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 \sec^2 \theta$$

$$= 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 (\tan^2 \theta + 1).$$

We now solve the quadratic for $\tan \theta$:

$$16\left(\frac{120}{44}\right)^2 \tan^2 \theta - 400 \tan \theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0$$

$$\tan \theta \approx 0.35185 \Rightarrow \theta \approx 19.4^\circ$$

Section 9.3 Parametric Equations and Calculus

1. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{2t} = \frac{-2}{t}$

3. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \cos t \sin t}{2 \sin t \cos t} = -1$

[Note: $x + y = 1 \Rightarrow y = 1 - x$ and $\frac{dy}{dx} = -1$]

5. $x = 2t$, $y = 3t - 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = 0 \text{ Line}$$

7. $x = t + 1$, $y = t^2 + 3t$

$$\frac{dy}{dx} = \frac{2t + 3}{1} = 1 \text{ when } t = -1.$$

$$\frac{d^2y}{dx^2} = 2 \text{ concave upwards}$$

9. $x = 2 \cos \theta$, $y = 2 \sin \theta$

$$\frac{dy}{dx} = \frac{2 \cos \theta}{-2 \sin \theta} = -\cot \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{\csc^2 \theta}{-2 \sin \theta} = \frac{-\csc^3 \theta}{2} = -\sqrt{2} \text{ when } \theta = \frac{\pi}{4}.$$

concave downward

11. $x = 2 + \sec \theta$, $y = 1 + 2 \tan \theta$

$$\frac{dy}{dx} = \frac{2 \sec^2 \theta}{\sec \theta \tan \theta}$$

$$= \frac{2 \sec \theta}{\tan \theta} = 2 \csc \theta = 4 \text{ when } \theta = \frac{\pi}{6}.$$

$$\frac{d^2y}{dx^2} = \frac{-2 \csc \theta \cot \theta}{\sec \theta \tan \theta}$$

$$= -2 \cot^3 \theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}.$$

concave downward

13. $x = \cos^3 \theta, y = \sin^3 \theta$

$$\frac{dy}{dx} = \frac{3 \sin^2 \theta \cos \theta}{-3 \cos^2 \theta \sin \theta}$$

$$= -\tan \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2 \theta}{-3 \cos^2 \theta \sin \theta} = \frac{1}{3 \cos^4 \theta \sin \theta}$$

$$= \frac{\sec^4 \theta \csc \theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

15. $x = 2 \cot \theta, y = 2 \sin^2 \theta$

$$\frac{dy}{dx} = \frac{4 \sin \theta \cos \theta}{-2 \csc^2 \theta} = -2 \sin^3 \theta \cos \theta$$

$$\text{At } \left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), \theta = \frac{2\pi}{3}, \text{ and } \frac{dy}{dx} = \frac{3\sqrt{3}}{8}.$$

$$\text{Tangent line: } y - \frac{3}{2} = \frac{3\sqrt{3}}{8} \left(x + \frac{2}{\sqrt{3}}\right)$$

$$3\sqrt{3}x - 8y + 18 = 0$$

$$\text{At } (0, 2), \theta = \frac{\pi}{2}, \text{ and } \frac{dy}{dx} = 0.$$

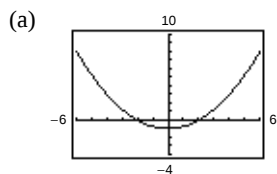
$$\text{Tangent line: } y - 2 = 0$$

$$\text{At } \left(2\sqrt{3}, \frac{1}{2}\right), \theta = \frac{\pi}{6}, \text{ and } \frac{dy}{dx} = -\frac{\sqrt{3}}{8}.$$

$$\text{Tangent line: } y - \frac{1}{2} = -\frac{\sqrt{3}}{8}(x - 2\sqrt{3})$$

$$\sqrt{3}x + 8y - 10 = 0$$

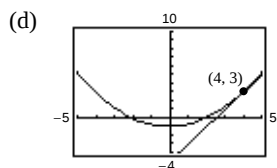
17. $x = 2t, y = t^2 - 1, t = 2$



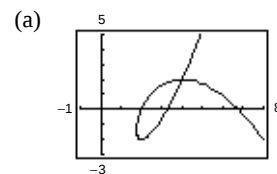
(b) At $t = 2, (x, y) = (4, 3)$, and

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 4, \frac{dy}{dx} = 2$$

(c) $\frac{dy}{dx} = 2$. At $(4, 3)$, $y - 3 = 2(x - 4)$
 $y = 2x - 5$



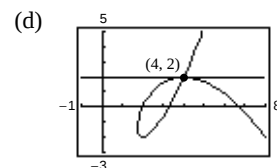
19. $x = t^2 - t + 2, y = t^3 - 3t, t = -1$



(b) At $t = -1, (x, y) = (4, 2)$, and

$$\frac{dx}{dt} = -3, \frac{dy}{dt} = 0, \frac{dy}{dx} = 0$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$
 $y = 2$



21. $x = 2 \sin 2t, y = 3 \sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3 \cos t}{4 \cos 2t}$$

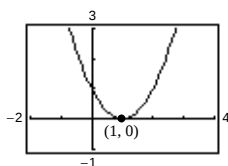
At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = -\frac{3}{4}x$ Tangent Line

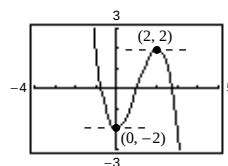
23. $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta \sin \theta = 0$ when $\theta = 0, \pi, 2\pi, 3\pi, \dots$ Points: $(-1, [2n-1]\pi)$, $(1, 2n\pi)$ where n is an integer.Points shown: $(1, 0)$, $(-1, \pi)$, $(1, -2\pi)$ Vertical tangents: $\frac{dx}{d\theta} = \theta \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ Points: $\left(\frac{(-1)^{n+1}(2n-1)\pi}{2}, (-1)^{n+1}\right)$ Points shown: $\left(\frac{\pi}{2}, 1\right)$, $\left(-\frac{3\pi}{2}, -1\right)$, $\left(\frac{5\pi}{2}, 1\right)$

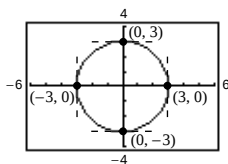
25. $x = 1 - t$, $y = t^2$

Horizontal tangents: $\frac{dy}{dt} = 2t = 0$ when $t = 0$.Point: $(1, 0)$ Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none

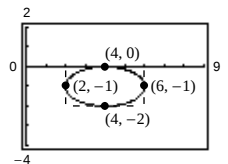
27. $x = 1 - t$, $y = t^3 - 3t$

Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.Points: $(0, -2)$, $(2, 2)$ Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none

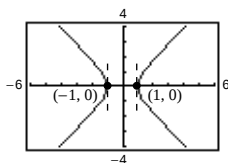
29. $x = 3 \cos \theta$, $y = 3 \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = 3 \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.Points: $(0, 3)$, $(0, -3)$ Vertical tangents: $\frac{dx}{d\theta} = -3 \sin \theta = 0$ when $\theta = 0, \pi$.Points: $(3, 0)$, $(-3, 0)$ 

31. $x = 4 + 2 \cos \theta$, $y = -1 + \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.Points: $(4, 0)$, $(4, -2)$ Vertical tangents: $\frac{dx}{d\theta} = -2 \sin \theta = 0$ when $\theta = 0, \pi$.Points: $(6, -1)$, $(2, -1)$ 

33. $x = \sec \theta$, $y = \tan \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \sec^2 \theta \neq 0$; noneVertical tangents: $\frac{dx}{d\theta} = \sec \theta \tan \theta = 0$ when $\theta = 0, \pi$.Points: $(1, 0)$, $(-1, 0)$ 

35. $x = t^2$, $y = 2t$, $0 \leq t \leq 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$\begin{aligned} s &= 2 \int_0^2 \sqrt{t^2 + 1} \, dt \\ &= \left[t\sqrt{t^2 + 1} + \ln|t + \sqrt{t^2 + 1}| \right]_0^2 \\ &= 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916 \end{aligned}$$

$$37. x = e^{-t} \cos t, y = e^{-t} \sin t, 0 \leq t \leq \frac{\pi}{2}$$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t), \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

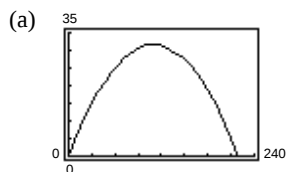
$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\pi/2} \sqrt{2e^{-2t}} dt = -\sqrt{2} \int_0^{\pi/2} e^{-t}(-1) dt \\ &= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12 \end{aligned}$$

$$41. x = a \cos^3 \theta, y = a \sin^3 \theta, \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta,$$

$$\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\begin{aligned} S &= 4 \int_0^{\pi/2} \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta \\ &= 12a \int_0^{\pi/2} \sin \theta \cos \theta \sqrt{\cos^2 \theta + \sin^2 \theta} d\theta \\ &= 6a \int_0^{\pi/2} \sin 2\theta d\theta = \left[-3a \cos 2\theta\right]_0^{\pi/2} = 6a \end{aligned}$$

$$45. x = (90 \cos 30^\circ)t, y = (90 \sin 30^\circ)t - 16t^2$$



(b) Range: 219.2 ft

$$(c) \frac{dx}{dt} = 90 \cos 30^\circ, \frac{dy}{dt} = 90 \sin 30^\circ - 32t.$$

$$y = 0 \text{ for } t = \frac{45}{16}.$$

$$\begin{aligned} s &= \int_0^{45/16} \sqrt{(90 \cos 30^\circ)^2 + (90 \sin 30^\circ - 32t)^2} dt \\ &= 230.8 \text{ ft} \end{aligned}$$

$$39. x = \sqrt{t}, y = 3t - 1, \frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 3$$

$$\begin{aligned} S &= \int_0^1 \sqrt{\frac{1}{4t} + 9} dt = \frac{1}{2} \int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}} dt \\ &= \frac{1}{6} \int_0^6 \sqrt{1 + u^2} du \\ &= \frac{1}{12} \left[\ln(\sqrt{1 + u^2} + u) + u\sqrt{1 + u^2} \right]_0^6 \\ &= \frac{1}{12} \left[\ln(\sqrt{37} + 6) + 6\sqrt{37} \right] \approx 3.249 \end{aligned}$$

$$u = 6\sqrt{t}, du = \frac{3}{\sqrt{t}} dt$$

$$43. x = a(\theta - \sin \theta), y = a(1 - \cos \theta),$$

$$\frac{dx}{d\theta} = a(1 - \cos \theta), \frac{dy}{d\theta} = a \sin \theta$$

$$\begin{aligned} S &= 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= 2\sqrt{2}a \int_0^\pi \sqrt{1 - \cos \theta} d\theta \\ &= 2\sqrt{2}a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta \\ &= \left[-4\sqrt{2}a\sqrt{1 + \cos \theta}\right]_0^\pi = 8a \end{aligned}$$

$$(d) y = 0 \Rightarrow (90 \sin \theta)t = 16t^2 \Rightarrow t = \frac{90}{16} \sin \theta$$

$$x = (90 \cos \theta)t = \frac{90^2}{16} \cos \theta \sin \theta = \frac{90^2}{32} \sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32} 2 \cos 2\theta = 0 \Rightarrow \theta = 45^\circ$$

By the First Derivative Test, $\theta = 45^\circ \left(\frac{\pi}{4}\right)$

maximizes the range.

$$\frac{dx}{dt} = 90 \cos \theta,$$

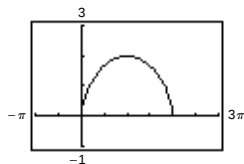
$$\frac{dy}{dt} = 90 \sin \theta - 32 = 90 \sin \theta - 32 \left(\frac{90}{16} \sin \theta\right) = -90 \sin \theta$$

$$\begin{aligned} s &= \int_0^{(90/16)\sin \theta} \sqrt{(90 \cos \theta)^2 + (-90 \sin \theta)^2} dt \\ &= \int_0^{(90/16)\sin \theta} 90 dt = 90t \Big|_0^{(90/16)\sin \theta} \\ &= \frac{90^2}{16} \sin \theta \end{aligned}$$

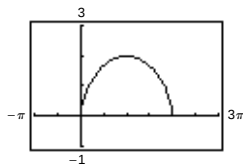
$$\frac{ds}{d\theta} = \frac{90^2}{16} \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

By the First Derivative Test, $\theta = 90^\circ$ maximizes the arc length.

47. (a) $x = t - \sin t$
 $y = 1 - \cos t$
 $0 \leq t \leq 2\pi$



$x = 2t - \sin(2t)$
 $y = 1 - \cos(2t)$
 $0 \leq t \leq \pi$



(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin(\frac{1}{2}t)$
 $y = 1 - \cos(\frac{1}{2}t)$

The time required for the particle to traverse the same path is $t = 4\pi$.

49. $x = t$, $y = 2t$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2$

(a) $S = 2\pi \int_0^4 2t\sqrt{1+4} dt = 4\sqrt{5}\pi \int_0^4 t dt$
 $= \left[2\sqrt{5}\pi t^2 \right]_0^4 = 32\pi\sqrt{5}$

(b) $S = 2\pi \int_0^4 t\sqrt{1+4} dt = 2\sqrt{5}\pi \int_0^4 t dt$
 $= \left[\sqrt{5}\pi t^2 \right]_0^4 = 16\pi\sqrt{5}$

51. $x = 4 \cos \theta$, $y = 4 \sin \theta$, $\frac{dx}{d\theta} = -4 \sin \theta$, $\frac{dy}{d\theta} = 4 \cos \theta$

$S = 2\pi \int_0^{\pi/2} 4 \cos \theta \sqrt{(-4 \sin \theta)^2 + (4 \cos \theta)^2} d\theta$
 $= 32\pi \int_0^{\pi/2} \cos \theta d\theta = \left[32\pi \sin \theta \right]_0^{\pi/2} = 32\pi$

53. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$, $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

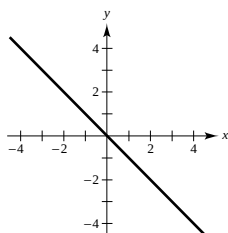
$S = 4\pi \int_0^{\pi/2} a \sin^3 \theta \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta = 12a^2\pi \int_0^{\pi/2} \sin^4 \theta \cos \theta d\theta = \frac{12\pi a^2}{5} \left[\sin^5 \theta \right]_0^{\pi/2} = \frac{12}{5}\pi a^2$

55. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

See Theorem 9.7, page 675.

57. One possible answer is the graph given by

$x = t$, $y = -t$.

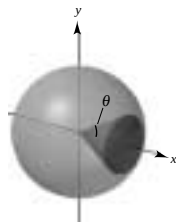


59. $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

See Theorem 9.8, page 678.

61. $x = r \cos \phi$, $y = r \sin \phi$

$S = 2\pi \int_0^\theta r \sin \phi \sqrt{r^2 \sin^2 \phi + r^2 \cos^2 \phi} d\phi$
 $= 2\pi r^2 \int_0^\theta \sin \phi d\phi$
 $= \left[-2\pi r^2 \cos \phi \right]_0^\theta$
 $= 2\pi r^2(1 - \cos \theta)$



63. $x = \sqrt{t}$, $y = 4 - t$, $0 \leq t \leq 4$

$$A = \int_0^4 (4 - t) \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \int_0^4 (4t^{-1/2} - t^{1/2}) dt = \left[\frac{1}{2} \left(8\sqrt{t} - \frac{2}{3}t\sqrt{t} \right) \right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{3}{16} \int_0^4 (4 - t) \sqrt{t} \left(\frac{1}{2\sqrt{t}} \right) dt = \frac{3}{32} \int_0^4 (4 - t) dt = \left[\frac{3}{32} \left(4t - \frac{t^2}{2} \right) \right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{3}{32} \int_0^4 (4 - t)^2 \frac{1}{2\sqrt{t}} dt = \frac{3}{64} \int_0^4 [16t^{-1/2} - 8t^{1/2} + t^{3/2}] dt = \frac{3}{64} \left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t} \right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5} \right)$$

65. $x = 3 \cos \theta$, $y = 3 \sin \theta$, $\frac{dx}{d\theta} = -3 \sin \theta$

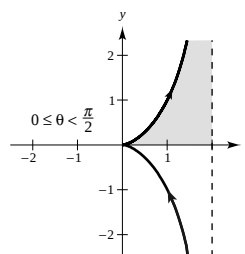
$$\begin{aligned} V &= 2\pi \int_{\pi/2}^0 (3 \sin \theta)^2 (-3 \sin \theta) d\theta \\ &= -54\pi \int_{\pi/2}^0 \sin^3 \theta d\theta \\ &= -54\pi \int_{\pi/2}^0 (1 - \cos^2 \theta) \sin \theta d\theta \\ &= -54\pi \left[-\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 = 36\pi \end{aligned}$$

67. $x = 2 \sin^2 \theta$

$$y = 2 \sin^2 \theta \tan \theta$$

$$\frac{dx}{d\theta} = 4 \sin \theta \cos \theta$$

$$\begin{aligned} A &= \int_0^{\pi/2} 2 \sin^2 \theta \tan \theta (4 \sin \theta \cos \theta) d\theta = 8 \int_0^{\pi/2} \sin^4 \theta d\theta \\ &= 8 \left[\frac{-\sin^3 \theta \cos \theta}{4} - \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta \right]_0^{\pi/2} = \frac{3\pi}{2} \end{aligned}$$



69. πab is area of ellipse (d).

71. $6\pi a^2$ is area of cardioid (f).

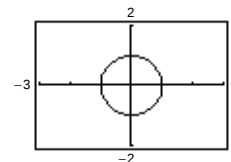
73. $\frac{8}{3}ab$ is area of hourglass (a).

75. (a) $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, $-20 \leq t \leq 20$

The graph is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

$$\text{Verify: } x^2 + y^2 = \left(\frac{1-t^2}{1+t^2} \right)^2 + \left(\frac{2t}{1+t^2} \right)^2 = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} = 1$$

(b) As t increases from -20 to 0 , the speed increases, and as t increases from 0 to 20 , the speed decreases.



77. False

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[\frac{g'(t)}{f'(t)} \right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

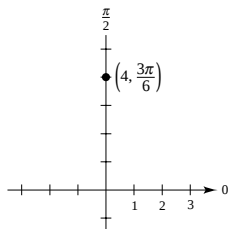
Section 9.4 Polar Coordinates and Polar Graphs

1. $\left(4, \frac{\pi}{2}\right)$

$$x = 4 \cos\left(\frac{\pi}{2}\right) = 0$$

$$y = 4 \sin\left(\frac{\pi}{2}\right) = 4$$

$$(x, y) = (0, 4)$$

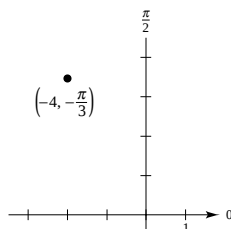


3. $\left(-4, -\frac{\pi}{3}\right)$

$$x = -4 \cos\left(-\frac{\pi}{3}\right) = -2$$

$$y = -4 \sin\left(-\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$(x, y) = (-2, 2\sqrt{3})$$

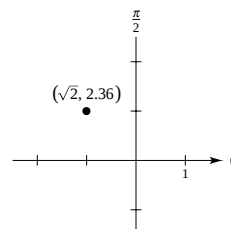


5. $(\sqrt{2}, 2.36)$

$$x = \sqrt{2} \cos(2.36) \approx -1.004$$

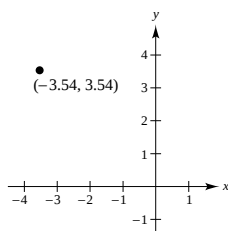
$$y = \sqrt{2} \sin(2.36) \approx 0.996$$

$$(x, y) = (-1.004, 0.996)$$



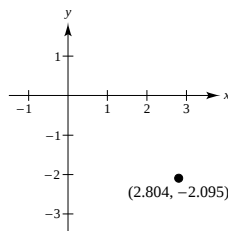
7. $(r, \theta) = \left(5, \frac{3\pi}{4}\right)$

$$(x, y) = (-3.5355, 3.5355)$$



9. $(r, \theta) = (-3.5, 2.5)$

$$(x, y) = (2.804, -2.095)$$

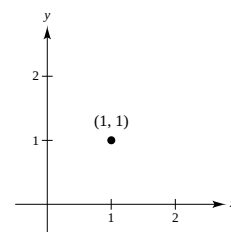


11. $(x, y) = (1, 1)$

$$r = \pm\sqrt{2}$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \left(\sqrt{2}, \frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{5\pi}{4}\right)$$

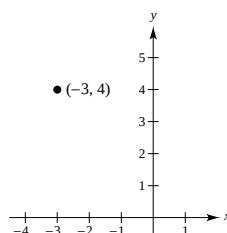


13. $(x, y) = (-3, 4)$

$$r = \pm\sqrt{9 + 16} = \pm 5$$

$$\tan \theta = -\frac{4}{3}$$

$$\theta \approx 2.214, 5.356, (5, 2.214), (-5, 5.356)$$



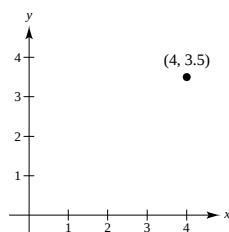
15. $(x, y) = (3, -2)$

$$(r, \theta) = (3.606, -0.588)$$

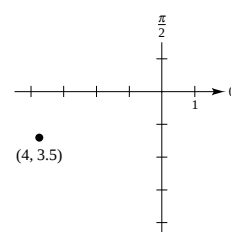
17. $(x, y) = \left(\frac{5}{2}, \frac{4}{3}\right)$

$$(r, \theta) = (2.833, 0.490)$$

19. (a) $(x, y) = (4, 3.5)$

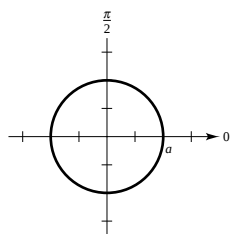


(b) $(r, \theta) = (4, 3.5)$



21. $x^2 + y^2 = a^2$

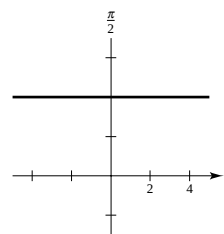
$$r = a$$



23. $y = 4$

$$r \sin \theta = 4$$

$$r = 4 \csc \theta$$

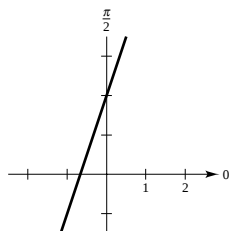


25. $3x - y + 2 = 0$

$$3r \cos \theta - r \sin \theta + 2 = 0$$

$$r(3 \cos \theta - \sin \theta) = -2$$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$

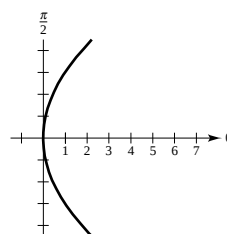


27. $y^2 = 9x$

$$r^2 \sin^2 \theta = 9r \cos \theta$$

$$r = \frac{9 \cos \theta}{\sin^2 \theta}$$

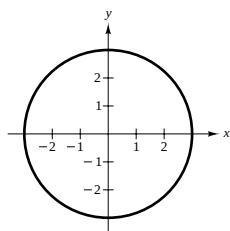
$$r = 9 \csc^2 \theta \cos \theta$$



29. $r = 3$

$$r^2 = 9$$

$$x^2 + y^2 = 9$$



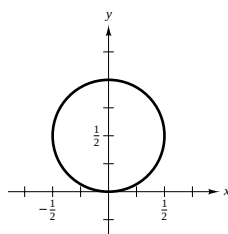
31. $r = \sin \theta$

$$r^2 = r \sin \theta$$

$$x^2 + y^2 = y$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$x^2 + y^2 - y = 0$$

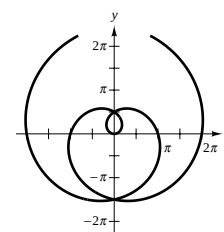


33. $r = \theta$

$$\tan r = \tan \theta$$

$$\tan \sqrt{x^2 + y^2} = \frac{y}{x}$$

$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$$

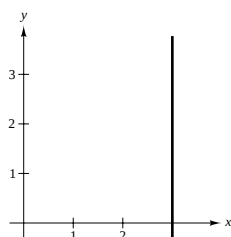


35. $r = 3 \sec \theta$

$$r \cos \theta = 3$$

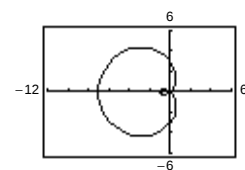
$$x = 3$$

$$x - 3 = 0$$



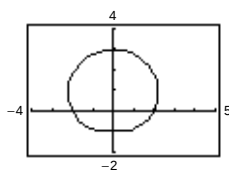
37. $r = 3 - 4 \cos \theta$

$$0 \leq \theta < 2\pi$$



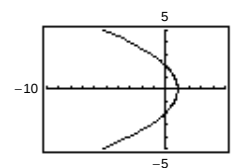
39. $r = 2 + \sin \theta$

$$0 \leq \theta < 2\pi$$



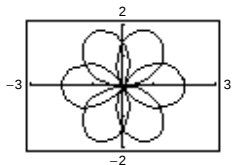
41. $r = \frac{2}{1 + \cos \theta}$

$$\text{Traced out once on } -\pi < \theta < \pi$$



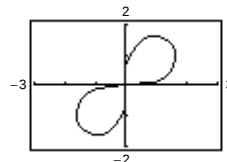
43. $r = 2 \cos\left(\frac{3\theta}{2}\right)$

$0 \leq \theta < 4\pi$



45. $r^2 = 4 \sin 2\theta$

$0 \leq \theta < \frac{\pi}{2}$



47.

$r = 2(h \cos \theta + k \sin \theta)$

Radius: $\sqrt{h^2 + k^2}$

$r^2 = 2r(h \cos \theta + k \sin \theta)$

Center: (h, k)

$r^2 = 2[h(r \cos \theta) + k(r \sin \theta)]$

$x^2 + y^2 = 2(hx + ky)$

$x^2 + y^2 - 2hx - 2ky = 0$

$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = 0 + h^2 + k^2$

$(x - h)^2 + (y - k)^2 = h^2 + k^2$

49. $\left(4, \frac{2\pi}{3}\right), \left(2, \frac{\pi}{6}\right)$

$$d = \sqrt{4^2 + 2^2 - 2(4)(2) \cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)}$$

$$= \sqrt{20 - 16 \cos \frac{\pi}{2}} = 2\sqrt{5} \approx 4.5$$

51. $(2, 0.5), (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7) \cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28 \cos(-0.7)} \approx 5.6$$

53. $r = 2 + 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + \cos \theta(2 + 3 \sin \theta)}{3 \cos \theta \cos \theta - \sin \theta(2 + 3 \sin \theta)}$$

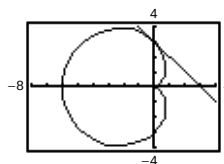
$$= \frac{2 \cos \theta(3 \sin \theta + 1)}{3 \cos 2\theta - 2 \sin \theta} = \frac{2 \cos \theta(3 \sin \theta + 1)}{6 \cos^2 \theta - 2 \sin \theta - 3}$$

At $\left(5, \frac{\pi}{2}\right), \frac{dy}{dx} = 0$.

At $(2, \pi), \frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right), \frac{dy}{dx} = 0$.

55. (a), (b) $r = 3(1 - \cos \theta)$



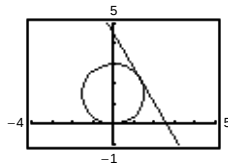
$$(r, \theta) = \left(3, \frac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$$

Tangent line: $y - 3 = -1(x - 0)$

$$y = -x + 3$$

(c) At $\theta = \frac{\pi}{2}, \frac{dy}{dx} = -1.0$.

57. (a), (b) $r = 3 \sin \theta$



$$(r, \theta) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) \Rightarrow (x, y) = \left(\frac{3\sqrt{3}}{4}, \frac{9}{4}\right)$$

Tangent line: $y - \frac{9}{4} = -\sqrt{3}\left(x - \frac{3\sqrt{3}}{4}\right)$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = -\sqrt{3} \approx -1.732$.

59. $r = 1 - \sin \theta$

$$\begin{aligned} \frac{dy}{d\theta} &= (1 - \sin \theta) \cos \theta - \cos \theta \sin \theta \\ &= \cos \theta (1 - 2 \sin \theta) = 0 \end{aligned}$$

$$\cos \theta = 0, \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{Horizontal tangents: } \left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$$

$$\begin{aligned} \frac{dx}{d\theta} &= (-1 + \sin \theta) \sin \theta - \cos \theta \cos \theta \\ &= -\sin \theta + \sin^2 \theta + \sin^2 \theta - 1 \\ &= 2 \sin^2 \theta - \sin \theta - 1 \\ &= (2 \sin \theta + 1)(\sin \theta - 1) = 0 \end{aligned}$$

$$\sin \theta = 1, \sin \theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\text{Vertical tangents: } \left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$$

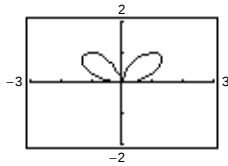
61. $r = 2 \csc \theta + 3$

$$\begin{aligned} \frac{dy}{d\theta} &= (2 \csc \theta + 3) \cos \theta + (-2 \csc \theta \cot \theta) \sin \theta \\ &= 3 \cos \theta = 0 \end{aligned}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{Horizontal: } \left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$$

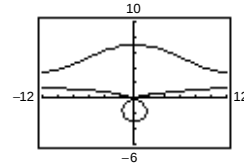
63. $r = 4 \sin \theta \cos^2 \theta$



Horizontal tangents:

$$(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)$$

65. $r = 2 \csc \theta + 5$



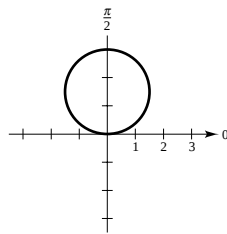
$$\text{Horizontal tangents: } \left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$$

67. $r = 3 \sin \theta$
 $r^2 = 3r \sin \theta$

$$\begin{aligned} x^2 + y^2 &= 3y \\ x^2 + \left(y - \frac{3}{2}\right)^2 &= \frac{9}{4} \end{aligned}$$

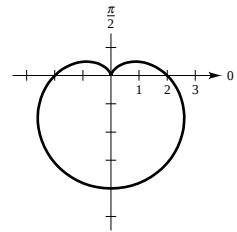
$$\text{Circle } r = \frac{3}{2}$$

$$\text{Center: } \left(0, \frac{3}{2}\right)$$

 Tangent at the pole: $\theta = 0$


69. $r = 2(1 - \sin \theta)$

Cardioid

 Symmetric to y-axis, $\theta = \frac{\pi}{2}$


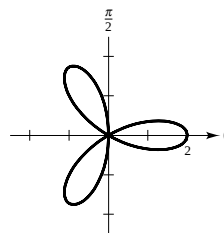
71. $r = 2 \cos(3\theta)$

Rose curve with three petals

Symmetric to the polar axis

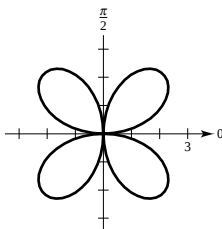
$$\text{Relative extrema: } (2, 0), \left(-2, \frac{\pi}{3}\right), \left(2, \frac{2\pi}{3}\right)$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	0	$-\sqrt{2}$	-2	0	2	0	-2

 Tangents at the pole: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$


73. $r = 3 \sin 2\theta$

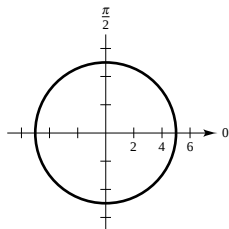
Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $\left(\pm 3, \frac{\pi}{4}\right), \left(\pm 3, \frac{5\pi}{4}\right)$ Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ $(\theta = \pi, 3\pi/2 \text{ give the same tangents.})$ 

75. $r = 5$

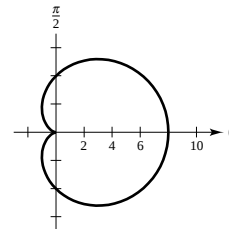
Circle radius: 5

$x^2 + y^2 = 25$



77. $r = 4(1 + \cos \theta)$

Cardioid

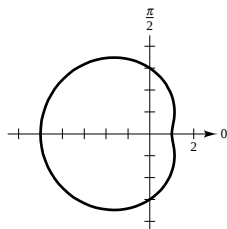


79. $r = 3 - 2 \cos \theta$

Limaçon

Symmetric to polar axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	2	3	4	5

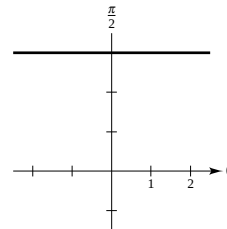


81. $r = 3 \csc \theta$

$r \sin \theta = 3$

$y = 3$

Horizontal line

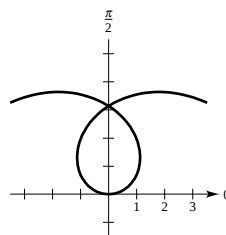


83. $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \frac{\pi}{2}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

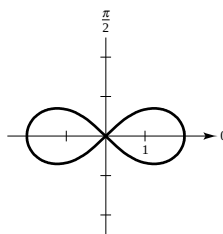
Tangent at the pole: $\theta = 0$

85. $r^2 = 4 \cos(2\theta)$

Lemniscate

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, 0)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 2	$\pm \sqrt{2}$	0

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 

87. Since

$$r = 2 - \sec \theta = 2 - \frac{1}{\cos \theta},$$

the graph has polar axis symmetry and the lengths at the pole are

$$\theta = \frac{\pi}{3}, \frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^-$$

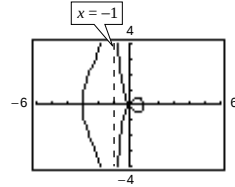
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^+.$$

$$\text{Also, } r = 2 - \frac{1}{\cos \theta} = 2 - \frac{r}{r \cos \theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1+x}.$$

Thus, $r \Rightarrow \pm\infty$ as $x \Rightarrow -1$.



89. $r = \frac{2}{\theta}$

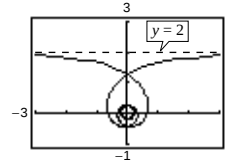
Hyperbolic spiral

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0$$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2 \sin \theta}{r \sin \theta} = \frac{2 \sin \theta}{y}$$

$$y = \frac{2 \sin \theta}{\theta}$$

$$\lim_{\theta \rightarrow 0} \frac{2 \sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{2 \cos \theta}{1} = 2$$



91. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y -axis to the point, and y is the directed distance from the x -axis to the point. Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x -axis, measured clockwise.

Point do not have a unique polar representation.

93. $r = a$ circle

$$\theta = b \text{ line}$$

95. $r = 2 \sin \theta$ circle

Matches (c)

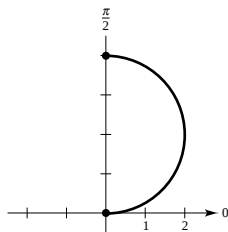
97. $r = 3(1 + \cos \theta)$

Cardioid

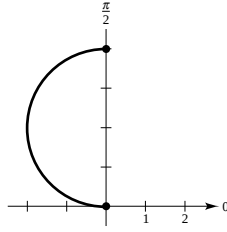
Matches (a)

99. $r = 4 \sin \theta$

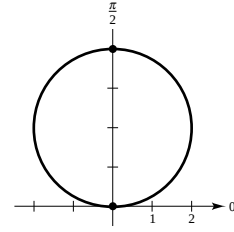
(a) $0 \leq \theta \leq \frac{\pi}{2}$



(b) $\frac{\pi}{2} \leq \theta \leq \pi$



(c) $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

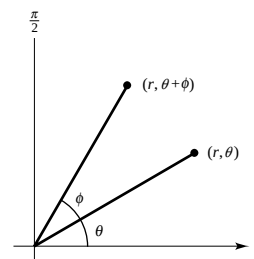


101. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

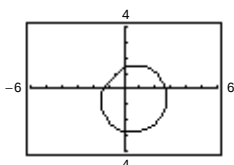
Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$

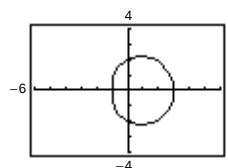


103. $r = 2 - \sin \theta$

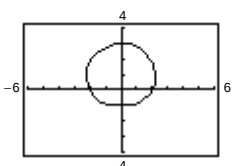
(a) $r = 2 - \sin\left(\theta - \frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2}(\sin \theta - \cos \theta)$



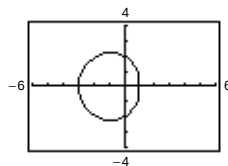
(b) $r = 2 - (-\cos \theta) = 2 + \cos \theta$



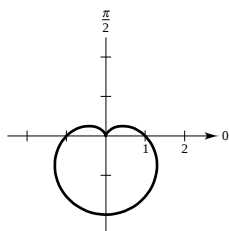
(c) $r = 2 - (-\sin \theta) = 2 + \sin \theta$



(d) $r = 2 - \cos \theta$



105. (a) $r = 1 - \sin \theta$

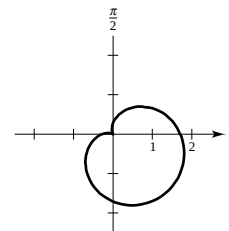


(b) $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

Rotate the graph of

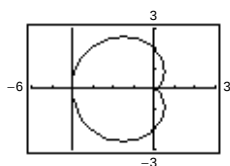
$$r = 1 - \sin \theta$$

through the angle $\pi/4$.



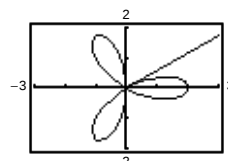
107. $\tan \psi = \frac{r}{dr/d\theta} = \frac{2(1 - \cos \theta)}{2 \sin \theta}$

At $\theta = \pi$, $\tan \psi$ is undefined $\Rightarrow \psi = \frac{\pi}{2}$.



109. $\tan \psi = \frac{r}{dr/d\theta} = \frac{2 \cos 3\theta}{-6 \sin 3\theta}$

At $\theta = \frac{\pi}{6}$, $\tan \psi = 0 \Rightarrow \psi = 0$.

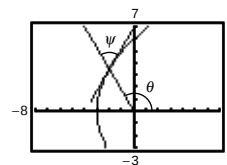


$$111. \quad r = \frac{6}{1 - \cos \theta} = 6(1 - \cos \theta)^{-1} \Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 - \cos \theta)^2}$$

$$\tan \psi = \frac{r}{\frac{dr}{d\theta}} = \frac{\frac{6}{(1 - \cos \theta)}}{\frac{6 \sin \theta}{(1 - \cos \theta)^2}} = \frac{1 - \cos \theta}{\sin \theta}$$

$$\text{At } \theta = \frac{2\pi}{3}, \tan \psi = \frac{1 - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} = \sqrt{3}.$$

$$\psi = \frac{\pi}{3}, (60^\circ)$$

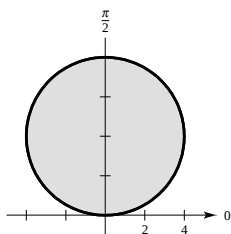


113. True

115. True

Section 9.5 Area and Arc Length in Polar Coordinates

$$1. \text{ (a) } r = 8 \sin \theta$$



$$A = \pi(4)^2 = 16\pi$$

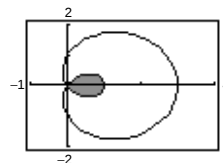
$$3. \quad A = 2 \left[\frac{1}{2} \int_0^{\pi/6} (2 \cos 3\theta)^2 d\theta \right] = 2 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = \frac{\pi}{3}$$

$$7. \quad A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \sin \theta)^2 d\theta \right] \\ = \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

$$\begin{aligned} \text{(b) } A &= 2 \left(\frac{1}{2} \right) \int_0^{\pi/2} [8 \sin \theta]^2 d\theta \\ &= 64 \int_0^{\pi/2} \sin^2 \theta d\theta \\ &= 32 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ &= 32 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 16\pi \end{aligned}$$

$$5. \quad A = 2 \left[\frac{1}{2} \int_0^{\pi/4} (\cos 2\theta)^2 d\theta \right] \\ = \frac{1}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{8}$$

$$9. \quad A = 2 \left[\frac{1}{2} \int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right] \\ = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$

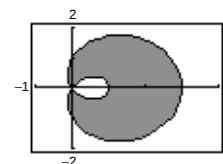


11. The area inside the outer loop is

$$2 \left[\frac{1}{2} \int_0^{2\pi/3} (1 + 2 \cos \theta)^2 d\theta \right] = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$$

From the result of Exercise 9, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2} \right) - \left(\frac{2\pi - 3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3}.$$



13. $r = 1 + \cos \theta$

$r = 1 - \cos \theta$

Solving simultaneously,

$$1 + \cos \theta = 1 - \cos \theta$$

$$2 \cos \theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos \theta = 1 - \cos \theta$, $\cos \theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right)$, $\left(1, \frac{3\pi}{2}\right)$, $(0, 0)$

17. $r = 4 - 5 \sin \theta$

$r = 3 \sin \theta$

Solving simultaneously,

$$4 - 5 \sin \theta = 3 \sin \theta$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$, $(0, 0)$

21. $r = 4 \sin 2\theta$

$r = 2$

$r = 4 \sin 2\theta$ is the equation of a rose curve with four petals and is symmetric to the polar axis, $\theta = \pi/2$, and the pole. Also, $r = 2$ is the equation of a circle of radius 2 centered at the pole. Solving simultaneously,

$$4 \sin 2\theta = 2$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$

Therefore, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. By symmetry, the other points of intersection are $(2, 7\pi/12)$, $(2, 11\pi/12)$, $(2, 13\pi/12)$, $(2, 17\pi/12)$, $(2, 19\pi/12)$, and $(2, 23\pi/12)$.

15. $r = 1 + \cos \theta$

$r = 1 - \sin \theta$

Solving simultaneously,

$$1 + \cos \theta = 1 - \sin \theta$$

$$\cos \theta = -\sin \theta$$

$$\tan \theta = -1$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos \theta = 1 - \sin \theta$, $\sin \theta + \cos \theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right)$, $\left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right)$, $(0, 0)$

19. $r = \frac{\theta}{2}$

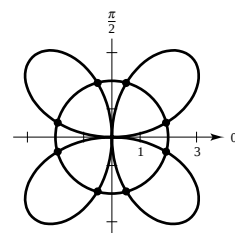
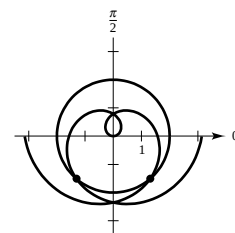
$r = 2$

Solving simultaneously, we have

$$\theta/2 = 2, \theta = 4.$$

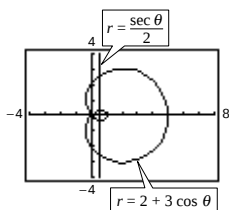
Points of intersection:

$$(2, 4), (-2, -4)$$



23. $r = 2 + 3 \cos \theta$

$$r = \frac{\sec \theta}{2}$$



The graph of $r = 2 + 3 \cos \theta$ is a limaçon with an inner loop ($b > a$) and is symmetric to the polar axis. The graph of $r = (\sec \theta)/2$ is the vertical line $x = 1/2$. Therefore, there are four points of intersection. Solving simultaneously,

$$2 + 3 \cos \theta = \frac{\sec \theta}{2}$$

$$6 \cos^2 \theta + 4 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection: $(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

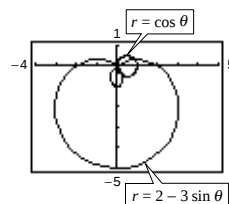
25. $r = \cos \theta$

$$r = 2 - 3 \sin \theta$$

Points of intersection:

$$(0, 0), (0.935, 0.363), (0.535, -1.006)$$

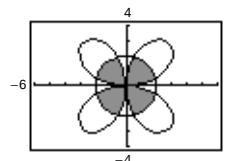
The graphs reach the pole at different times (θ values).



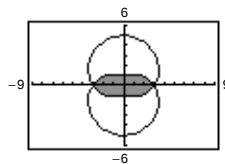
27. From Exercise 21, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/12} (4 \sin 2\theta)^2 d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2)^2 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} (4 \sin 2\theta)^2 d\theta \\ &= 16 \int_0^{\pi/12} \sin^2(2\theta) d\theta + 2 \int_{\pi/12}^{5\pi/12} d\theta \quad (\text{by symmetry of the petal}) \\ &= 8 \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/12} + \left[2\theta \right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}. \end{aligned}$$

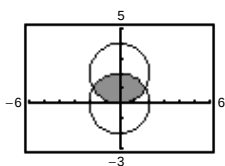
$$\text{Total area} = 4 \left(\frac{4\pi}{3} - \sqrt{3} \right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}(4\pi - 3\sqrt{3})$$



$$\begin{aligned} 29. A &= 4 \left[\frac{1}{2} \int_0^{\pi/2} (3 - 2 \sin \theta)^2 d\theta \right] \\ &= 2 \left[11\theta + 12 \cos \theta - \sin(2\theta) \right]_0^{\pi/2} = 11\pi - 24 \end{aligned}$$

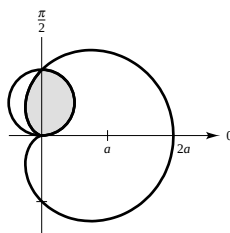


$$\begin{aligned} 31. A &= 2 \left[\frac{1}{2} \int_0^{\pi/6} (4 \sin \theta)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} (2)^2 d\theta \right] \\ &= 16 \left[\frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/6} + \left[4\theta \right]_{\pi/6}^{\pi/2} \\ &= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}(4\pi - 3\sqrt{3}) \end{aligned}$$



$$\begin{aligned} 33. A &= 2 \left[\frac{1}{2} \int_0^{\pi} [a(1 + \cos \theta)]^2 d\theta \right] - \frac{a^2\pi}{4} \\ &= a^2 \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\pi} - \frac{a^2\pi}{4} \\ &= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4} \end{aligned}$$

$$\begin{aligned}
 35. A &= \frac{\pi a^2}{8} + \frac{1}{2} \int_{\pi/2}^{\pi} [a(1 + \cos \theta)]^2 d\theta \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{\cos 2\theta}{2} \right) d\theta \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_{\pi/2}^{\pi} \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2 \right] = \frac{a^2}{2} [\pi - 2]
 \end{aligned}$$

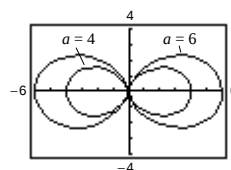


$$37. (a) \quad r = a \cos^2 \theta$$

$$r^3 = ar^2 \cos^2 \theta$$

$$(x^2 + y^2)^{3/2} = ax^2$$

(b)



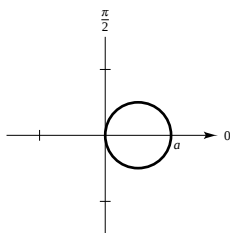
$$\begin{aligned}
 (c) \quad A &= 4 \left(\frac{1}{2} \right) \int_0^{\pi/2} [(6 \cos^2 \theta)^2 - (4 \cos^2 \theta)^2] d\theta = 40 \int_0^{\pi/2} \cos^4 \theta d\theta = 10 \int_0^{\pi/2} (1 + \cos 2\theta)^2 d\theta \\
 &= 10 \int_0^{\pi/2} \left(1 + 2 \cos 2\theta + \frac{1 - \cos 4\theta}{2} \right) d\theta = 10 \left[\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi/2} = \frac{15\pi}{2}
 \end{aligned}$$

$$39. \quad r = a \cos(n\theta)$$

For $n = 1$:

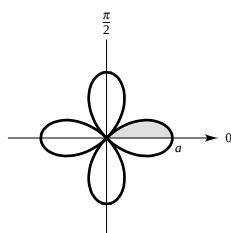
$$r = a \cos \theta$$

$$A = \pi \left(\frac{a}{2} \right)^2 = \frac{\pi a^2}{4}$$

For $n = 2$:

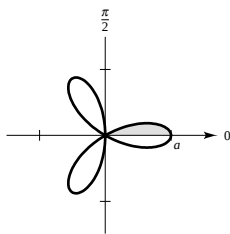
$$r = a \cos 2\theta$$

$$A = 8 \left(\frac{1}{2} \right) \int_0^{\pi/4} (a \cos 2\theta)^2 d\theta = \frac{\pi a^2}{2}$$

For $n = 3$:

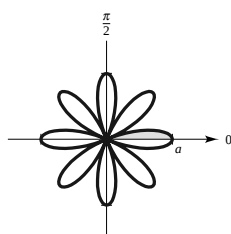
$$r = a \cos 3\theta$$

$$A = 6 \left(\frac{1}{2} \right) \int_0^{\pi/6} (a \cos 3\theta)^2 d\theta = \frac{\pi a^2}{4}$$

For $n = 4$:

$$r = a \cos 4\theta$$

$$A = 16 \left(\frac{1}{2} \right) \int_0^{\pi/8} (a \cos 4\theta)^2 d\theta = \frac{\pi a^2}{2}$$



In general, the area of the region enclosed by $r = a \cos(n\theta)$ for $n = 1, 2, 3, \dots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

41. $r = a$

$$r' = 0$$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2} d\theta = \left[a\theta \right]_0^{2\pi} = 2\pi a$$

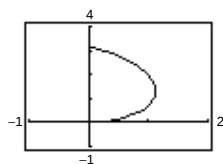
(circumference of circle of radius a)

43. $r = 1 + \sin \theta$

$$r' = \cos \theta$$

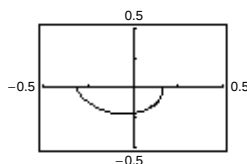
$$\begin{aligned} s &= 2 \int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin \theta)^2 + (\cos \theta)^2} d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin \theta} d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{-\cos \theta}{\sqrt{1 - \sin \theta}} d\theta \\ &= \left[4\sqrt{2} \sqrt{1 - \sin \theta} \right]_{\pi/2}^{3\pi/2} \\ &= 4\sqrt{2} (\sqrt{2} - 0) = 8 \end{aligned}$$

45. $r = 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$



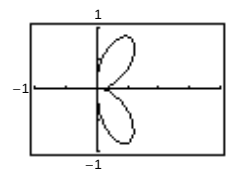
Length ≈ 4.16

47. $r = \frac{1}{\theta}, \pi \leq \theta \leq 2\pi$



Length ≈ 0.71

49. $r = \sin(3 \cos \theta), 0 \leq \theta \leq \pi$



Length ≈ 4.39

51. $r = 6 \cos \theta$

$$r' = -6 \sin \theta$$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} 6 \cos \theta \sin \theta \sqrt{36 \cos^2 \theta + 36 \sin^2 \theta} d\theta \\ &= 72\pi \int_0^{\pi/2} \sin \theta \cos \theta d\theta \\ &= \left[36\pi \sin^2 \theta \right]_0^{\pi/2} \\ &= 36\pi \end{aligned}$$

53. $r = e^{a\theta}$

$$r' = ae^{a\theta}$$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} e^{a\theta} \cos \theta \sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2} d\theta \\ &= 2\pi \sqrt{1 + a^2} \int_0^{\pi/2} e^{2a\theta} \cos \theta d\theta \\ &= 2\pi \sqrt{1 + a^2} \left[\frac{e^{2a\theta}}{4a^2 + 1} (2a \cos \theta + \sin \theta) \right]_0^{\pi/2} \\ &= \frac{2\pi \sqrt{1 + a^2}}{4a^2 + 1} (e^{\pi a} - 2a) \end{aligned}$$

55. $r = 4 \cos 2\theta$

$$r' = -8 \sin 2\theta$$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/4} 4 \cos 2\theta \sin \theta \sqrt{16 \cos^2 2\theta + 64 \sin^2 2\theta} d\theta \\ &= 32\pi \int_0^{\pi/4} \cos 2\theta \sin \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta \approx 21.87 \end{aligned}$$

57. $\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

$$\text{Arc length} = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta$$

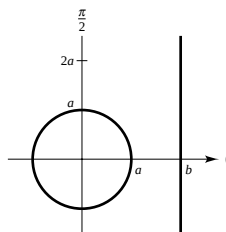
59. (a) is correct: $s \approx 33.124$.

61. Revolve
- $r = a$
- about the line
- $r = b \sec \theta$
- where
- $b > a > 0$
- .

$$f(\theta) = a$$

$$f'(\theta) = 0$$

$$\begin{aligned} S &= 2\pi \int_0^{2\pi} [b - a \cos \theta] \sqrt{a^2 + 0^2} d\theta \\ &= 2\pi a \left[b\theta - a \sin \theta \right]_0^{2\pi} \\ &= 2\pi a(2\pi b) = 4\pi^2 ab \end{aligned}$$



63. False.
- $f(\theta) = 1$
- and
- $g(\theta) = -1$
- have the same graphs.

65. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Using θ instead of t , we have $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$. Thus,

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \text{ and } \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

$$\text{Therefore, } s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$$

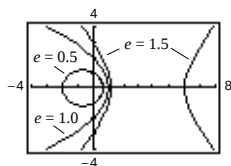
Section 9.6 Polar Equations of Conics and Kepler's Laws

$$1. r = \frac{2e}{1 + e \cos \theta}$$

$$(a) e = 1, r = \frac{2}{1 + \cos \theta}, \text{ parabola}$$

$$(b) e = 0.5, r = \frac{1}{1 + 0.5 \cos \theta} = \frac{2}{2 + \cos \theta}, \text{ ellipse}$$

$$(c) e = 1.5, r = \frac{3}{1 + 1.5 \cos \theta} = \frac{6}{2 + 3 \cos \theta}, \text{ hyperbola}$$

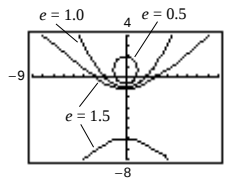


$$3. r = \frac{2e}{1 - e \sin \theta}$$

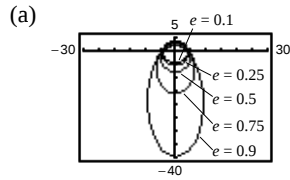
$$(a) e = 1, r = \frac{2}{1 - \sin \theta}, \text{ parabola}$$

$$(b) e = 0.5, r = \frac{1}{1 - 0.5 \sin \theta} = \frac{2}{2 - \sin \theta}, \text{ ellipse}$$

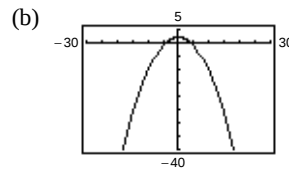
$$(c) e = 1.5, r = \frac{3}{1 - 1.5 \sin \theta} = \frac{6}{2 - 3 \sin \theta}, \text{ hyperbola}$$



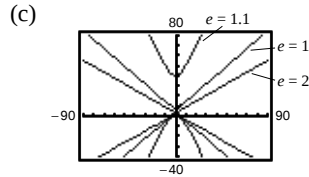
$$5. r = \frac{4}{1 + e \sin \theta}$$



The conic is an ellipse. As $e \rightarrow 1^-$, the ellipse becomes more elliptical, and as $e \rightarrow 0^+$, it becomes more circular.



The conic is a parabola.



The conic is a hyperbola. As $e \rightarrow 1^+$, the hyperbolas open more slowly, and as $e \rightarrow \infty$, they open more rapidly.

7. Parabola; Matches (c)

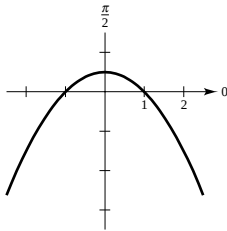
9. Hyperbola; Matches (a)

11. Ellipse; Matches (b)

$$13. r = \frac{-1}{1 - \sin \theta}$$

Parabola since $e = 1$

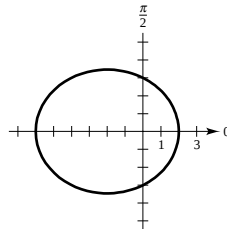
Vertex: $\left(-\frac{1}{2}, \frac{3\pi}{2}\right)$



$$15. r = \frac{6}{2 + \cos \theta} = \frac{3}{1 + (1/2) \cos \theta}$$

Ellipse since $e = \frac{1}{2} < 1$

Vertices: $(2, 0), (6, \pi)$



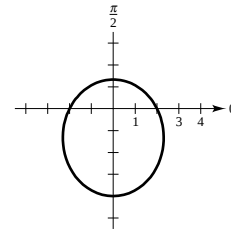
$$17. r(2 + \sin \theta) = 4$$

$$r = \frac{4}{2 + \sin \theta}$$

$$= \frac{2}{1 + (1/2) \sin \theta}$$

Ellipse since $e = \frac{1}{2} < 1$

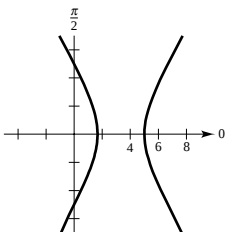
Vertices: $\left(\frac{4}{3}, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$



$$19. r = \frac{5}{-1 + 2 \cos \theta} = \frac{-5}{1 - 2 \cos \theta}$$

Hyperbola since $e = 2 > 1$

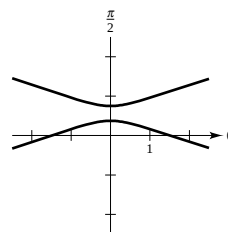
Vertices: $(5, 0), \left(-\frac{5}{3}, \pi\right)$



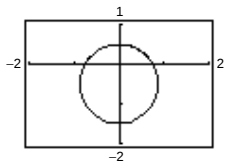
$$21. r = \frac{3}{2 + 6 \sin \theta} = \frac{3/2}{1 + 3 \sin \theta}$$

Hyperbola since $e = 3 > 1$

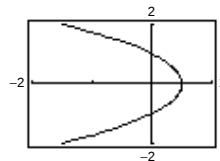
Vertices: $\left(\frac{3}{8}, \frac{\pi}{2}\right), \left(-\frac{3}{4}, \frac{3\pi}{2}\right)$



23. Ellipse



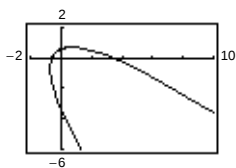
25. Parabola



$$27. r = \frac{-1}{1 - \sin\left(\theta - \frac{\pi}{4}\right)}$$

Rotate the graph of

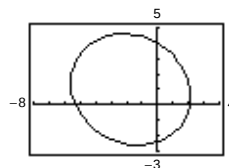
$$r = \frac{-1}{1 - \sin \theta}$$

counterclockwise through the angle $\frac{\pi}{4}$.

$$29. r = \frac{6}{2 + \cos\left(\theta + \frac{\pi}{6}\right)}$$

Rotate the graph of

$$r = \frac{6}{2 + \cos \theta}$$

clockwise through the angle $\frac{\pi}{6}$.

$$31. \text{ Change } \theta \text{ to } \theta + \frac{\pi}{4}; r = \frac{5}{5 + 3 \cos\left(\theta + \frac{\pi}{4}\right)}.$$

33. Parabola

$$e = 1, x = -1, d = 1$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{1}{1 - \cos \theta}$$

35. Ellipse

$$e = \frac{1}{2}, y = 1, d = 1$$

$$r = \frac{ed}{1 + e \sin \theta}$$

$$= \frac{1/2}{1 + (1/2) \sin \theta}$$

$$= \frac{1}{2 + \sin \theta}$$

37. Hyperbola

$$e = 2, x = 1, d = 1$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{2}{1 + 2 \cos \theta}$$

39. Parabola

$$\text{Vertex: } \left(1, -\frac{\pi}{2}\right)$$

$$e = 1, d = 2, r = \frac{2}{1 - \sin \theta}$$

41. Ellipse

Vertices: $(2, 0), (8, \pi)$

$$e = \frac{3}{5}, d = \frac{16}{3}$$

$$r = \frac{ed}{1 + e \cos \theta}$$

$$= \frac{16/5}{1 + (3/5) \cos \theta}$$

$$= \frac{16}{5 + 3 \cos \theta}$$

43. Hyperbola

$$\text{Vertices: } \left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$$

$$e = \frac{5}{4}, d = \frac{9}{5}$$

$$r = \frac{ed}{1 - e \sin \theta}$$

$$= \frac{9/4}{1 - (5/4) \sin \theta}$$

$$= \frac{9}{4 - 5 \sin \theta}$$

45. Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

47. (a) Hyperbola ($e = 2 > 1$)

 (b) Ellipse ($e = \frac{1}{2} < 1$)

 (c) Parabola ($e = 1$)

 (d) Rotated hyperbola ($e = 3$)

49. $a = 5, c = 4, e = \frac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25) \cos^2 \theta}$$

51. $a = 3, b = 4, c = 5, e = \frac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9) \cos^2 \theta}$$

53.
$$A = 2 \left[\frac{1}{2} \int_0^\pi \left(\frac{3}{2 - \cos \theta} \right)^2 d\theta \right]$$

$$= 9 \int_0^\pi \frac{1}{(2 - \cos \theta)^2} d\theta \approx 10.88$$

 55. Vertices: (126,000, 0), (4119, π)

$$a = \frac{126,000 + 4119}{2} = 65,059.5, c = 65,059.5 - 4119 = 60,940.5, e = \frac{c}{a} = \frac{40,627}{43,373}, d = 4119 \left(\frac{84,000}{40,627} \right)$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{4119(84,000/43,373)}{1 - (40,627/43,373) \cos \theta} = \frac{345,996,000}{43,373 - 40,627 \cos \theta}$$

$$\text{When } \theta = 60^\circ, r = \frac{345,996,000}{23,059.5} \approx 15,004.49.$$

 Distance between the surface of the earth and the satellite is $r - 4000 = 11,004.49$ miles.

 57. $a = 92.957 \times 10^6$ mi, $e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{92,931,075.2223}{1 - 0.0167 \cos \theta}$$

 Perihelion distance: $a(1 - e) \approx 91,404,618$ mi

 Aphelion distance: $a(1 + e) \approx 94,509,382$ mi

 59. $a = 5.900 \times 10^9$ km, $e = 0.2481$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta}$$

 Perihelion distance: $a(1 - e) = 4.436 \times 10^9$ km

 Aphelion distance: $a(1 + e) = 7.364 \times 10^9$ km

61. $r = \frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta}$

(a) $A = \frac{1}{2} \int_0^{\pi/9} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \approx 9.341 \times 10^{18} \text{ km}^2$

$$248 \left[\frac{1}{2} \int_0^{\pi/9} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \right] \approx 21.867 \text{ yr}$$

$$\left[\frac{1}{2} \int_0^{2\pi} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \right] \approx 21.867 \text{ yr}$$

(b) $\frac{1}{2} \int_\pi^{\alpha - \pi} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta = 9.341 \times 10^{18}$

$$\alpha \approx \pi + 0.8995 \text{ rad}$$

In part (a) the ray swept through a smaller angle to generate the same area since the length of the ray is longer than in part (b).

(c) $r' = \frac{(-5.537 \times 10^9)(0.2481 \sin \theta)}{(1 - 0.2481 \cos \theta)^2}$

$$s = \int_0^{\pi/9} \sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right)^2 + \left[\frac{-1.3737297 \times 10^9 \sin \theta}{(1 - 0.2481 \cos \theta)^2} \right]^2} d\theta \approx 2.559 \times 10^9 \text{ km}$$

$$\frac{2.559 \times 10^9 \text{ km}}{21.867 \text{ yr}} \approx 1.17 \times 10^8 \text{ km/yr}$$

$$s = \int_\pi^{\pi + 0.899} \sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right)^2 + \left[\frac{-1.3737297 \times 10^9 \sin \theta}{(1 - 0.2481 \cos \theta)^2} \right]^2} d\theta \approx 4.119 \times 10^9 \text{ km}$$

$$\frac{4.119 \times 10^9 \text{ km}}{21.867 \text{ yr}} \approx 1.88 \times 10^8 \text{ km/yr}$$

$$63. r_1 = \frac{ed}{1 + \sin \theta} \text{ and } r_2 = \frac{ed}{1 - \sin \theta}$$

Points of intersection: $(ed, 0), (ed, \pi)$

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin \theta}\right)(\cos \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 + \sin \theta}\right)(\sin \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0), \frac{dy}{dx} = -1$. At $(ed, \pi), \frac{dy}{dx} = 1$.

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin \theta}\right)(\cos \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 - \sin \theta}\right)(\sin \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0), \frac{dy}{dx} = 1$. At $(ed, \pi), \frac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

$$5. 16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

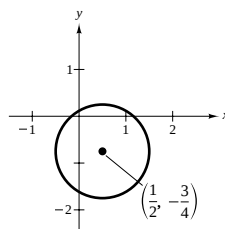
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



3. Matches (a) - parabola

$$7. 3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

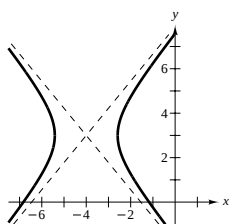
$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $(-4 \pm \sqrt{2}, 3)$

Asymptotes: $y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$



$$9. 3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

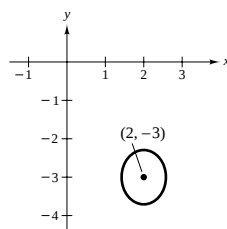
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



11. Vertex: (0, 2)

Directrix: $x = -3$

Parabola opens to the right

 $p = 3$

$$(y - 2)^2 = 4(3)(x - 0)$$

$$y^2 - 4y - 12x + 4 = 0$$

13. Vertices: $(-3, 0), (7, 0)$ Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$$a = 5, c = 2, b = \sqrt{21}$$

$$\frac{(x - 2)^2}{25} + \frac{y^2}{21} = 1$$

15. Vertices: $(\pm 4, 0)$ Foci: $(\pm 6, 0)$ Center: $(0, 0)$

Horizontal transverse axis

$$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

$$17. \frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$$

By Example 5 of Section 9.1,

$$C = 12 \int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right) \sin^2 \theta} d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1 .

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

$$\text{Perpendicular line: } y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$$

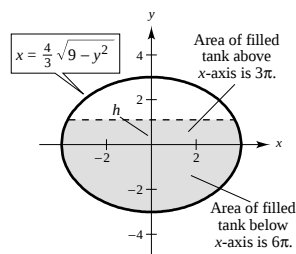
$$4x + 4y - 7 = 0$$

$$21. (a) V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$$

$$\begin{aligned} (b) F &= 2(62.4) \int_{-3}^3 (3 - y) \frac{4}{3} \sqrt{9 - y^2} dy = \frac{8}{3}(62.4) \left[3 \int_{-3}^3 \sqrt{9 - y^2} dy - \int_{-3}^3 y \sqrt{9 - y^2} dy \right] \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(y \sqrt{9 - y^2} + 9 \arcsin \frac{y}{3} \right) + \frac{1}{3} (9 - y^2)^{3/2} \right]_{-3}^3 \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(\frac{9\pi}{2} \right) - \frac{3}{2} \left(-\frac{9\pi}{2} \right) \right] = \frac{8}{3}(62.4) \left(\frac{27\pi}{2} \right) \approx 7057.274 \end{aligned}$$

(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$\begin{aligned} 2 \int_0^h \frac{4}{3} \sqrt{9 - y^2} dy &= 3\pi \\ \int_0^h \sqrt{9 - y^2} dy &= \frac{9\pi}{8} \\ \frac{1}{2} \left[y \sqrt{9 - y^2} + 9 \arcsin \left(\frac{y}{3} \right) \right]_0^h &= \frac{9\pi}{8} \\ h \sqrt{9 - h^2} + 9 \arcsin \left(\frac{h}{3} \right) &= \frac{9\pi}{4}. \end{aligned}$$

By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

$$(d) \text{Area of ends} = 2(12\pi) = 24\pi$$

Area of sides = (Perimeter)(Length)

$$\begin{aligned} &= 16 \int_0^{\pi/2} \left(\sqrt{1 - \left(\frac{7}{16}\right) \sin^2 \theta} \right) d\theta (16) \text{ [from Example 5 of Section 9.1]} \\ &\approx 256 \left(\frac{\pi/2}{12} \right) \left[\sqrt{1 - \left(\frac{7}{16}\right) \sin^2(0)} + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{8}\right)} + 2 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{4}\right)} \right. \\ &\quad \left. + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{2}\right)} \right] \approx 353.65 \end{aligned}$$

$$\text{Total area} = 24\pi + 353.65 \approx 429.05$$

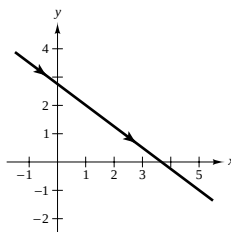
23. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

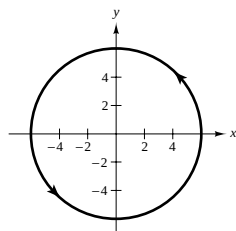


25. $x = 6 \cos \theta, y = 6 \sin \theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

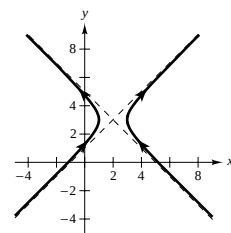


27. $x = 2 + \sec \theta, y = 3 + \tan \theta$

$$(x-2)^2 = \sec^2 \theta = 1 + \tan^2 \theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$$y = 2 + (2 - 6)t = 2 - 4t$$

(other answers possible)

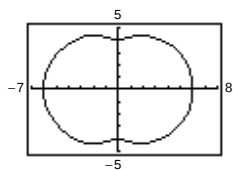
31. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$

$$\text{Let } \frac{(x+3)^2}{16} = \cos^2 \theta \text{ and } \frac{(y-4)^2}{9} = \sin^2 \theta.$$

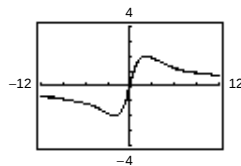
$$\text{Then } x = -3 + 4 \cos \theta \text{ and } y = 4 + 3 \sin \theta.$$

33. $x = \cos 3\theta + 5 \cos \theta$

$$y = \sin 3\theta + 5 \sin \theta$$



35. (a) $x = 2 \cot \theta, y = 4 \sin \theta \cos \theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4 \cot^2 \theta)4 \sin \theta \cos \theta$

$$= 16 \csc^2 \theta \cdot \sin \theta \cdot \cos \theta$$

$$= 16 \frac{\cos \theta}{\sin \theta}$$

$$= 8(2 \cot \theta)$$

$$= 8x$$

37. $x = 1 + 4t$

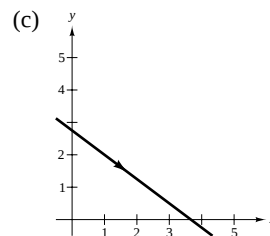
$$y = 2 - 3t$$

(a) $\frac{dy}{dx} = -\frac{3}{4}$

No horizontal tangents

(b) $t = \frac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$



39. $x = \frac{1}{t}$

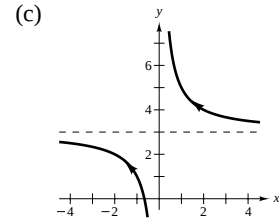
$y = 2t + 3$

(a) $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

No horizontal tangents
($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{2}{x} + 3$



41. $x = \frac{1}{2t + 1}$

$y = \frac{1}{t^2 - 2t}$

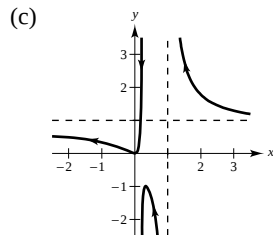
(a) $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(\frac{1}{3}, -1)$

(b) $2t + 1 = \frac{1}{x} \Rightarrow t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1}{x} - 1\right)\left[\frac{1}{2}\left(\frac{1}{x} - 1\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$



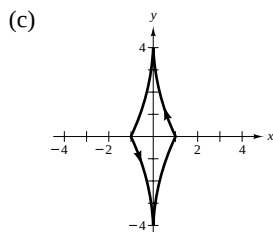
45. $x = \cos^3 \theta$

$y = 4 \sin^3 \theta$

(a) $\frac{dy}{dx} = \frac{12 \sin^2 \theta \cos \theta}{3 \cos^2 \theta (-\sin \theta)} = \frac{-4 \sin \theta}{\cos \theta} = -4 \tan \theta = 0$ when $\theta = 0, \pi$.

But, $\frac{dy}{dt} = \frac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\frac{y}{4}\right)^{2/3} = 1$



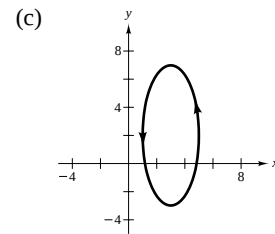
43. $x = 3 + 2 \cos \theta$

$y = 2 + 5 \sin \theta$

(a) $\frac{dy}{dx} = \frac{5 \cos \theta}{-2 \sin \theta} = -2.5 \cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

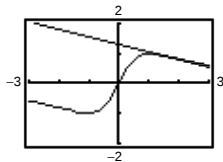
(b) $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{25} = 1$



47. $x = \cot \theta$

$$y = \sin 2\theta = 2 \sin \theta \cos \theta$$

(a), (c)



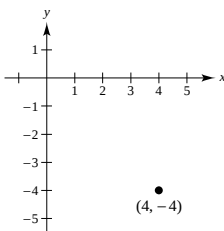
(b) At $\theta = \frac{\pi}{6}$, $\frac{dx}{d\theta} = -4$, $\frac{dy}{d\theta} = 1$, and $\frac{dy}{dx} = -\frac{1}{4}$

51. $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = 7\frac{\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$



53. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

55. $r = -2(1 + \cos \theta)$

$$r^2 = -2r(1 + \cos \theta)$$

$$x^2 + y^2 = -2(\pm\sqrt{x^2 + y^2}) - 2x$$

$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

57. $r^2 = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

59. $r = 4 \cos 2\theta \sec \theta$

$$= 4(2 \cos^2 \theta - 1) \left(\frac{1}{\cos \theta} \right)$$

$$r \cos \theta = 8 \cos^2 \theta - 4$$

$$x = 8 \left(\frac{x^2}{x^2 + y^2} \right) - 4$$

$$x^3 + xy^2 = 4x^2 - 4y^2$$

$$y^2 = x^2 \left(\frac{4-x}{4+x} \right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2 \cos^2 \theta)(r \sin \theta)$$

$$r = a \cos^2 \theta \sin \theta$$

63. $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x} \right)^2$

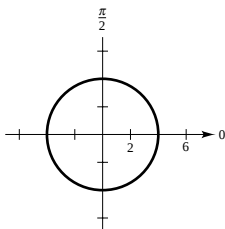
$$r^2 = a^2 \theta^2$$

65. $r = 4$

Circle of radius 4

Centered at the pole

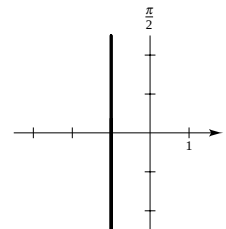
Symmetric to polar axis,

 $\theta = \pi/2$, and pole

67. $r = -\sec \theta = \frac{-1}{\cos \theta}$

$$r \cos \theta = -1, x = -1$$

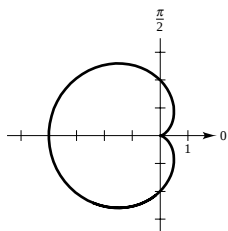
Vertical line



69. $r = -2(1 + \cos \theta)$

Cardioid

Symmetric to polar axis

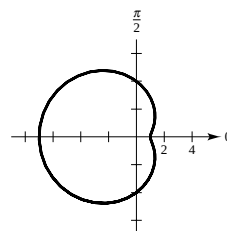


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3 \cos \theta$

Limaçon

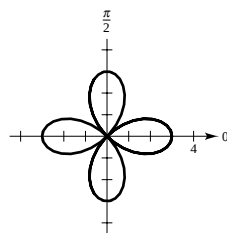
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

73. $r = -3 \cos(2\theta)$

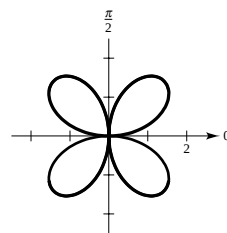
Rose curve with four petals

Symmetric to polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(-3, 0)$, $(3, \frac{\pi}{2})$, $(-3, \pi)$, $(3, \frac{3\pi}{2})$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 

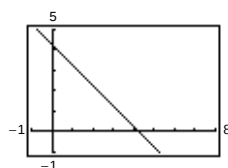
75. $r^2 = 4 \sin^2(2\theta)$

$r = \pm 2 \sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, \frac{\pi}{4})$, $(\pm 2, \frac{3\pi}{4})$ Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

77. $r = \frac{3}{\cos[\theta - (\pi/4)]}$

Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$ 

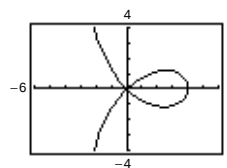
79. $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow -\infty$ as $\theta \Rightarrow \frac{\pi^-}{2}$

$r \Rightarrow -\infty$ as $\theta \Rightarrow \frac{-\pi^+}{2}$



81. $r = 1 - 2 \cos \theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

$$(b) \frac{dy}{dx} = \frac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$$

$$\text{Horizontal tangents: } -4 \cos^2 \theta + \cos \theta + 2 = 0, \cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\text{When } \cos \theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2 \left(\frac{1 \pm \sqrt{33}}{8} \right) = \frac{3 \mp \sqrt{33}}{4},$$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, -2.206).$$

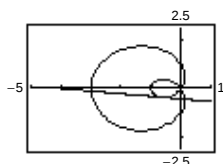
Vertical tangents:

$$\sin \theta (4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm \arccos \left(\frac{1}{4} \right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm \arccos \frac{1}{4} \right) \approx (0.5, \pm 1.318)$$

(c)



83. Circle: $r = 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + 3 \sin \theta \cos \theta}{3 \cos \theta \cos \theta - 3 \sin \theta \sin \theta} = \frac{\sin 2\theta}{\cos^2 \theta - \sin^2 \theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5 \sin \theta$

$$\frac{dy}{dx} = \frac{-5 \cos \theta \sin \theta + (4 - 5 \sin \theta) \cos \theta}{-5 \cos \theta \cos \theta - (4 - 5 \sin \theta) \sin \theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan \alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

$$\text{Therefore, } \alpha = \arctan \left(\frac{2\sqrt{3}}{3} \right) \approx 49.1^\circ.$$

85. $r = 1 + \cos \theta, r = 1 - \cos \theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos \theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos \theta(1 + \cos \theta)}{-\sin \theta \cos \theta - \sin \theta(1 + \cos \theta)}.$$

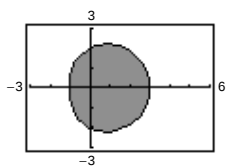
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos \theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2 \theta + \cos \theta(1 - \cos \theta)}{\sin \theta \cos \theta - \sin \theta(1 - \cos \theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

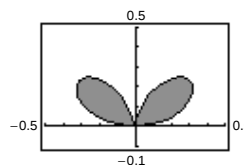
87. $r = 2 + \cos \theta$

$$A = 2 \left[\frac{1}{2} \int_0^\pi (2 + \cos \theta)^2 d\theta \right] \approx 14.14 \quad \left(\frac{9\pi}{2} \right)$$



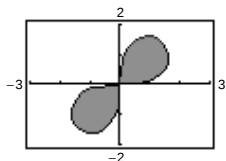
89. $r = \sin \theta \cdot \cos^2 \theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} (\sin \theta \cos^2 \theta)^2 d\theta \right] \approx 0.10 \quad \left(\frac{\pi}{32} \right)$$



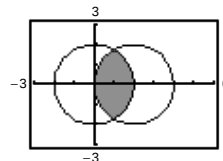
91. $r^2 = 4 \sin 2\theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} 4 \sin 2\theta d\theta \right] = 4$$



93. $r = 4 \cos \theta, r = 2$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/3} 4 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4 \cos \theta)^2 d\theta \right] \approx 4.91$$

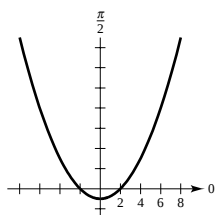


95. $s = 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \cos \theta} d\theta = 2\sqrt{2} a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta = \left[-4\sqrt{2} a(1 + \cos \theta)^{1/2} \right]_0^\pi = 8a$$

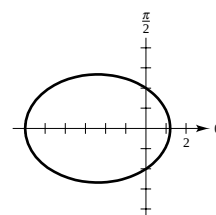
97. $r = \frac{2}{1 - \sin \theta}, e = 1$

Parabola



99. $r = \frac{6}{3 + 2 \cos \theta} = \frac{2}{1 + (2/3) \cos \theta}, e = \frac{2}{3}$

Ellipse



$$63. r_1 = \frac{ed}{1 + \sin \theta} \text{ and } r_2 = \frac{ed}{1 - \sin \theta}$$

Points of intersection: $(ed, 0), (ed, \pi)$

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin \theta}\right)(\cos \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 + \sin \theta}\right)(\sin \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0)$, $\frac{dy}{dx} = -1$. At (ed, π) , $\frac{dy}{dx} = 1$.

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin \theta}\right)(\cos \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 - \sin \theta}\right)(\sin \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0)$, $\frac{dy}{dx} = 1$. At (ed, π) , $\frac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

$$5. 16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

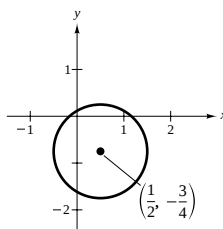
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



3. Matches (a) - parabola

$$7. 3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

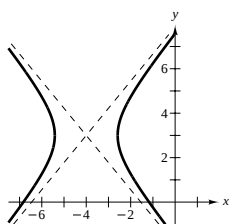
$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $(-4 \pm \sqrt{2}, 3)$

Asymptotes: $y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$



$$9. 3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

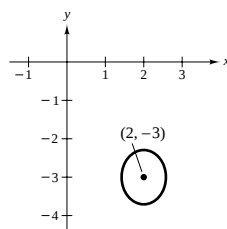
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



11. Vertex: (0, 2)

Directrix: $x = -3$

Parabola opens to the right

 $p = 3$

$$(y - 2)^2 = 4(3)(x - 0)$$

$$y^2 - 4y - 12x + 4 = 0$$

13. Vertices: $(-3, 0), (7, 0)$ Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$$a = 5, c = 2, b = \sqrt{21}$$

$$\frac{(x - 2)^2}{25} + \frac{y^2}{21} = 1$$

15. Vertices: $(\pm 4, 0)$ Foci: $(\pm 6, 0)$ Center: $(0, 0)$

Horizontal transverse axis

$$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

$$17. \frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$$

By Example 5 of Section 9.1,

$$C = 12 \int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right) \sin^2 \theta} d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1 .

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

$$\text{Perpendicular line: } y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$$

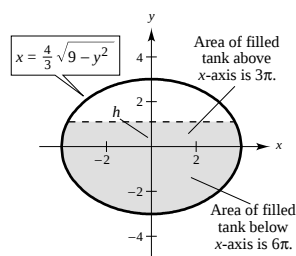
$$4x + 4y - 7 = 0$$

$$21. (a) V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$$

$$\begin{aligned} (b) F &= 2(62.4) \int_{-3}^3 (3 - y) \frac{4}{3} \sqrt{9 - y^2} dy = \frac{8}{3}(62.4) \left[3 \int_{-3}^3 \sqrt{9 - y^2} dy - \int_{-3}^3 y \sqrt{9 - y^2} dy \right] \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(y \sqrt{9 - y^2} + 9 \arcsin \frac{y}{3} \right) + \frac{1}{3} (9 - y^2)^{3/2} \right]_{-3}^3 \\ &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(\frac{9\pi}{2} \right) - \frac{3}{2} \left(-\frac{9\pi}{2} \right) \right] = \frac{8}{3}(62.4) \left(\frac{27\pi}{2} \right) \approx 7057.274 \end{aligned}$$

(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$\begin{aligned} 2 \int_0^h \frac{4}{3} \sqrt{9 - y^2} dy &= 3\pi \\ \int_0^h \sqrt{9 - y^2} dy &= \frac{9\pi}{8} \\ \frac{1}{2} \left[y \sqrt{9 - y^2} + 9 \arcsin \left(\frac{y}{3} \right) \right]_0^h &= \frac{9\pi}{8} \\ h \sqrt{9 - h^2} + 9 \arcsin \left(\frac{h}{3} \right) &= \frac{9\pi}{4}. \end{aligned}$$

By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

$$(d) \text{Area of ends} = 2(12\pi) = 24\pi$$

Area of sides = (Perimeter)(Length)

$$\begin{aligned} &= 16 \int_0^{\pi/2} \left(\sqrt{1 - \left(\frac{7}{16}\right) \sin^2 \theta} \right) d\theta (16) \text{ [from Example 5 of Section 9.1]} \\ &\approx 256 \left(\frac{\pi/2}{12} \right) \left[\sqrt{1 - \left(\frac{7}{16}\right) \sin^2(0)} + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{8}\right)} + 2 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{4}\right)} \right. \\ &\quad \left. + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{2}\right)} \right] \approx 353.65 \end{aligned}$$

$$\text{Total area} = 24\pi + 353.65 \approx 429.05$$

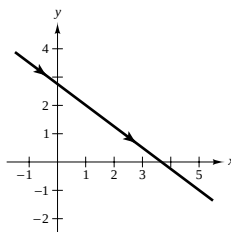
23. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

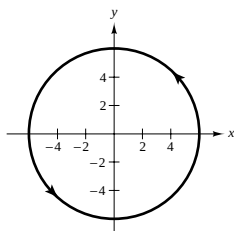


25. $x = 6 \cos \theta, y = 6 \sin \theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

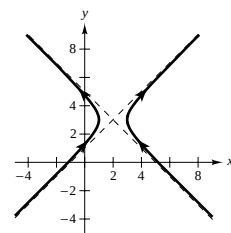


27. $x = 2 + \sec \theta, y = 3 + \tan \theta$

$$(x-2)^2 = \sec^2 \theta = 1 + \tan^2 \theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$$y = 2 + (2 - 6)t = 2 - 4t$$

(other answers possible)

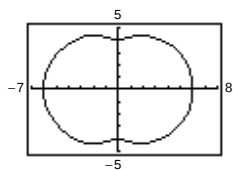
31. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$

$$\text{Let } \frac{(x+3)^2}{16} = \cos^2 \theta \text{ and } \frac{(y-4)^2}{9} = \sin^2 \theta.$$

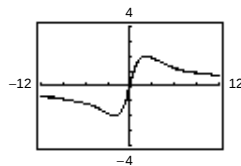
$$\text{Then } x = -3 + 4 \cos \theta \text{ and } y = 4 + 3 \sin \theta.$$

33. $x = \cos 3\theta + 5 \cos \theta$

$$y = \sin 3\theta + 5 \sin \theta$$



35. (a) $x = 2 \cot \theta, y = 4 \sin \theta \cos \theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4 \cot^2 \theta)4 \sin \theta \cos \theta$

$$= 16 \csc^2 \theta \cdot \sin \theta \cdot \cos \theta$$

$$= 16 \frac{\cos \theta}{\sin \theta}$$

$$= 8(2 \cot \theta)$$

$$= 8x$$

37. $x = 1 + 4t$

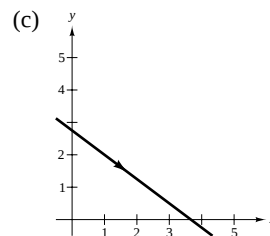
$$y = 2 - 3t$$

(a) $\frac{dy}{dx} = -\frac{3}{4}$

No horizontal tangents

(b) $t = \frac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$



39. $x = \frac{1}{t}$

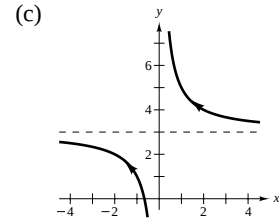
$y = 2t + 3$

(a) $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

No horizontal tangents
($t \neq 0$)

(b) $t = \frac{1}{x}$

$y = \frac{2}{x} + 3$



41. $x = \frac{1}{2t + 1}$

$y = \frac{1}{t^2 - 2t}$

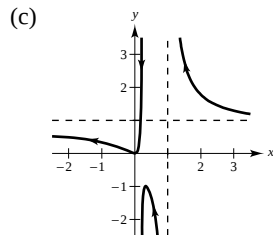
(a) $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

Point of horizontal tangency: $(\frac{1}{3}, -1)$

(b) $2t + 1 = \frac{1}{x} \Rightarrow t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1}{x} - 1\right)\left[\frac{1}{2}\left(\frac{1}{x} - 1\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}$$



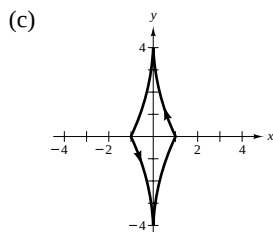
45. $x = \cos^3 \theta$

$y = 4 \sin^3 \theta$

(a) $\frac{dy}{dx} = \frac{12 \sin^2 \theta \cos \theta}{3 \cos^2 \theta (-\sin \theta)} = \frac{-4 \sin \theta}{\cos \theta} = -4 \tan \theta = 0$ when $\theta = 0, \pi$.

But, $\frac{dy}{dt} = \frac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\frac{y}{4}\right)^{2/3} = 1$



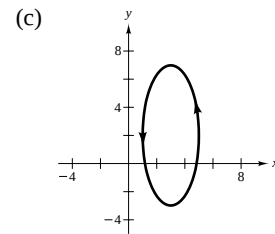
43. $x = 3 + 2 \cos \theta$

$y = 2 + 5 \sin \theta$

(a) $\frac{dy}{dx} = \frac{5 \cos \theta}{-2 \sin \theta} = -2.5 \cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

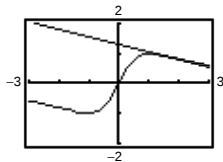
(b) $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{25} = 1$



47. $x = \cot \theta$

$$y = \sin 2\theta = 2 \sin \theta \cos \theta$$

(a), (c)



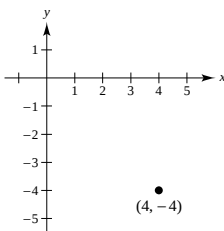
(b) At $\theta = \frac{\pi}{6}$, $\frac{dx}{d\theta} = -4$, $\frac{dy}{d\theta} = 1$, and $\frac{dy}{dx} = -\frac{1}{4}$

51. $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = 7\frac{\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$



53. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

55. $r = -2(1 + \cos \theta)$

$$r^2 = -2r(1 + \cos \theta)$$

$$x^2 + y^2 = -2(\pm\sqrt{x^2 + y^2}) - 2x$$

$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

57. $r^2 = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

59. $r = 4 \cos 2\theta \sec \theta$

$$= 4(2 \cos^2 \theta - 1) \left(\frac{1}{\cos \theta} \right)$$

$$r \cos \theta = 8 \cos^2 \theta - 4$$

$$x = 8 \left(\frac{x^2}{x^2 + y^2} \right) - 4$$

$$x^3 + xy^2 = 4x^2 - 4y^2$$

$$y^2 = x^2 \left(\frac{4 - x}{4 + x} \right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2 \cos^2 \theta)(r \sin \theta)$$

$$r = a \cos^2 \theta \sin \theta$$

63. $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x} \right)^2$

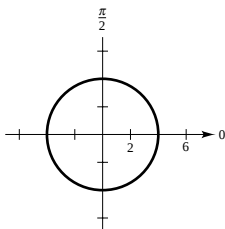
$$r^2 = a^2 \theta^2$$

65. $r = 4$

Circle of radius 4

Centered at the pole

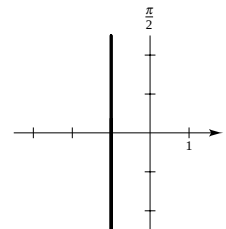
Symmetric to polar axis,

 $\theta = \pi/2$, and pole


67. $r = -\sec \theta = \frac{-1}{\cos \theta}$

$$r \cos \theta = -1, x = -1$$

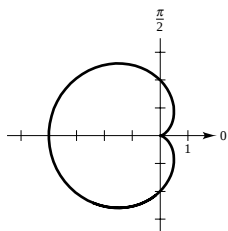
Vertical line



69. $r = -2(1 + \cos \theta)$

Cardioid

Symmetric to polar axis

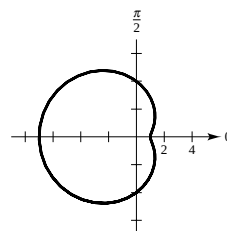


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3 \cos \theta$

Limaçon

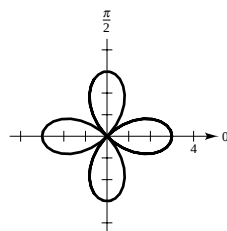
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

73. $r = -3 \cos(2\theta)$

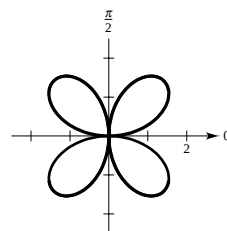
Rose curve with four petals

Symmetric to polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(-3, 0)$, $(3, \frac{\pi}{2})$, $(-3, \pi)$, $(3, \frac{3\pi}{2})$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 

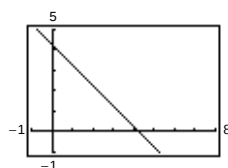
75. $r^2 = 4 \sin^2(2\theta)$

$r = \pm 2 \sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, \frac{\pi}{4})$, $(\pm 2, \frac{3\pi}{4})$ Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

77. $r = \frac{3}{\cos[\theta - (\pi/4)]}$

Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$ 

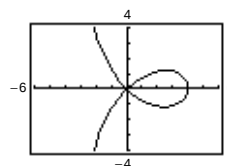
79. $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow -\infty$ as $\theta \Rightarrow \frac{\pi^-}{2}$

$r \Rightarrow -\infty$ as $\theta \Rightarrow \frac{-\pi^+}{2}$



81. $r = 1 - 2 \cos \theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

$$(b) \frac{dy}{dx} = \frac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$$

$$\text{Horizontal tangents: } -4 \cos^2 \theta + \cos \theta + 2 = 0, \cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\text{When } \cos \theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2 \left(\frac{1 \pm \sqrt{33}}{8} \right) = \frac{3 \mp \sqrt{33}}{4},$$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, 2.206)$$

$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, -2.206).$$

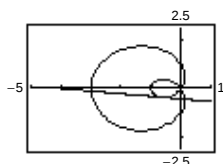
Vertical tangents:

$$\sin \theta (4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm \arccos \left(\frac{1}{4} \right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm \arccos \frac{1}{4} \right) \approx (0.5, \pm 1.318)$$

(c)



83. Circle: $r = 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + 3 \sin \theta \cos \theta}{3 \cos \theta \cos \theta - 3 \sin \theta \sin \theta} = \frac{\sin 2\theta}{\cos^2 \theta - \sin^2 \theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5 \sin \theta$

$$\frac{dy}{dx} = \frac{-5 \cos \theta \sin \theta + (4 - 5 \sin \theta) \cos \theta}{-5 \cos \theta \cos \theta - (4 - 5 \sin \theta) \sin \theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan \alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

$$\text{Therefore, } \alpha = \arctan \left(\frac{2\sqrt{3}}{3} \right) \approx 49.1^\circ.$$

85. $r = 1 + \cos \theta, r = 1 - \cos \theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos \theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos \theta(1 + \cos \theta)}{-\sin \theta \cos \theta - \sin \theta(1 + \cos \theta)}.$$

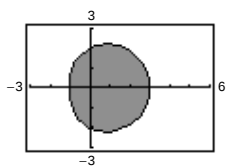
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos \theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2 \theta + \cos \theta(1 - \cos \theta)}{\sin \theta \cos \theta - \sin \theta(1 - \cos \theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

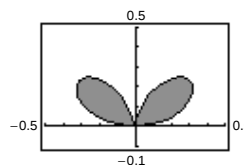
87. $r = 2 + \cos \theta$

$$A = 2 \left[\frac{1}{2} \int_0^\pi (2 + \cos \theta)^2 d\theta \right] \approx 14.14 \quad \left(\frac{9\pi}{2} \right)$$



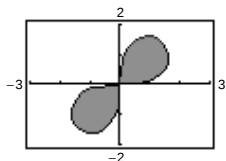
89. $r = \sin \theta \cdot \cos^2 \theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} (\sin \theta \cos^2 \theta)^2 d\theta \right] \approx 0.10 \quad \left(\frac{\pi}{32} \right)$$



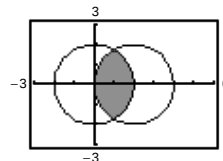
91. $r^2 = 4 \sin 2\theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} 4 \sin 2\theta d\theta \right] = 4$$



93. $r = 4 \cos \theta, r = 2$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/3} 4 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4 \cos \theta)^2 d\theta \right] \approx 4.91$$

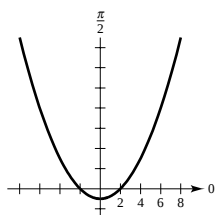


95. $s = 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \cos \theta} d\theta = 2\sqrt{2} a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta = \left[-4\sqrt{2} a(1 + \cos \theta)^{1/2} \right]_0^\pi = 8a$$

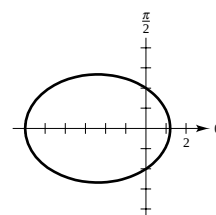
97. $r = \frac{2}{1 - \sin \theta}, e = 1$

Parabola



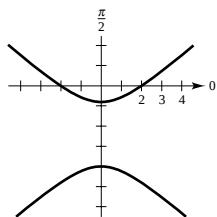
99. $r = \frac{6}{3 + 2 \cos \theta} = \frac{2}{1 + (2/3) \cos \theta}, e = \frac{2}{3}$

Ellipse



$$101. r = \frac{4}{2 - 3 \sin \theta} = \frac{2}{1 - (3/2) \sin \theta}, e = \frac{3}{2}$$

Hyperbola



105. Parabola

Vertex: $(2, \pi)$ Focus: $(0, 0)$

$$e = 1, d = 4$$

$$r = \frac{4}{1 - \cos \theta}$$

103. Circle

Center: $\left(5, \frac{\pi}{2}\right) = (0, 5)$ in rectangular coordinatesSolution point: $(0, 0)$

$$x^2 + (y - 5)^2 = 25$$

$$x^2 + y^2 - 10y = 0$$

$$r^2 - 10r \sin \theta = 0$$

$$r = 10 \sin \theta$$

107. Ellipse

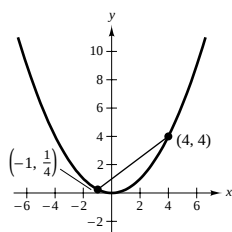
Vertices: $(5, 0), (1, \pi)$ Focus: $(0, 0)$

$$a = 3, c = 2, e = \frac{2}{3}, d = \frac{5}{2}$$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right) \cos \theta} = \frac{5}{3 - 2 \cos \theta}$$

Problem Solving for Chapter 9

1. (a)



$$(b) x^2 = 4y$$

$$2x = 4y'$$

$$y' = \frac{1}{2}x$$

$$y - 4 = 2(x - 4) \Rightarrow y = 2x - 4 \quad \text{Tangent line at } (4, 4)$$

$$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{4} \quad \text{Tangent line at } \left(-1, \frac{1}{4}\right)$$

Tangent lines have slopes of 2 and $-1/2 \Rightarrow$ perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$

$$8x - 16 = -2x - 1$$

$$10x = 15$$

$$x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, -1\right)$$

Point of intersection, $(3/2, -1)$, is on directrix $y = -1$.

3. Consider $x^2 = 4py$ with focus $(0, p)$.

Let $P(a, b)$ be point on parabola.

$$2x = 4py' \Rightarrow y' = \frac{x}{2p}$$

$$y - b = \frac{a}{2p}(x - a) \quad \text{Tangent line}$$

$$\text{For } x = 0, y = b + \frac{a}{2p}(-a) = b - \frac{a^2}{2p} = b - \frac{4pb}{2p} = -b.$$

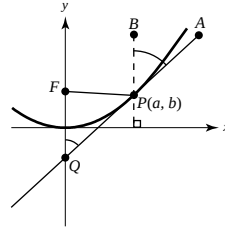
Thus, $Q = (0, -b)$.

$\triangle FQP$ is isosceles because

$$|FQ| = p + b$$

$$\begin{aligned} |FP| &= \sqrt{(a-0)^2 + (b-p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2} \\ &= \sqrt{4pb + b^2 - 2bp + p^2} \\ &= \sqrt{(b+p)^2} \\ &= b + p. \end{aligned}$$

Thus, $\angle FQP = \angle BPA = \angle FPQ$.



5. (a) In $\triangle OCB$, $\cos \theta = \frac{2a}{OB} \Rightarrow OB = 2a \cdot \sec \theta$.

$$\text{In } \triangle OAC, \cos \theta = \frac{OA}{2a} \Rightarrow OA = 2a \cdot \cos \theta.$$

$$\begin{aligned} r = OP = AB = OB - OA &= 2a(\sec \theta - \cos \theta) \\ &= 2a\left(\frac{1}{\cos \theta} - \cos \theta\right) \\ &= 2a \cdot \frac{\sin^2 \theta}{\cos \theta} \\ &= 2a \cdot \tan \theta \sin \theta \end{aligned}$$

$$(b) x = r \cos \theta = (2a \tan \theta \sin \theta) \cos \theta = 2a \sin^2 \theta$$

$$y = r \sin \theta = (2a \tan \theta \sin \theta) \sin \theta = 2a \tan \theta \cdot \sin^2 \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Let $t = \tan \theta$, $-\infty < t < \infty$.

$$\text{Then } \sin^2 \theta = \frac{t^2}{1+t^2} \text{ and } x = 2a \frac{t^2}{1+t^2}, y = 2a \frac{t^3}{1+t^2}.$$

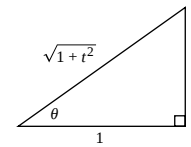
$$(c) \quad r = 2a \tan \theta \sin \theta$$

$$r \cos \theta = 2a \sin^2 \theta$$

$$r^3 \cos \theta = 2a r^2 \sin^2 \theta$$

$$(x^2 + y^2)x = 2ay^2$$

$$y^2 = \frac{x^3}{(2a-x)}$$



7. $y = a(1 - \cos \theta) \Rightarrow \cos \theta = \frac{a-y}{a}$

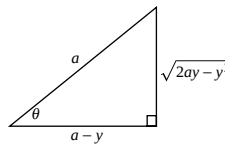
$$\theta = \arccos\left(\frac{a-y}{a}\right)$$

$$x = a(\theta - \sin \theta)$$

$$= a\left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right)\right)$$

$$= a\left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay-y^2}}{a}\right)$$

$$x = a \cdot \arccos\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}, 0 \leq y \leq 2a$$



9. For $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$

$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \dots$$

Hence, the curve has length greater than

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \dots \\ &= \frac{2}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \right) \\ &> \frac{2}{\pi} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \right) \\ &= \infty. \end{aligned}$$

13. If a dog is located at (r, θ) , then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$$(x, y) = (r \cos \theta, r \sin \theta) \text{ and } (x, y) = (-r \sin \theta, r \cos \theta).$$

The slope joining these points is

$$\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \text{slope of tangent line at } (r, \theta).$$

$$\frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r$$

$$\frac{dr}{r} = -d\theta$$

$$\ln r = -\theta + C_1$$

$$r = e^{-\theta + C_1}$$

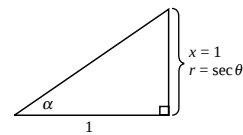
$$r = Ce^{-\theta}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}} e^{\pi/4}$$

$$\text{Finally, } r = \frac{d}{\sqrt{2}} e^{((\pi/4) - \theta)}.$$

11. (a) $\text{Area} = \int_0^\alpha \frac{1}{2} r^2 d\theta$

$$= \frac{1}{2} \int_0^\alpha \sec^2 \theta d\theta$$



(b) $\tan \alpha = \frac{h}{1} \Rightarrow \text{Area} = \frac{1}{2}(1)\tan \alpha$

$$\Rightarrow \tan \alpha = \int_0^\alpha \sec^2 \theta d\theta$$

(c) Differentiating, $\frac{d}{d\alpha}(\tan \alpha) = \sec^2 \alpha$.

15. (a) The first plane makes an angle of 70° with the positive x -axis, and is 150 miles from P :

$$x_1 = \cos 70^\circ(150 - 375t)$$

$$y_1 = \sin 70^\circ(150 - 375t)$$

Similarly for the second plane,

$$x_2 = \cos 135^\circ(190 - 450t)$$

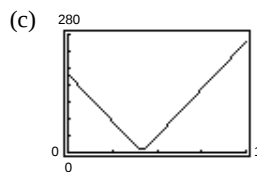
$$= \cos 45^\circ(-190 + 450t)$$

$$y_2 = \sin 135^\circ(190 - 450t)$$

$$= \sin 45^\circ(190 - 450t)$$

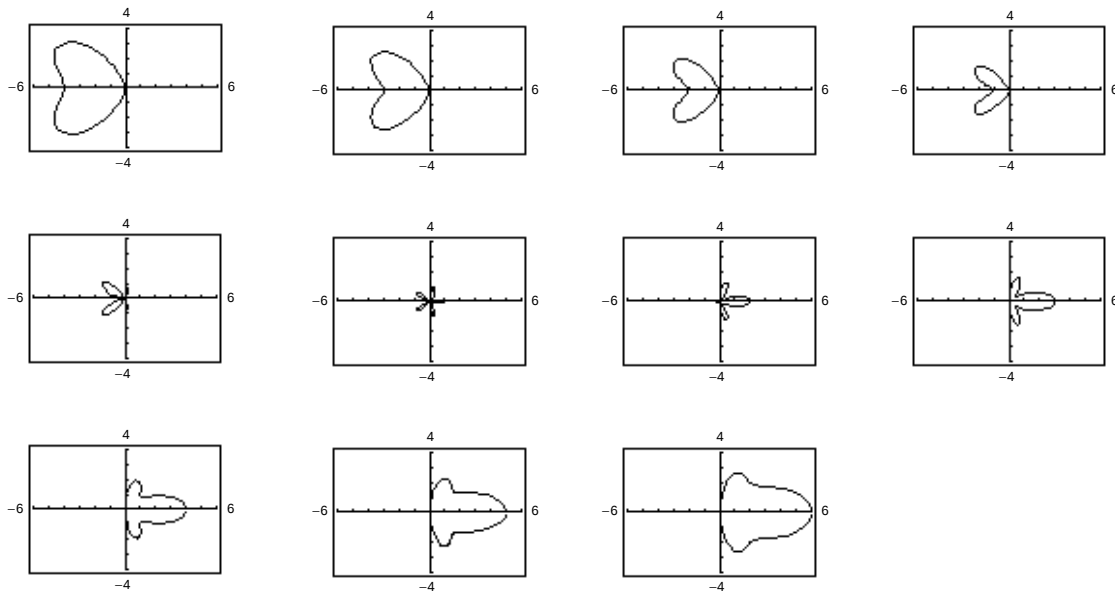
(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= [\cos 45^\circ(-190 + 450t) - \cos 70^\circ(150 - 375t)]^2 + [\sin 45^\circ(190 - 450t) - \sin 70^\circ(150 - 375t)]^2]^{1/2}$$



The minimum distance is 7.59 miles when $t = 0.4145$.

17.



$n = 1, 2, 3, 4, 5$ produce “bells”; $n = -1, -2, -3, -4, -5$ produce “hearts”.

PART I

CHAPTER P Preparation for Calculus

Section P.1	Graphs and Models	2
Section P.2	Linear Models and Rates of Change	7
Section P.3	Functions and Their Graphs	14
Section P.4	Fitting Models to Data	18
	Review Exercises	19
	Problem Solving	23

CHAPTER P

Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Odd-Numbered Exercises

1. $y = -\frac{1}{2}x + 2$

x-intercept: (4, 0)

y-intercept: (0, 2)

Matches graph (b)

3. $y = 4 - x^2$

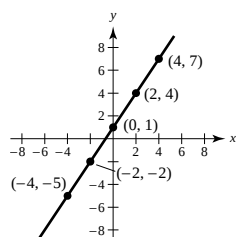
x-intercepts: (2, 0), (-2, 0)

y-intercept: (0, 4)

Matches graph (a)

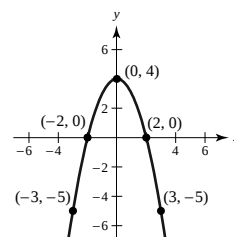
5. $y = \frac{3}{2}x + 1$

x	-4	-2	0	2	4
y	-5	-2	1	4	7



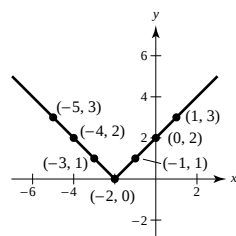
7. $y = 4 - x^2$

x	-3	-2	0	2	3
y	-5	0	4	0	-5



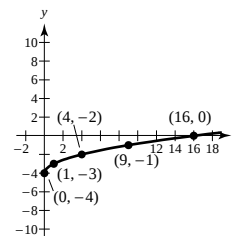
9. $y = |x + 2|$

x	-5	-4	-3	-2	-1	0	1
y	3	2	1	0	1	2	3



11. $y = \sqrt{x} - 4$

x	0	1	4	9	16
y	-4	-3	-2	-1	0



13.
$$\begin{array}{l} \text{Xmin} = -3 \\ \text{Xmax} = 5 \\ \text{Xscl} = 1 \\ \text{Ymin} = -3 \\ \text{Ymax} = 5 \\ \text{Yscl} = 1 \end{array}$$

Note that $y = 4$ when $x = 0$.

17. $y = x^2 + x - 2$

y-intercept: $y = 0^2 + 0 - 2$

$y = -2; (0, -2)$

x-intercepts: $0 = x^2 + x - 2$

$0 = (x + 2)(x - 1)$

$x = -2, 1; (-2, 0), (1, 0)$

21. $y = \frac{3(2 - \sqrt{x})}{x}$

y-intercept: None. x cannot equal 0.

x-intercepts: $0 = \frac{3(2 - \sqrt{x})}{x}$

$0 = 2 - \sqrt{x}$

$x = 4; (4, 0)$

25. Symmetric with respect to the y-axis since

$y = (-x)^2 - 2 = x^2 - 2.$

29. Symmetric with respect to the origin since

$(-x)(-y) = xy = 4.$

33. Symmetric with respect to the origin since

$-y = \frac{-x}{(-x)^2 + 1}$

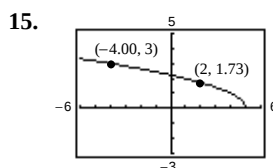
$y = \frac{x}{x^2 + 1}.$

37. $y = -3x + 2$

Intercepts:

$(\frac{2}{3}, 0), (0, 2)$

Symmetry: none



(a) $(2, y) = (2, 1.73) \quad (y = \sqrt{5 - 2} = \sqrt{3} \approx 1.73)$

(b) $(x, 3) = (-4, 3) \quad (3 = \sqrt{5 - (-4)})$

19. $y = x^2 \sqrt{25 - x^2}$

y-intercept: $y = 0^2 \sqrt{25 - 0^2}$

$y = 0; (0, 0)$

x-intercepts: $0 = x^2 \sqrt{25 - x^2}$

$0 = x^2 \sqrt{(5 - x)(5 + x)}$

$x = 0, \pm 5; (0, 0); (\pm 5, 0)$

23. $x^2y - x^2 + 4y = 0$

y-intercept:

$0^2(y) - 0^2 + 4y = 0$

$y = 0; (0, 0)$

x-intercept:

$x^2(0) - x^2 + 4(0) = 0$

$x = 0; (0, 0)$

27. Symmetric with respect to the x-axis since

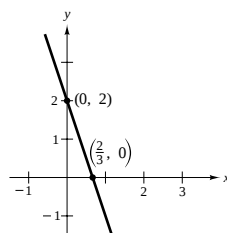
$(-y)^2 = y^2 = x^3 - 4x.$

31. $y = 4 - \sqrt{x + 3}$

No symmetry with respect to either axis or the origin.

35. $y = |x^3 + x|$ is symmetric with respect to the y-axis

since $y = |(-x)^3 + (-x)| = |-(x^3 + x)| = |x^3 + x|.$

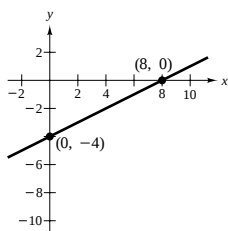


39. $y = \frac{x}{2} - 4$

Intercepts:

$(8, 0), (0, -4)$

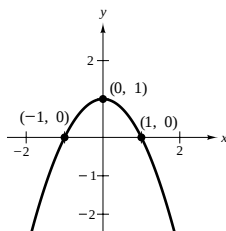
Symmetry: none



41. $y = 1 - x^2$

Intercepts:

$(1, 0), (-1, 0), (0, 1)$

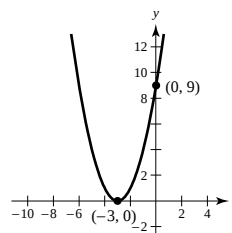
Symmetry: y -axis

43. $y = (x + 3)^2$

Intercepts:

$(-3, 0), (0, 9)$

Symmetry: none

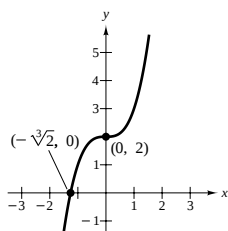


45. $y = x^3 + 2$

Intercepts:

$(-\sqrt[3]{2}, 0), (0, 2)$

Symmetry: none

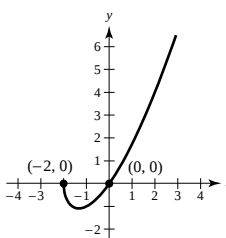


47. $y = x\sqrt{x+2}$

Intercepts:

$(0, 0), (-2, 0)$

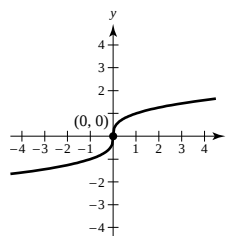
Symmetry: none

Domain: $x \geq -2$ 

49. $x = y^3$

Intercepts: $(0, 0)$

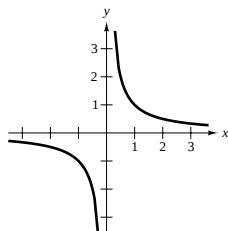
Symmetry: origin



51. $y = \frac{1}{x}$

Intercepts: none

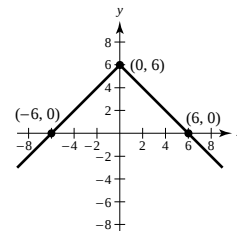
Symmetry: origin



53. $y = 6 - |x|$

Intercepts:

$(0, 6), (-6, 0), (6, 0)$

Symmetry: y -axis

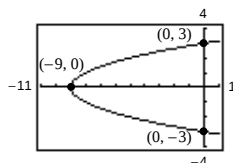
55. $y^2 - x = 9$

$y^2 = x + 9$

$y = \pm\sqrt{x+9}$

Intercepts:

$(0, 3), (0, -3), (-9, 0)$

Symmetry: x -axis

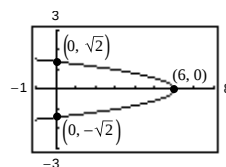
57. $x + 3y^2 = 6$

$3y^2 = 6 - x$

$y = \pm\sqrt{2 - \frac{x}{3}}$

Intercepts:

$(6, 0), (0, \sqrt{2}), (0, -\sqrt{2})$

Symmetry: x -axis

59. $y = (x + 2)(x - 4)(x - 6)$ (other answers possible)

63. $x + y = 2 \Rightarrow y = 2 - x$

$$2x - y = 1 \Rightarrow y = 2x - 1$$

$$2 - x = 2x - 1$$

$$3 = 3x$$

$$1 = x$$

The corresponding y -value is $y = 1$.

Point of intersection: $(1, 1)$

67. $x^2 + y = 6 \Rightarrow y = 6 - x^2$

$$x + y = 4 \Rightarrow y = 4 - x$$

$$6 - x^2 = 4 - x$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2, -1$$

The corresponding y -values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$).

Points of intersection: $(2, 2), (-1, 5)$

71. $y = x^3$

$$y = x$$

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = 0, x = -1, \text{ or } x = 1$$

The corresponding y -values are $y = 0, y = -1$, and $y = 1$.

Points of intersection: $(0, 0), (-1, -1), (1, 1)$

61. Some possible equations:

$$y = x$$

$$y = x^3$$

$$y = 3x^3 - x$$

$$y = \sqrt[3]{x}$$

65. $x + y = 7 \Rightarrow y = 7 - x$

$$3x - 2y = 11 \Rightarrow y = \frac{3x - 11}{2}$$

$$7 - x = \frac{3x - 11}{2}$$

$$14 - 2x = 3x - 11$$

$$-5x = -25$$

$$x = 5$$

The corresponding y -value is $y = 2$.

Point of intersection: $(5, 2)$

69. $x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$$x - y = 1 \Rightarrow y = x - 1$$

$$5 - x^2 = (x - 1)^2$$

$$5 - x^2 = x^2 - 2x + 1$$

$$0 = 2x^2 - 2x - 4 = 2(x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding y -values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

73. $y = x^3 - 2x^2 + x - 1$

$$y = -x^2 + 3x - 1$$

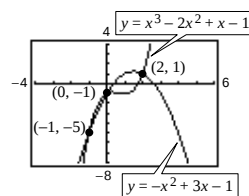
$$x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$$

$$x^3 - x^2 - 2x = 0$$

$$x(x - 2)(x + 1) = 0$$

$$x = -1, 0, 2$$

$(-1, -5), (0, -1), (2, 1)$



75. $5.5\sqrt{x} + 10,000 = 3.29x$

$$(5.5\sqrt{x})^2 = (3.29x - 10,000)^2$$

$$30.25x = 10.8241x^2 - 65,800x + 100,000,000$$

$$0 = 10.8241x^2 - 65,830.25x + 100,000,000 \quad \text{Use the Quadratic Formula.}$$

$$x \approx 3133 \text{ units}$$

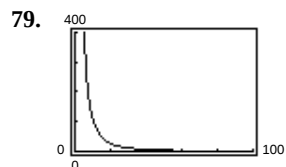
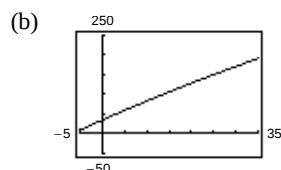
The other root, $x \approx 2949$, does not satisfy the equation $R = C$.

This problem can also be solved by using a graphing utility and finding the intersection of the graphs of C and R .

77. (a) Using a graphing utility, you obtain

$$y = -0.0153t^2 + 4.9971t + 34.9405$$

(c) For the year 2004, $t = 34$ and
 $y \approx 187.2$ CPI.



If the diameter is doubled, the resistance is changed by approximately a factor of $(1/4)$. For instance, $y(20) \approx 26.555$ and $y(40) \approx 6.36125$.

81. False; x -axis symmetry means that if $(1, -2)$ is on the graph, then $(1, 2)$ is also on the graph.

83. True; the x -intercepts are

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0 \right).$$

85. Distance to the origin $= K \times$ Distance to $(2, 0)$

$$\sqrt{x^2 + y^2} = K\sqrt{(x - 2)^2 + y^2}, K \neq 1$$

$$x^2 + y^2 = K^2(x^2 - 4x + 4 + y^2)$$

$$(1 - K^2)x^2 + (1 - K^2)y^2 + 4K^2x - 4K^2 = 0$$

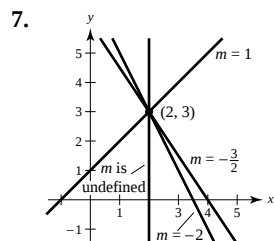
Note: This is the equation of a circle!

Section P.2 Linear Models and Rates of Change

1. $m = 1$

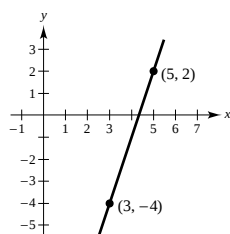
3. $m = 0$

5. $m = -12$



9.
$$m = \frac{2 - (-4)}{5 - 3}$$

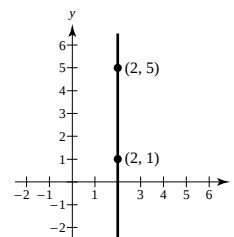
$$= \frac{6}{2} = 3$$



11.
$$m = \frac{5 - 1}{2 - 2}$$

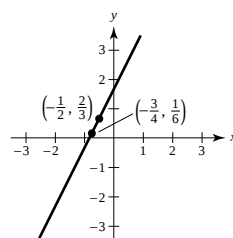
$$= \frac{4}{0}$$

undefined



13.
$$m = \frac{2/3 - 1/6}{-1/2 - (-3/4)}$$

$$= \frac{1/2}{1/4} = 2$$



15. Since the slope is 0, the line is horizontal and its equation is $y = 1$. Therefore, three additional points are $(0, 1)$, $(1, 1)$, and $(3, 1)$.

17. The equation of this line is

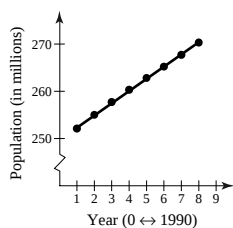
$$y - 7 = -3(x - 1)$$

$$y = -3x + 10.$$

Therefore, three additional points are $(0, 10)$, $(2, 4)$, and $(3, 1)$.

19. Given a line L , you can use any two distinct points to calculate its slope. Since a line is straight, the ratio of the change in y -values to the change in x -values will always be the same. See Section P.2 Exercise 93 for a proof.

21. (a)



(b) The slopes of the line segments are

$$\frac{255.0 - 252.1}{2 - 1} = 2.9$$

$$\frac{257.7 - 255.0}{3 - 2} = 2.7$$

$$\frac{260.3 - 257.7}{4 - 3} = 2.6$$

$$\frac{262.8 - 260.3}{5 - 4} = 2.5$$

$$\frac{265.2 - 262.8}{6 - 5} = 2.4$$

$$\frac{267.7 - 265.2}{7 - 6} = 2.5$$

$$\frac{270.3 - 267.7}{8 - 7} = 2.6$$

The population increased most rapidly from 1991 to 1992.

 $(m = 2.9)$

23. $x + 5y = 20$

$$y = -\frac{1}{5}x + 4$$

Therefore, the slope is $m = -\frac{1}{5}$ and the y -intercept is $(0, 4)$.

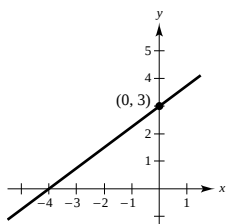
25. $x = 4$

The line is vertical. Therefore, the slope is undefined and there is no y -intercept.

27. $y = \frac{3}{4}x + 3$

$$4y = 3x + 12$$

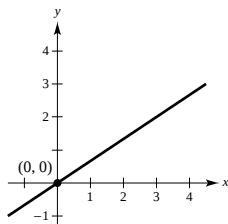
$$0 = 3x - 4y + 12$$



29. $y = \frac{2}{3}x$

$$3y = 2x$$

$$2x - 3y = 0$$

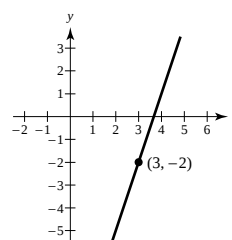


31. $y + 2 = 3(x - 3)$

$$y + 2 = 3x - 9$$

$$y = 3x - 11$$

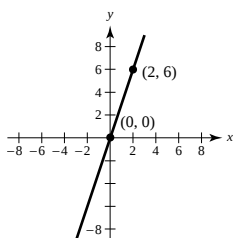
$$y - 3x + 11 = 0$$



33. $m = \frac{6 - 0}{2 - 0} = 3$

$$y - 0 = 3(x - 0)$$

$$y = 3x$$

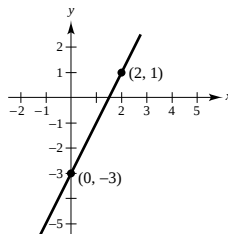


35. $m = \frac{1 - (-3)}{2 - 0} = 2$

$$y - 1 = 2(x - 2)$$

$$y - 1 = 2x - 4$$

$$0 = 2x - y - 3$$

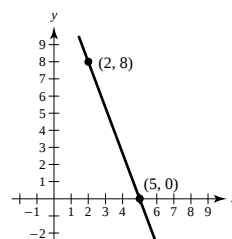


37. $m = \frac{8 - 0}{2 - 5} = -\frac{8}{3}$

$$y - 0 = -\frac{8}{3}(x - 5)$$

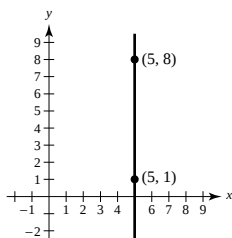
$$y = -\frac{8}{3}x + \frac{40}{3}$$

$$3y + 8x - 40 = 0$$



$$39. m = \frac{8-1}{5-5} \text{ Undefined.}$$

Vertical line $x = 5$

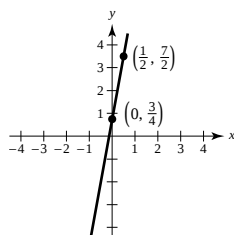


$$41. m = \frac{7/2 - 3/4}{1/2 - 0} = \frac{11/4}{1/2} = \frac{11}{2}$$

$$y - \frac{3}{4} = \frac{11}{2}(x - 0)$$

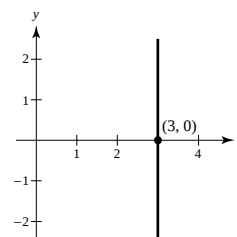
$$y = \frac{11}{2}x + \frac{3}{4}$$

$$22x - 4y + 3 = 0$$



$$43. x = 3$$

$$x - 3 = 0$$



$$45. \frac{x}{2} + \frac{y}{3} = 1$$

$$3x + 2y - 6 = 0$$

$$47. \frac{x}{a} + \frac{y}{a} = 1$$

$$\frac{1}{a} + \frac{2}{a} = 1$$

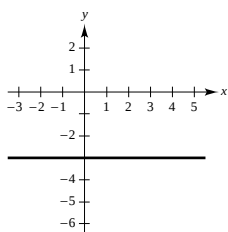
$$\frac{3}{a} = 1$$

$$a = 3 \Rightarrow x + y = 3$$

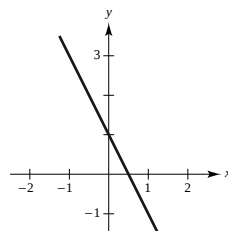
$$x + y - 3 = 0$$

$$49. y = -3$$

$$y + 3 = 0$$



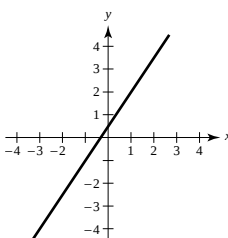
$$51. y = -2x + 1$$



$$53. y - 2 = \frac{3}{2}(x - 1)$$

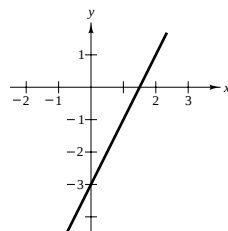
$$y = \frac{3}{2}x + \frac{1}{2}$$

$$2y - 3x - 1 = 0$$

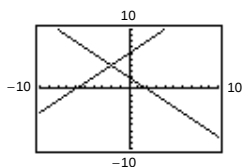


$$55. 2x - y - 3 = 0$$

$$y = 2x - 3$$

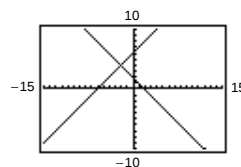


57.



The lines do not appear perpendicular.

The lines are perpendicular because their slopes 1 and -1 are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.



The lines appear perpendicular.

59. $4x - 2y = 3$

$$y = 2x - \frac{3}{2}$$

$$m = 2$$

(a) $y - 1 = 2(x - 2)$

$$y - 1 = 2x - 4$$

$$2x - y - 3 = 0$$

(b) $y - 1 = -\frac{1}{2}(x - 2)$

$$2y - 2 = -x + 2$$

$$x + 2y - 4 = 0$$

61. $5x - 3y = 0$

$$y = \frac{5}{3}x$$

$$m = \frac{5}{3}$$

(a) $y - \frac{7}{8} = \frac{5}{3}(x - \frac{3}{4})$

$$24y - 21 = 40x - 30$$

$$24y - 40x + 9 = 0$$

(b) $y - \frac{7}{8} = -\frac{3}{5}(x - \frac{3}{4})$

$$40y - 35 = -24x + 18$$

$$40y + 24x - 53 = 0$$

63. (a) $x = 2 \Rightarrow x - 2 = 0$

(b) $y = 5 \Rightarrow y - 5 = 0$

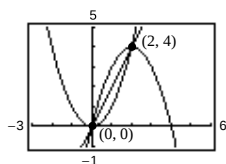
65. The slope is 125. Hence, $V = 125(t - 1) + 2540$

$$= 125t + 2415$$

67. The slope is -2000 . Hence, $V = -2000(t - 1) + 20,400$

$$= -2000t + 22,400$$

69.



You can use the graphing utility to determine that the points of intersection are $(0, 0)$ and $(2, 4)$. Analytically,

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x - 2) = 0$$

$$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$$

$$x = 2 \Rightarrow y = 4 \Rightarrow (2, 4).$$

The slope of the line joining $(0, 0)$ and $(2, 4)$ is $m = (4 - 0)/(2 - 0) = 2$. Hence, an equation of the line is

$$y - 0 = 2(x - 0)$$

$$y = 2x.$$

$$71. m_1 = \frac{1 - 0}{-2 - (-1)} = -1$$

$$m_2 = \frac{-2 - 0}{2 - (-1)} = -\frac{2}{3}$$

$$m_1 \neq m_2$$

The points are not collinear.

73. Equations of perpendicular bisectors:

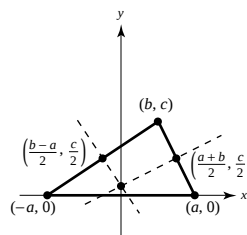
$$y - \frac{c}{2} = \frac{a - b}{c} \left(x - \frac{a + b}{2} \right)$$

$$y - \frac{c}{2} = \frac{a + b}{-c} \left(x - \frac{b - a}{2} \right)$$

Letting $x = 0$ in either equation gives the point of intersection:

$$\left(0, \frac{-a^2 + b^2 + c^2}{2c} \right).$$

This point lies on the third perpendicular bisector, $x = 0$.



75. Equations of altitudes:

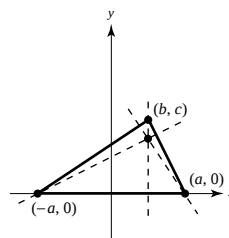
$$y = \frac{a - b}{c}(x + a)$$

$$x = b$$

$$y = -\frac{a + b}{c}(x - a)$$

Solving simultaneously, the point of intersection is

$$\left(b, \frac{a^2 - b^2}{c} \right).$$



77. Find the equation of the line through the points (0, 32) and (100, 212).

$$m = \frac{180}{100} = \frac{9}{5}$$

$$F - 32 = \frac{9}{5}(C - 0)$$

$$F = \frac{9}{5}C + 32$$

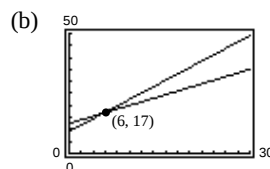
$$5F - 9C - 160 = 0$$

For $F = 72^\circ$, $C \approx 22.2^\circ$.

79. (a) $W_1 = 0.75x + 12.50$

$$W_2 = 1.30x + 9.20$$

(c) Both jobs pay \$17 per hour if 6 units are produced. For someone who can produce more than 6 units per hour, the second offer would pay more. For a worker who produces less than 6 units per hour, the first offer pays more.



Using a graphing utility, the point of intersection is approximately (6, 17). Analytically,

$$0.75x + 12.50 = 1.30x + 9.20$$

$$3.3 = 0.55x \Rightarrow x = 6$$

$$y = 0.75(6) + 12.50 = 17.$$

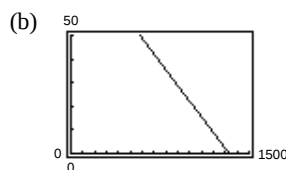
81. (a) Two points are (50, 580) and (47, 625). The slope is

$$m = \frac{625 - 580}{47 - 50} = -15.$$

$$p - 580 = -15(x - 50)$$

$$p = -15x + 750 + 580 = -15x + 1330$$

$$\text{or } x = \frac{1}{15}(1330 - p)$$



$$\text{If } p = 655, x = \frac{1}{15}(1330 - 655) = 45 \text{ units.}$$

$$(c) \text{ If } p = 595, x = \frac{1}{15}(1330 - 595) = 49 \text{ units.}$$

$$83. 4x + 3y - 10 = 0 \Rightarrow d = \frac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}} = \frac{10}{5} = 2$$

$$85. x - y - 2 = 0 \Rightarrow d = \frac{|1(-2) + (-1)(1) - 2|}{\sqrt{1^2 + 1^2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

87. A point on the line
- $x + y = 1$
- is (0, 1). The distance from the point (0, 1) to
- $x + y - 5 = 0$
- is

$$d = \frac{|1(0) + 1(1) - 5|}{\sqrt{1^2 + 1^2}} = \frac{|1 - 5|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

89. If
- $A = 0$
- , then
- $By + C = 0$
- is the horizontal line
- $y = -C/B$
- . The distance to
- (x_1, y_1)
- is

$$d = \left| y_1 - \left(\frac{-C}{B} \right) \right| = \frac{|By_1 + C|}{|B|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then $Ax + C = 0$ is the vertical line $x = -C/A$. The distance to (x_1, y_1) is

$$d = \left| x_1 - \left(\frac{-C}{A} \right) \right| = \frac{|Ax_1 + C|}{|A|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(Note that A and B cannot both be zero.)

The slope of the line $Ax + By + C = 0$ is $-A/B$. The equation of the line through (x_1, y_1) perpendicular to $Ax + By + C = 0$ is:

$$y - y_1 = \frac{B}{A}(x - x_1)$$

$$Ay - Ay_1 = Bx - Bx_1$$

$$Bx_1 - Ay_1 = Bx - Ay$$

The point of intersection of these two lines is:

$$Ax + By = -C \quad \Rightarrow \quad A^2x + AB y = -AC \quad (1)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \frac{B^2x - AB y}{A^2 + B^2} = \frac{B^2x_1 - AB y_1}{A^2 + B^2} \quad (2)$$

$$(A^2 + B^2)x = -AC + B^2x_1 - AB y_1 \quad (\text{By adding equations (1) and (2)})$$

$$x = \frac{-AC + B^2x_1 - AB y_1}{A^2 + B^2}$$

$$Ax + By = -C \quad \Rightarrow \quad ABx + B^2y = -BC \quad (3)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \frac{-ABx + A^2y}{A^2 + B^2} = \frac{-ABx_1 + A^2y_1}{A^2 + B^2} \quad (4)$$

$$(A^2 + B^2)y = -BC - ABx_1 + A^2y_1 \quad (\text{By adding equations (3) and (4)})$$

$$y = \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}$$

89. —CONTINUED—

$$\left(\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2}, \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} \right) \text{ point of intersection}$$

The distance between (x_1, y_1) and this point gives us the distance between (x_1, y_1) and the line $Ax + By + C = 0$.

$$\begin{aligned} d &= \sqrt{\left[\frac{-AC + B^2x_1 - ABy_1}{A^2 + B^2} - x_1 \right]^2 + \left[\frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} - y_1 \right]^2} \\ &= \sqrt{\left[\frac{-AC - ABx_1 - A^2y_1}{A^2 + B^2} \right]^2 + \left[\frac{-BC - ABx_1 - B^2y_1}{A^2 + B^2} \right]^2} \\ &= \sqrt{\left[\frac{-A(C + Bx_1 + Ay_1)}{A^2 + B^2} \right]^2 + \left[\frac{-B(C + Ax_1 + By_1)}{A^2 + B^2} \right]^2} \\ &= \sqrt{\frac{(A^2 + B^2)(C + Ax_1 + By_1)^2}{(A^2 + B^2)^2}} \\ &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

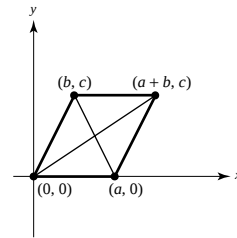
91. For simplicity, let the vertices of the rhombus be $(0, 0)$, $(a, 0)$, (b, c) , and $(a + b, c)$, as shown in the figure. The slopes of the diagonals are then

$$m_1 = \frac{c}{a + b} \text{ and } m_2 = \frac{c}{b - a}.$$

Since the sides of the Rhombus are equal, $a^2 = b^2 + c^2$, and we have

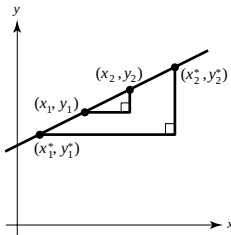
$$m_1 m_2 = \frac{c}{a + b} \cdot \frac{c}{b - a} = \frac{c^2}{b^2 - a^2} = \frac{c^2}{-c^2} = -1.$$

Therefore, the diagonals are perpendicular.



93. Consider the figure below in which the four points are collinear. Since the triangles are similar, the result immediately follows.

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$



95. True.

$$ax + by = c_1 \Rightarrow y = -\frac{a}{b}x + \frac{c_1}{b} \Rightarrow m_1 = -\frac{a}{b}$$

$$bx - ay = c_2 \Rightarrow y = \frac{b}{a}x - \frac{c_2}{a} \Rightarrow m_2 = \frac{b}{a}$$

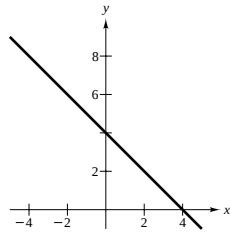
$$m_2 = -\frac{1}{m_1}$$

Section P.3 Functions and Their Graphs

1. (a) $f(0) = 2(0) - 3 = -3$
 (b) $f(-3) = 2(-3) - 3 = -9$
 (c) $f(b) = 2b - 3$
 (d) $f(x - 1) = 2(x - 1) - 3 = 2x - 5$
3. (a) $g(0) = 3 - 0^2 = 3$
 (b) $g(\sqrt{3}) = 3 - (\sqrt{3})^2 = 3 - 3 = 0$
 (c) $g(-2) = 3 - (-2)^2 = 3 - 4 = -1$
 (d) $g(t - 1) = 3 - (t - 1)^2 = -t^2 + 2t + 2$
5. (a) $f(0) = \cos(2(0)) = \cos 0 = 1$
 (b) $f\left(-\frac{\pi}{4}\right) = \cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$
 (c) $f\left(\frac{\pi}{3}\right) = \cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$
7. $\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2, \Delta x \neq 0$
9. $\frac{f(x) - f(2)}{x - 2} = \frac{(1/\sqrt{x-1}) - 1}{x - 2} = \frac{1 - \sqrt{x-1}}{(x-2)\sqrt{x-1}} \cdot \frac{1 + \sqrt{x-1}}{1 + \sqrt{x-1}} = \frac{2 - x}{(x-2)\sqrt{x-1}(1 + \sqrt{x-1})} = \frac{-1}{\sqrt{x-1}(1 + \sqrt{x-1})}, x \neq 2$
11. $h(x) = -\sqrt{x+3}$
 Domain: $x + 3 \geq 0 \Rightarrow [-3, \infty)$
 Range: $(-\infty, 0]$
13. $f(t) = \sec \frac{\pi t}{4}$
 $\frac{\pi t}{4} \neq \frac{(2k+1)\pi}{2} \Rightarrow t \neq 4k + 2$
 Domain: all $t \neq 4k + 2, k$ an integer
 Range: $(-\infty, -1], [1, \infty)$
15. $f(x) = \frac{1}{x}$
 Domain: $(-\infty, 0), (0, \infty)$
 Range: $(-\infty, 0), (0, \infty)$
17. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
 (a) $f(-1) = 2(-1) + 1 = -1$
 (b) $f(0) = 2(0) + 2 = 2$
 (c) $f(2) = 2(2) + 2 = 6$
 (d) $f(t^2 + 1) = 2(t^2 + 1) = 2t^2 + 4$
 (Note: $t^2 + 1 \geq 0$ for all t)
 Domain: $(-\infty, \infty)$
 Range: $(-\infty, 1), [2, \infty)$
19. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$
 (a) $f(-3) = |-3| + 1 = 4$
 (b) $f(1) = -1 + 1 = 0$
 (c) $f(3) = -3 + 1 = -2$
 (d) $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$
 Domain: $(-\infty, \infty)$
 Range: $(-\infty, 0] \cup [1, \infty)$

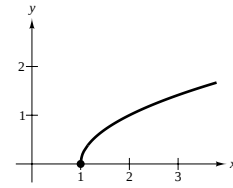
21. $f(x) = 4 - x$

 Domain: $(-\infty, \infty)$

 Range: $(-\infty, \infty)$


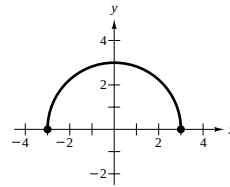
23. $h(x) = \sqrt{x - 1}$

 Domain: $[1, \infty)$

 Range: $[0, \infty)$


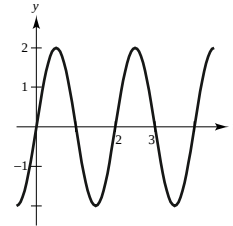
25. $f(x) = \sqrt{9 - x^2}$

 Domain: $[-3, 3]$

 Range: $[0, 3]$


27. $g(t) = 2 \sin \pi t$

 Domain: $(-\infty, \infty)$

 Range: $[-2, 2]$


29. $x - y^2 = 0 \Rightarrow y = \pm \sqrt{x}$

y is not a function of x . Some vertical lines intersect the graph twice.

33. $x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}$

y is not a function of x since there are two values of y for some x .

31. y is a function of x . Vertical lines intersect the graph at most once.

35. $y^2 = x^2 - 1 \Rightarrow y = \pm \sqrt{x^2 - 1}$

y is not a function of x since there are two values of y for some x .

37. $f(x) = |x| + |x - 2|$

If $x < 0$, then $f(x) = -x - (x - 2) = -2x + 2 = 2(1 - x)$.

If $0 \leq x < 2$, then $f(x) = x - (x - 2) = 2$.

If $x \geq 2$, then $f(x) = x + (x - 2) = 2x - 2 = 2(x - 1)$.

Thus,

$$f(x) = \begin{cases} 2(1 - x), & x < 0 \\ 2, & 0 \leq x < 2. \\ 2(x - 1), & x \geq 2. \end{cases}$$

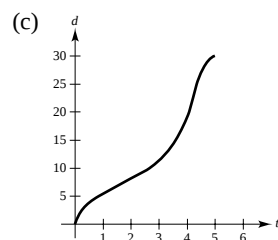
39. The function is $g(x) = cx^2$. Since $(1, -2)$ satisfies the equation, $c = -2$. Thus, $g(x) = -2x^2$.

41. The function is $r(x) = c/x$, since it must be undefined at $x = 0$. Since $(1, 32)$ satisfies the equation, $c = 32$. Thus, $r(x) = 32/x$.

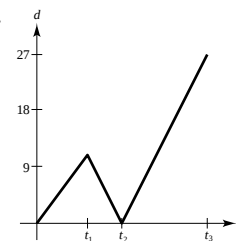
43. (a) For each time t , there corresponds a depth d .

(b) Domain: $0 \leq t \leq 5$

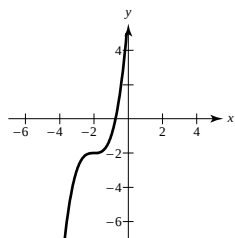
Range: $0 \leq d \leq 30$



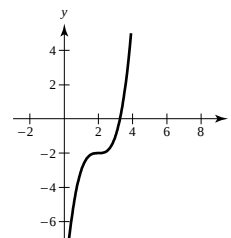
45.



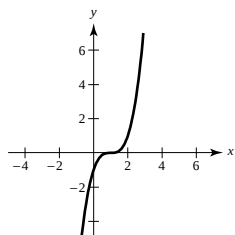
47. (a) The graph is shifted 3 units to the left.



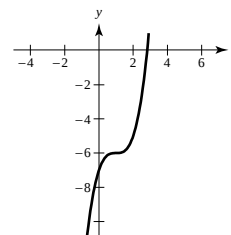
- (b) The graph is shifted 1 unit to the right.



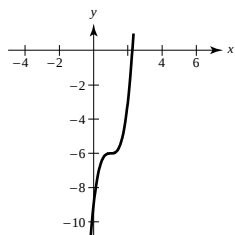
- (c) The graph is shifted 2 units upward.



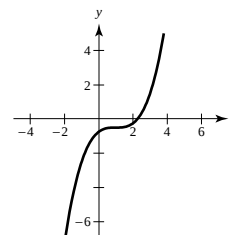
- (d) The graph is shifted 4 units downward.



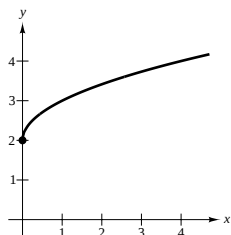
- (e) The graph is stretched vertically by a factor of 3.



- (f) The graph is stretched vertically by a factor of $\frac{1}{4}$.

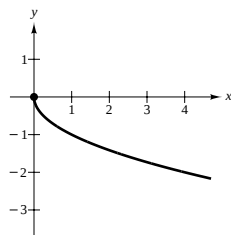


49. (a) $y = \sqrt{x} + 2$



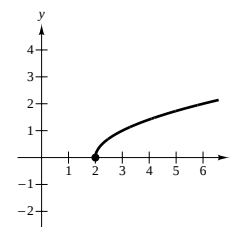
Vertical shift 2 units upward

- (b) $y = -\sqrt{x}$



Reflection about the x -axis

- (c) $y = \sqrt{x - 2}$



Horizontal shift 2 units to the right

51. (a) $T(4) = 16^\circ$, $T(15) \approx 23^\circ$

(b) If $H(t) = T(t - 1)$, then the program would turn on (and off) one hour later.

(c) If $H(t) = T(t) - 1$, then the overall temperature would be reduced 1 degree.

53. $f(x) = x^2$, $g(x) = \sqrt{x}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, \quad x \geq 0$$

Domain: $[0, \infty)$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

Domain: $(-\infty, \infty)$

No. Their domains are different. $(f \circ g) = (g \circ f)$ for $x \geq 0$.

55. $f(x) = \frac{3}{x}$, $g(x) = x^2 - 1$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \frac{3}{x^2 - 1}$$

Domain: all $x \neq \pm 1$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{3}{x}\right) = \left(\frac{3}{x}\right)^2 - 1 = \frac{9}{x^2} - 1 = \frac{9 - x^2}{x^2}$$

Domain: all $x \neq 0$

No, $f \circ g \neq g \circ f$.

$$57. (A \circ r)(t) = A(r(t)) = A(0.6t) = \pi(0.6t)^2 = 0.36\pi t^2$$

$(A \circ r)(t)$ represents the area of the circle at time t .

$$61. f(-x) = (-x) \cos(-x) = -x \cos x = -f(x)$$

Odd

$$63. (a) \text{ If } f \text{ is even, then } \left(\frac{3}{2}, 4\right) \text{ is on the graph.}$$

$$(b) \text{ If } f \text{ is odd, then } \left(\frac{3}{2}, -4\right) \text{ is on the graph.}$$

$$\begin{aligned} 65. f(-x) &= a_{2n+1}(-x)^{2n+1} + \cdots + a_3(-x)^3 + a_1(-x) \\ &= -[a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x] \\ &= -f(x) \end{aligned}$$

Odd

$$67. \text{ Let } F(x) = f(x)g(x) \text{ where } f \text{ and } g \text{ are even. Then}$$

$$F(-x) = f(-x)g(-x) = f(x)g(x) = F(x).$$

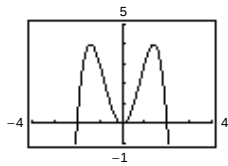
Thus, $F(x)$ is even. Let $F(x) = f(x)g(x)$ where f and g are odd. Then

$$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x).$$

Thus, $F(x)$ is even.

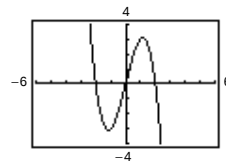
$$69. f(x) = x^2 + 1 \text{ and } g(x) = x^4 \text{ are even.}$$

$$f(x)g(x) = (x^2 + 1)(x^4) = x^6 + x^4 \text{ is even.}$$



$$f(x) = x^3 - x \text{ is odd and } g(x) = x^2 \text{ is even.}$$

$$f(x)g(x) = (x^3 - x)(x^2) = x^5 - x^3 \text{ is odd.}$$



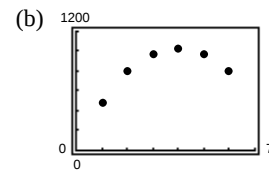
71. (a)

x	length and width	volume V
1	$24 - 2(1)$	484
2	$24 - 2(2)$	800
3	$24 - 2(3)$	972
4	$24 - 2(4)$	1024
5	$24 - 2(5)$	980
6	$24 - 2(6)$	864

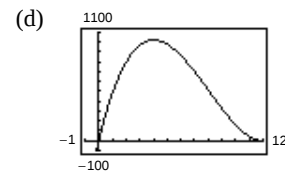
The maximum volume appears to be 1024 cm^3 .

$$(c) V = x(24 - 2x)^2 = 4x(12 - x)^2$$

Domain: $0 < x < 12$



Yes, V is a function of x .



Maximum volume is $V = 1024 \text{ cm}^3$ for box having dimensions $4 \times 16 \times 16 \text{ cm}$.

$$73. \text{ False; let } f(x) = x^2.$$

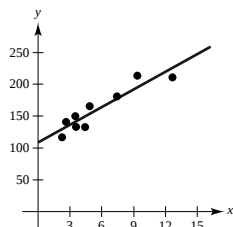
$$\text{Then } f(-3) = f(3) = 9, \text{ but } -3 \neq 3.$$

$$75. \text{ True, the function is even.}$$

Section P.4 Fitting Models to Data

1. Quadratic function

5. (a), (b)



Yes. The cancer mortality increases linearly with increased exposure to the carcinogenic substance.

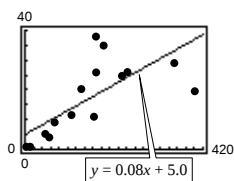
(c) If $x = 3$, then $y \approx 136$.

9. (a) Let x = per capita energy usage (in millions of Btu)
 y = per capita gross national product (in thousands)

$$y = 0.0764x + 4.9985 \approx 0.08x + 5.0$$

$$r = 0.7052$$

(b)



(c) Denmark, Japan, and Canada

(d) Deleting the data for the three countries above,

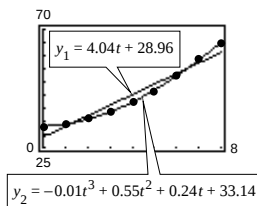
$$y = 0.0959x + 1.0539$$

($r = 0.9202$ is much closer to 1.)

13. (a) $y_1 = 4.0367t + 28.9644$

$$y_2 = -0.0099t^3 + 0.5488t^2 + 0.2399t + 33.1414$$

(b)

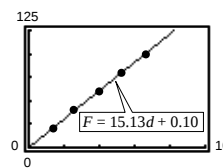


(c) The cubic model is better.

3. Linear function

7. (a) $d = 0.066F$ or $F = 15.1d + 0.1$

(b)



The model fits well.

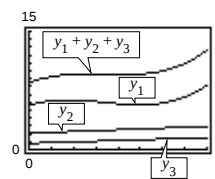
(c) If $F = 55$, then $d \approx 0.066(55) = 3.63$ cm.

11. (a) $y_1 = 0.0343t^3 - 0.3451t^2 + 0.8837t + 5.6061$

$$y_2 = 0.1095t + 2.0667$$

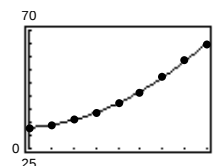
$$y_3 = 0.0917t + 0.7917$$

(b)



For 2002, $t = 12$ and $y_1 + y_2 + y_3 \approx 31.06$ cents/mile

(d) $y_3 = 0.4297t^2 + 0.5994t + 32.9745$

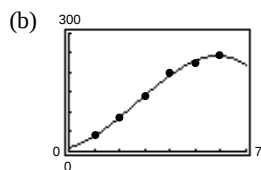


(e) The slope represents the average increase per year in the number of people (in millions) in HMOs.

(f) For 2000, $t = 10$, and $y_1 \approx 69.3$ million. (linear)

$$y_2 \approx 80.5 \text{ million (cubic)}$$

15. (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$



(c) If $x = 4.5$, $y \approx 214$ horsepower.

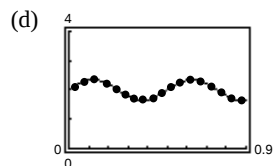
17. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.

19. Answers will vary.

Review Exercises for Chapter P

1. $y = 2x - 3$

$x = 0 \Rightarrow y = 2(0) - 3 = -3 \Rightarrow (0, -3) \quad y\text{-intercept}$

$y = 0 \Rightarrow 0 = 2x - 3 \Rightarrow x = \frac{3}{2} \Rightarrow (\frac{3}{2}, 0) \quad x\text{-intercept}$

3. $y = \frac{x-1}{x-2}$

$x = 0 \Rightarrow y = \frac{0-1}{0-2} = \frac{1}{2} \Rightarrow (0, \frac{1}{2}) \quad y\text{-intercept}$

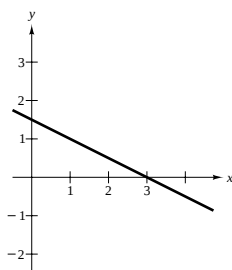
$y = 0 \Rightarrow 0 = \frac{x-1}{x-2} \Rightarrow x = 1 \Rightarrow (1, 0) \quad x\text{-intercept}$

5. Symmetric with respect to y -axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$

$$x^2y - x^2 + 4y = 0.$$

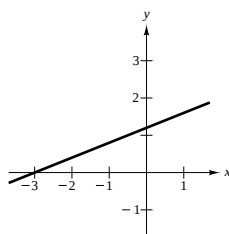
7. $y = -\frac{1}{2}x + \frac{3}{2}$



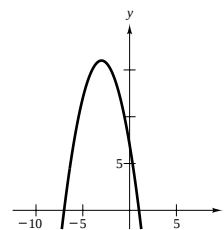
9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

$$-\frac{2}{5}x + y = \frac{6}{5}$$

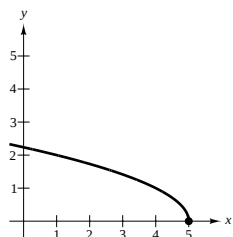
$$y = \frac{2}{5}x + \frac{6}{5}$$

Slope: $\frac{2}{5}$ y -intercept: $\frac{6}{5}$ 

11. $y = 7 - 6x - x^2$



13. $y = \sqrt{5 - x}$

Domain: $(-\infty, 5]$ 

15. $y = 4x^2 - 25$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $3x - 4y = 8$

$$\frac{4x + 4y = 20}{7x} = 28$$

$$7x = 28$$

$$x = 4$$

$$y = 1$$

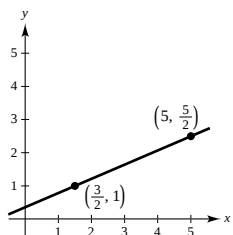
Point: (4, 1)

19. You need factors $(x + 2)$ and $(x - 2)$. Multiply by x to obtain origin symmetry

$$y = x(x + 2)(x - 2).$$

$$= x^3 - 4x.$$

21.



$$\text{Slope} = \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$$

23. $\frac{1 - t}{1 - 0} = \frac{1 - 5}{1 - (-2)}$

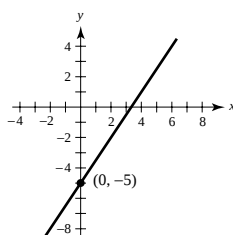
$$1 - t = -\frac{4}{3}$$

$$t = \frac{7}{3}$$

25. $y - (-5) = \frac{3}{2}(x - 0)$

$$y = \frac{3}{2}x - 5$$

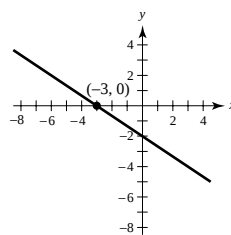
$$2y - 3x + 10 = 0$$



27. $y - 0 = -\frac{2}{3}(x - (-3))$

$$y = -\frac{2}{3}x - 2$$

$$3y + 2x + 6 = 0$$



29. (a) $y - 4 = \frac{7}{16}(x + 2)$

$$16y - 64 = 7x + 14$$

$$0 = 7x - 16y + 78$$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$$y = -2x$$

$$2x + y = 0$$

(b) Slope of line is $\frac{5}{3}$.

$$y - 4 = \frac{5}{3}(x + 2)$$

$$3y - 12 = 5x + 10$$

$$0 = 5x - 3y + 22$$

(d) $x = -2$

$$x + 2 = 0$$

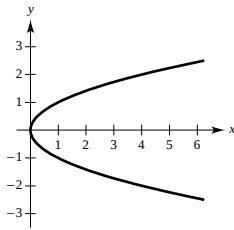
31. The slope is -850 . $V = -850t + 12,500$.

$$V(3) = -850(3) + 12,500 = \$9950$$

33. $x - y^2 = 0$

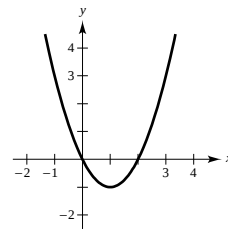
$$y = \pm \sqrt{x}$$

Not a function of x since there are two values of y for some x .



35. $y = x^2 - 2x$

Function of x since there is one value of y for each x .



37. $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b)
$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$$

$$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$$

39. (a) Domain: $36 - x^2 \geq 0 \Rightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

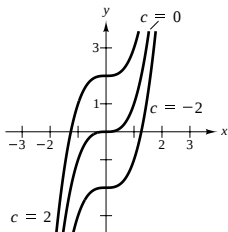
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

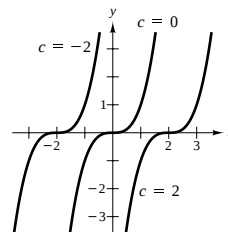
(c) Domain: all x or $(-\infty, \infty)$

Range: all y or $(-\infty, \infty)$

41. (a) $f(x) = x^3 + c, c = -2, 0, 2$



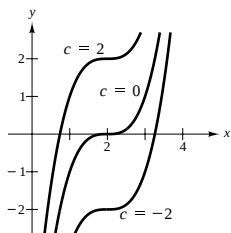
(b) $f(x) = (x - c)^3, c = -2, 0, 2$



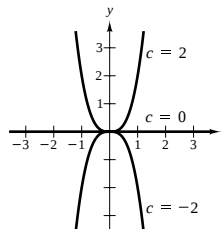
—CONTINUED—

41. —CONTINUED—

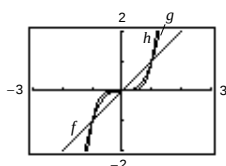
(c) $f(x) = (x - 2)^3 + c$, $c = -2, 0, 2$



(d) $f(x) = cx^3$, $c = -2, 0, 2$



43. (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

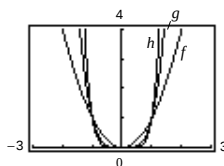


The graphs of f , g , and h all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

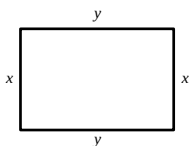
$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$



The graphs of f , g , and h all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

45. (a)

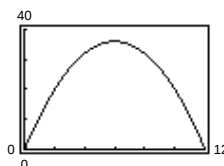


$2x + 2y = 24$

$y = 12 - x$

$A = xy = x(12 - x) = 12x - x^2$

(b) Domain: $0 < x < 12$



(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

47. (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

(d) 5, positive leading coefficient

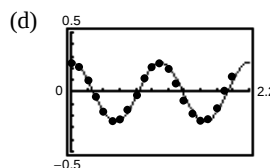
49. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$(0.25 - (-0.25))/2 = 0.25$.

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4} \cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4} \cos(5.7t)$



PART II

CHAPTER P Preparation for Calculus

Section P.1	Graphs and Models	282
Section P.2	Linear Models and Rates of Change	287
Section P.3	Functions and Their Graphs	292
Section P.4	Fitting Models to Data	296
	Review Exercises	297
	Problem Solving	300

CHAPTER P

Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Even-Numbered Exercises

2. $y = \sqrt{9 - x^2}$

x -intercepts: $(-3, 0), (3, 0)$

y -intercept: $(0, 3)$

Matches graph (d)

4. $y = x^3 - x$

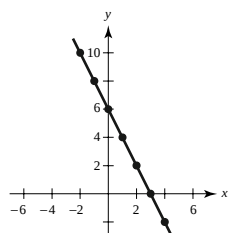
x -intercepts: $(0, 0), (-1, 0), (1, 0)$

y -intercept: $(0, 0)$

Matches graph (c)

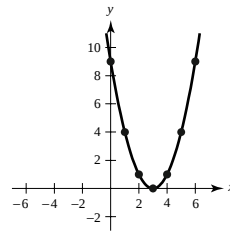
6. $y = 6 - 2x$

x	-2	-1	0	1	2	3	4
y	10	8	6	4	2	0	-2



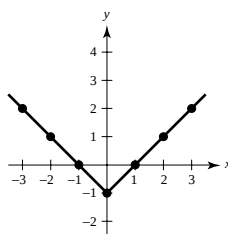
8. $y = (x - 3)^2$

x	0	1	2	3	4	5	6
y	9	4	1	0	1	4	9



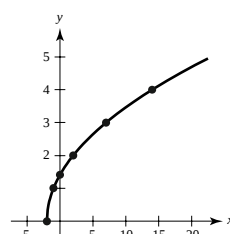
10. $y = |x| - 1$

x	-3	-2	-1	0	1	2	3
y	2	1	0	-1	0	1	2



12. $y = \sqrt{x + 2}$

x	-2	-1	0	2	7	14
y	0	1	$\sqrt{2}$	2	3	4



14. $\begin{array}{l} \text{Xmin} = -30 \\ \text{Xmax} = 30 \\ \text{Xscl} = 5 \\ \text{Ymin} = -10 \\ \text{Ymax} = 40 \\ \text{Yscl} = 5 \end{array}$

Note that $y = 10$ when $x = 0$ or $x = 10$.

18. $y^2 = x^3 - 4x$

y -intercept: $y^2 = 0^3 - 4(0)$

$y = 0; (0, 0)$

x -intercepts: $0 = x^3 - 4x$

$0 = x(x - 2)(x + 2)$

$x = 0, \pm 2; (0, 0), (\pm 2, 0)$

22. $y = \frac{x^2 + 3x}{(3x + 1)^2}$

y -intercept: $y = \frac{0^2 + 3(0)}{[3(0) + 1]^2}$

$y = 0; (0, 0)$

x -intercepts: $0 = \frac{x^2 + 3x}{(3x + 1)^2}$

$0 = \frac{x(x + 3)}{(3x + 1)^2}$

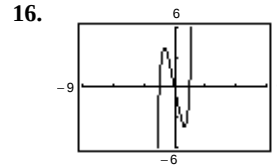
$x = 0, -3; (0, 0), (-3, 0)$

26. $y = x^2 - x$

No symmetry with respect to either axis or the origin.

30. Symmetric with respect to the x -axis since

$x(-y)^2 = xy^2 = -10.$



(a) $(-0.5, y) = (-0.5, 2.47)$

(b) $(x, -4) = (-1.65, -4)$ and $(x, -4) = (1, -4)$

20. $y = (x - 1)\sqrt{x^2 + 1}$

y -intercept: $y = (0 - 1)\sqrt{0^2 + 1}$

$y = -1; (0, -1)$

x -intercepts: $0 = (x - 1)\sqrt{x^2 + 1}$

$x = 1; (1, 0)$

24. $y = 2x - \sqrt{x^2 + 1}$

y -intercept: $y = 2(0) - \sqrt{0^2 + 1}$

$y = -1; (0, -1)$

x -intercepts: $0 = 2x - \sqrt{x^2 + 1}$

$2x = \sqrt{x^2 + 1}$

$4x^2 = x^2 + 1$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$

$x = \frac{\sqrt{3}}{3}; \left(\frac{\sqrt{3}}{3}, 0\right)$

Note: $x = -\sqrt{3}/3$ is an extraneous solution.

28. Symmetric with respect to the origin since

$(-y) = (-x)^3 + (-x)$

$-y = -x^3 - x$

$y = x^3 + x.$

32. Symmetric with respect to the origin since

$(-x)(-y) - \sqrt{4 - (-x)^2} = 0$

$xy - \sqrt{4 - x^2} = 0.$

34. $y = \frac{x^2}{x^2 + 1}$ is symmetric with respect to the y -axis

since $y = \frac{(-x)^2}{(-x)^2 + 1} = \frac{x^2}{x^2 + 1}$.

36. $|y| - x = 3$ is symmetric with respect to the x -axis

since $|-y| - x = 3$

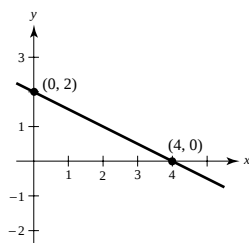
$|y| - x = 3$.

38. $y = -\frac{x}{2} + 2$

Intercepts:

$(4, 0), (0, 2)$

Symmetry: none

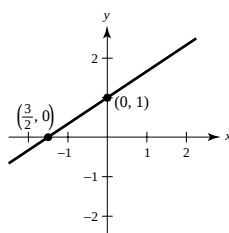


40. $y = \frac{2}{3}x + 1$

Intercepts:

$(0, 1), (-\frac{3}{2}, 0)$

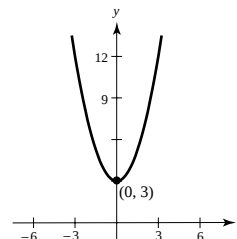
Symmetry: none



42. $y = x^2 + 3$

Intercept: $(0, 3)$

Symmetry: y -axis

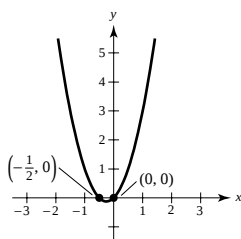


44. $y = 2x^2 + x = x(2x + 1)$

Intercepts:

$(0, 0), (-\frac{1}{2}, 0)$

Symmetry: none

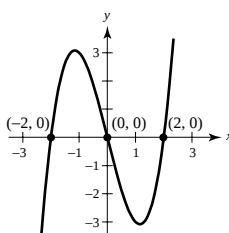


46. $y = x^3 - 4x$

Intercepts:

$(0, 0), (2, 0), (-2, 0)$

Symmetry: origin



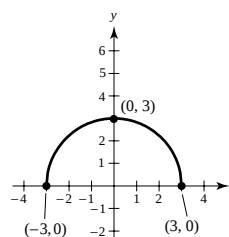
48. $y = \sqrt{9 - x^2}$

Intercepts:

$(-3, 0), (3, 0), (0, 3)$

Symmetry: y -axis

Domain: $[-3, 3]$

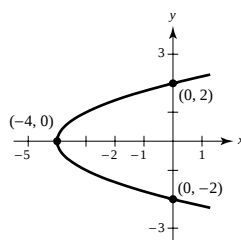


50. $x = y^2 - 4$

Intercepts:

$(0, 2), (0, -2), (-4, 0)$

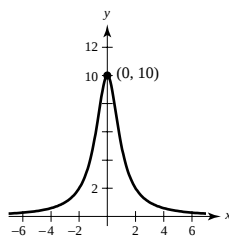
Symmetry: x -axis



52. $y = \frac{10}{x^2 + 1}$

Intercepts: $(0, 10)$

Symmetry: y -axis

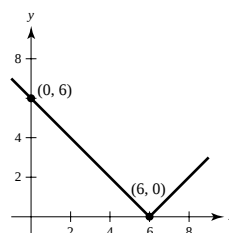


54. $y = |6 - x|$

Intercepts:

$(0, 6), (6, 0)$

Symmetry: none



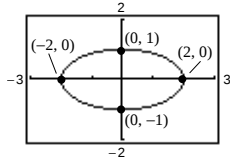
$$56. x^2 + 4y^2 = 4 \Rightarrow y = \pm \frac{\sqrt{4 - x^2}}{2}$$

Intercepts:

$$(-2, 0), (2, 0), (0, -1), (0, 1)$$

Symmetry: origin and both axes

Domain: $[-2, 2]$



$$60. y = \left(x + \frac{5}{2}\right)(x - 2)\left(x - \frac{3}{2}\right) \text{ (other answers possible)}$$

$$64. 2x - 3y = 13 \Rightarrow y = \frac{2x - 13}{3}$$

$$5x + 3y = 1 \Rightarrow y = \frac{1 - 5x}{3}$$

$$\frac{2x - 13}{3} = \frac{1 - 5x}{3}$$

$$2x - 13 = 1 - 5x$$

$$7x = 14$$

$$x = 2$$

The corresponding y-value is $y = -3$.

Point of intersection: $(2, -3)$

$$68. x = 3 - y^2 \Rightarrow y^2 = 3 - x$$

$$y = x - 1$$

$$3 - x = (x - 1)^2$$

$$3 - x = x^2 - 2x + 1$$

$$0 = x^2 - x - 2 = (x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding y-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

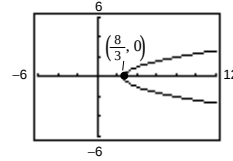
$$58. 3x - 4y^2 = 8$$

$$4y^2 = 3x - 8$$

$$y = \pm \sqrt{\frac{3}{4}x - 2}$$

Intercept: $\left(\frac{8}{3}, 0\right)$

Symmetry: x-axis



62. Some possible equations:

$$x = y^2$$

$$x = |y|$$

$$x = y^4 + 1$$

$$x^2 + y^2 = 25$$

$$66. 5x - 6y = 9 \Rightarrow y = \frac{5x - 9}{6}$$

$$-7x + 3y = -18 \Rightarrow y = \frac{7x - 18}{3}$$

$$\frac{5x - 9}{6} = \frac{7x - 18}{3}$$

$$5x - 9 = 14x - 36$$

$$27 = 9x$$

$$x = 3$$

The corresponding y-value is $y = 1$.

Point of intersection: $(3, 1)$

$$70. x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2$$

$$2x + y = 10 \Rightarrow y = 10 - 2x$$

$$25 - x^2 = (10 - 2x)^2$$

$$25 - x^2 = 100 - 40x + 4x^2$$

$$0 = 5x^2 - 40x + 75 = 5(x - 3)(x - 5)$$

$$x = 3 \text{ or } x = 5$$

The corresponding y-values are $y = 4$ and $y = 0$.

Points of intersection: $(3, 4), (5, 0)$

72. $y = x^3 - 4x$

$y = -(x + 2)$

$x^3 - 4x = -(x + 2)$

$x^3 - 3x + 2 = 0$

$(x - 1)^2(x + 2) = 0$

$x = 1 \quad \text{or} \quad x = -2$

The corresponding y -values are $y = -3$ and $y = 0$.

Points of intersection: $(1, -3), (-2, 0)$

74. $y = x^4 - 2x^2 + 1$

$y = 1 - x^2$

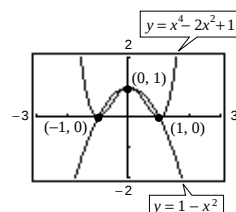
$1 - x^2 = x^4 - 2x^2 + 1$

$0 = x^4 - x^2$

$0 = x^2(x + 1)(x - 1)$

$x = -1, 0, 1$

$(-1, 0), (0, 1), (1, 0)$



76. $y = kx + 5$ matches (b).

Use $(1, 7)$: $7 = k(1) + 5 \Rightarrow k = 2$, thus, $y = 2x + 5$.

$y = x^2 + k$ matches (d).

Use $(1, -9)$: $-9 = (1)^2 + k \Rightarrow k = -10$, thus, $y = x^2 - 10$.

$y = kx^{3/2}$ matches (a).

Use $(1, 3)$: $3 = k(1)^{3/2} \Rightarrow k = 3$, thus, $y = 3x^{3/2}$.

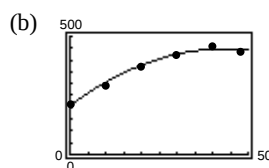
$xy = k$ matches (c).

Use $(1, 36)$: $(1)(36) = k \Rightarrow k = 36$, thus, $xy = 36$.

78. (a) Using a graphing utility, you obtain

$y = -0.1283t^2 + 11.0988t + 207.1116$

(c) For the year 2004, $t = 54$ and
 $y \approx 432.3$ acres per farm.



80. (a) If (x, y) is on the graph, then so is $(-x, y)$ by y -axis symmetry. Since $(-x, y)$ is on the graph, then so is $(-x, -y)$ by x -axis symmetry. Hence, the graph is symmetric with respect to the origin. The converse is not true. For example, $y = x^3$ has origin symmetry but is not symmetric with respect to either the x -axis or the y -axis.

(b) Assume that the graph has x -axis and origin symmetry. If (x, y) is on the graph, so is $(x, -y)$ by x -axis symmetry. Since $(x, -y)$ is on the graph, then so is $(-x, -(-y)) = (-x, y)$ by origin symmetry. Therefore, the graph is symmetric with respect to the y -axis. The argument is similar for y -axis and origin symmetry.

82. True

84. True; the x -intercept is

$$\left(-\frac{b}{2a}, 0\right).$$

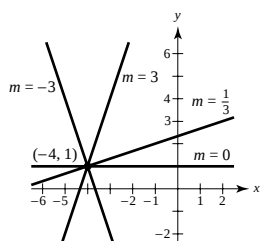
Section P.2 Linear Models and Rates of Change

2. $m = 2$

4. $m = -1$

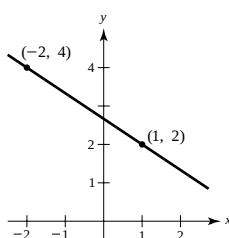
6. $m = \frac{40}{3}$

8.

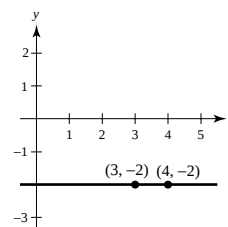


10.
$$m = \frac{4 - 2}{-2 - 1}$$

$$= -\frac{2}{3}$$

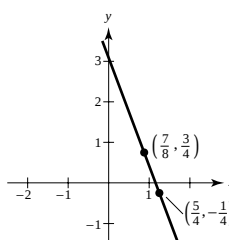


12.
$$m = \frac{-2 - (-2)}{4 - 1} = 0$$



14.
$$m = \frac{(3/4) - (-1/4)}{(7/8) - (5/4)}$$

$$= \frac{1}{-3/8} = -\frac{8}{3}$$



16. Since the slope is undefined, the line is vertical and its equation is $x = -3$. Therefore, three additional points are $(-3, 2)$, $(-3, 3)$, and $(-3, 5)$.

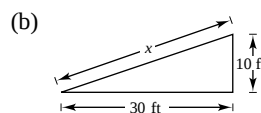
18. The equation of this line is

$$y + 2 = 2(x + 2)$$

$$y = 2x + 2.$$

Therefore, three additional points are $(-3, -4)$, $(-1, 0)$, and $(0, 2)$.

20. (a) $\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{1}{3}$



By the Pythagorean Theorem,

$$x^2 = 30^2 + 10^2 = 1000$$

$$x \approx 31.623 \text{ feet.}$$

22. (a) $m = 400$ indicates that the revenues increase by 400 in one day.

(b) $m = 100$ indicates that the revenues increase by 100 in one day.

(c) $m = 0$ indicates that the revenues do not change from one day to the next.

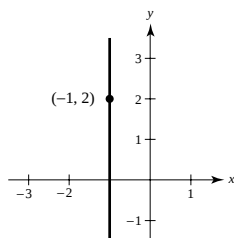
24. $6x - 5y = 15$

$$y = \frac{6}{5}x - 3$$

Therefore, the slope is $m = \frac{6}{5}$ and the y-intercept is $(0, -3)$.

28. $x = -1$

$$x + 1 = 0$$

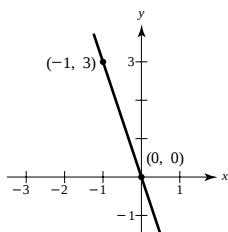


34. $m = \frac{3 - 0}{-1 - 0} = -3$

$$y - 0 = -3(x - 0)$$

$$y = -3x$$

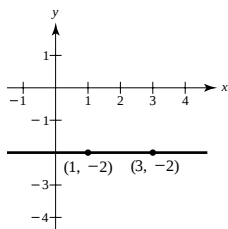
$$3x + y = 0$$



40. $m = 0$

$$y = -2$$

$$y + 2 = 0$$

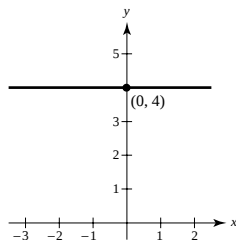


26. $y = -1$

The line is horizontal. Therefore, the slope is $m = 0$ and the y-intercept is $(0, -1)$.

30. $y = 4$

$$y - 4 = 0$$

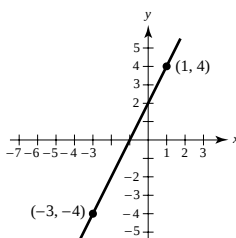


36. $m = \frac{4 - (-4)}{1 - (-3)} = \frac{8}{4} = 2$

$$y - 4 = 2(x - 1)$$

$$y - 4 = 2x - 2$$

$$0 = 2x - y + 2$$



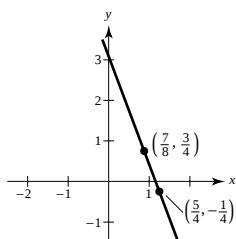
42. $m = \frac{(3/4) - (-1/4)}{(7/8) - (5/4)}$

$$= \frac{1}{-3/8} = -\frac{8}{3}$$

$$y + \frac{1}{4} = -\frac{8}{3}\left(x - \frac{5}{4}\right)$$

$$12y + 3 = -32x + 40$$

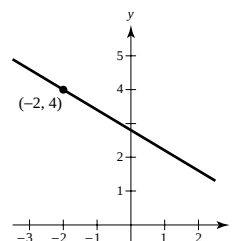
$$32x + 12y - 37 = 0$$



32. $y - 4 = -\frac{3}{5}(x + 2)$

$$5y - 20 = -3x - 6$$

$$3x + 5y - 14 = 0$$

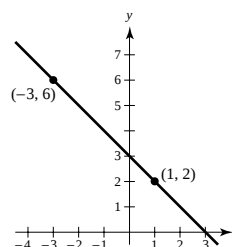


38. $m = \frac{6 - 2}{-3 - 1} = \frac{4}{-4} = -1$

$$y - 2 = -1(x - 1)$$

$$y - 2 = -x + 1$$

$$x + y - 3 = 0$$

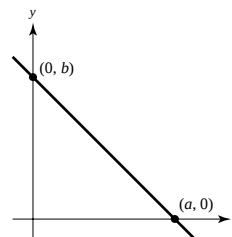


44. $m = -\frac{b}{a}$

$$y = -\frac{b}{a}x + b$$

$$\frac{b}{a}x + y = b$$

$$\frac{x}{a} + \frac{y}{b} = 1$$



46. $\frac{x}{-2/3} + \frac{y}{-2} = 1$

$$\frac{-3x}{2} - \frac{y}{2} = 1$$

$$3x + y = -2$$

$$3x + y + 2 = 0$$

48. $\frac{x}{a} + \frac{y}{a} = 1$

$$\frac{-3}{a} + \frac{4}{a} = 1$$

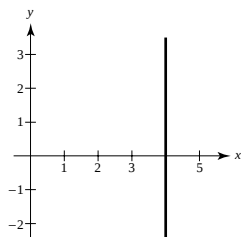
$$\frac{1}{a} = 1$$

$$a = 1 \Rightarrow x + y = 1$$

$$x + y - 1 = 0$$

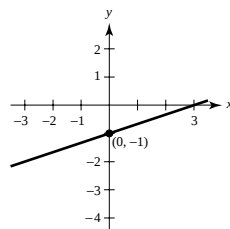
50. $x = 4$

$$x - 4 = 0$$



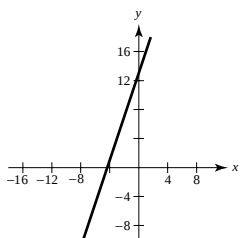
52. $y = \frac{1}{3}x - 1$

$$3y - x + 3 = 0$$



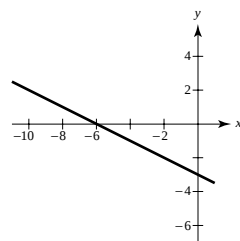
54. $y - 1 = 3(x + 4)$

$$y = 3x + 13$$

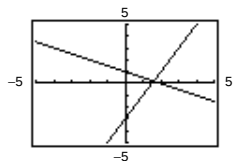


56. $x + 2y + 6 = 0$

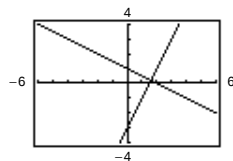
$$y = -\frac{1}{2}x - 3$$



58.



The lines do not appear perpendicular.



The lines appear perpendicular.

The lines are perpendicular because their slopes 2 and $-\frac{1}{2}$ are negative reciprocals of each other. You must use a square setting in order for perpendicular lines to appear perpendicular.

60. $x + y = 7$

$$y = -x + 7$$

$$m = -1$$

(a) $y - 2 = -1(x + 3)$

$$y - 2 = -x - 3$$

$$x + y + 1 = 0$$

(b) $y - 2 = 1(x + 3)$

$$y - 2 = x + 3$$

$$x - y + 5 = 0$$

62. $3x + 4y = 7$

$$y = -\frac{3}{4}x + \frac{7}{4}$$

$$m = -\frac{3}{4}$$

(a) $y - 4 = -\frac{3}{4}(x + 6)$

$$4y - 16 = -3x - 18$$

$$3x + 4y + 2 = 0$$

(b) $y - 4 = \frac{4}{3}(x + 6)$

$$3y - 12 = 4x + 24$$

$$4x - 3y + 36 = 0$$

64. (a) $y = 0$

(b) $x = -1 \Rightarrow x + 1 = 0$

66. The slope is 4.50.

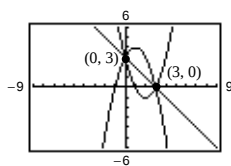
$$\text{Hence, } V = 4.5(t - 1) + 156$$

$$= 4.5t + 151.5$$

68. The slope is -5600 . Hence, $V = -5600(t - 1) + 245,000$

$$= -5600t + 250,600$$

70.



You can use the graphing utility to determine that the points of intersection are $(0, 3)$ and $(3, 0)$. Analytically,

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0 \Rightarrow y = 3 \Rightarrow (0, 3)$$

$$x = 3 \Rightarrow y = 0 \Rightarrow (3, 0).$$

The slope of the line joining $(0, 3)$ and $(3, 0)$ is $m = (0 - 3)/(3 - 0) = -1$. Hence, an equation of the line is

$$y - 3 = -1(x - 0)$$

$$y = -x + 3.$$

72. $m_1 = \frac{-6 - 4}{7 - 0} = -\frac{10}{7}$

$$m_2 = \frac{11 - 4}{-5 - 0} = -\frac{7}{5}$$

$$m_1 \neq m_2$$

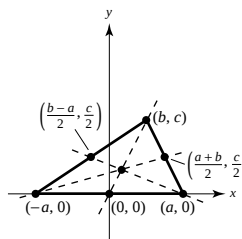
The points are not collinear.

74. Equations of medians:

$$y = \frac{c}{b}x$$

$$y = \frac{c}{3a+b}(x+a)$$

$$y = \frac{c}{-3a+b}(x-a)$$



Solving simultaneously, the point of intersection is

$$\left(\frac{b}{3}, \frac{c}{3}\right).$$

76. The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(b, \frac{a^2 - b^2}{c}\right)$ is:

$$m_1 = \frac{[(a^2 - b^2)/c] - (c/3)}{b - (b/3)} = \frac{(3a^2 - 3b^2 - c^2)/(3c)}{(2b)/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

The slope of the line segment from $\left(\frac{b}{3}, \frac{c}{3}\right)$ to $\left(0, \frac{-a^2 + b^2 + c^2}{2c}\right)$ is:

$$m_2 = \frac{[(-a^2 + b^2 + c^2)/(2c)] - (c/3)}{0 - (b/3)} = \frac{(-3a^2 + 3b^2 + 3c^2 - 2c^2)/(6c)}{-b/3} = \frac{3a^2 - 3b^2 - c^2}{2bc}$$

$$m_1 = m_2$$

Therefore, the points are collinear.

78. $C = 0.34x + 150$. If $x = 137$, $C = 0.34(137) + 150 = \$196.58$

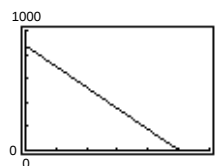
80. (a) Depreciation per year:

$$\frac{875}{5} = \$175$$

$$y = 875 - 175x$$

where $0 \leq x \leq 5$.

$$(b) y = 875 - 175(2) = \$525$$

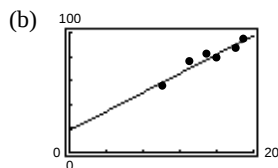


$$(c) 200 = 875 - 175x$$

$$175x = 675$$

$$x \approx 3.86 \text{ years}$$

82. (a) $y = 18.91 + 3.97x$ (x = quiz score, y = test score)



$$(c) \text{ If } x = 17, y = 18.91 + 3.97(17) = 86.4.$$

(d) The slope shows the average increase in exam score for each unit increase in quiz score.

(e) The points would shift vertically upward 4 units. The new regression line would have a y -intercept 4 greater than before: $y = 22.91 + 3.97x$.

$$84. 4x + 3y - 10 = 0 \Rightarrow d = \frac{|4(2) + 3(3) - 10|}{\sqrt{4^2 + 3^2}} = \frac{7}{5}$$

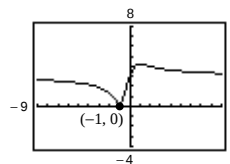
$$86. x + 1 = 0 \Rightarrow d = \frac{|1(6) + (0)(2) + 1|}{\sqrt{1^2 + 0^2}} = 7$$

88. A point on the line $3x - 4y = 1$ is $(-1, -1)$. The distance from the point $(-1, -1)$ to $3x - 4y - 10 = 0$ is

$$d = \frac{|-3 + 4 - 10|}{5} = \frac{9}{5}.$$

90. $y = mx + 4 \Rightarrow mx + (-1)y + 4 = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m(3) + (-1)(1) + 4|}{\sqrt{m^2 + (-1)^2}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$



The distance is 0 when $m = -1$. In this case, the line $y = -x + 4$ contains the point $(3, 1)$.

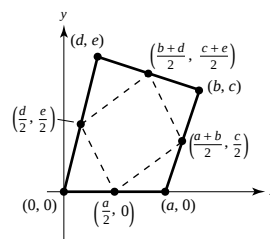
92. For simplicity, let the vertices of the quadrilateral be $(0, 0)$, $(a, 0)$, (b, c) , and (d, e) , as shown in the figure. The midpoints of the sides are

$$\left(\frac{a}{2}, 0\right), \left(\frac{a+b}{2}, \frac{c}{2}\right), \left(\frac{b+d}{2}, \frac{c+e}{2}\right), \text{ and } \left(\frac{d}{2}, \frac{e}{2}\right).$$

The slope of the opposite sides are equal:

$$\begin{aligned} \frac{\frac{c}{2} - 0}{\frac{a+b}{2} - \frac{a}{2}} &= \frac{\frac{c+e}{2} - \frac{e}{2}}{\frac{b+d}{2} - \frac{d}{2}} = \frac{c}{b} \\ 0 - \frac{e}{2} &= \frac{\frac{c}{2} - \frac{c+e}{2}}{\frac{a+b}{2} - \frac{b+d}{2}} = -\frac{e}{a-d} \end{aligned}$$

Therefore, the figure is a parallelogram.



94. If $m_1 = -1/m_2$, then $m_1 m_2 = -1$. Let L_3 be a line with slope m_3 that is perpendicular to L_1 . Then $m_1 m_3 = -1$. Hence, $m_2 = m_3 \Rightarrow L_2$ and L_3 are parallel. Therefore, L_2 and L_1 are also perpendicular.

96. False; if m_1 is positive, then $m_2 = -1/m_1$ is negative.

Section P.3 Functions and Their Graphs

2. (a) $f(-2) = \sqrt{-2+3} = \sqrt{1} = 1$

(b) $f(6) = \sqrt{6+3} = \sqrt{9} = 3$

(c) $f(c) = \sqrt{c+3}$

(d) $f(x + \Delta x) = \sqrt{x + \Delta x + 3}$

4. (a) $g(4) = 4^2(4-4) = 0$

(b) $g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2\left(\frac{3}{2}-4\right) = \frac{9}{4}\left(-\frac{5}{2}\right) = -\frac{45}{8}$

(c) $g(c) = c^2(c-4) = c^3 - 4c^2$

(d) $g(t+4) = (t+4)^2(t+4-4) = (t+4)^2 t = t^3 + 8t^2 + 16t$

6. (a) $f(\pi) = \sin \pi = 0$

(c) $f\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

(b) $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

8. $\frac{f(x) - f(1)}{x - 1} = \frac{3x - 1 - (3 - 1)}{x - 1} = \frac{3(x - 1)}{x - 1} = 3, x \neq 1$

10. $\frac{f(x) - f(1)}{x - 1} = \frac{x^3 - x - 0}{x - 1} = \frac{x(x+1)(x-1)}{x-1} = x(x+1), x \neq 1$

12. $g(x) = x^2 - 5$

 Domain: $(-\infty, \infty)$

 Range: $[-5, \infty)$

16. $g(x) = \frac{2}{x-1}$

 Domain: $(-\infty, 1), (1, \infty)$

 Range: $(-\infty, 0), (0, \infty)$

18. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2) = (-2)^2 + 2 = 6$

(b) $f(0) = 0^2 + 2 = 2$

(c) $f(1) = 1^2 + 2 = 3$

(d) $f(s^2 + 2) = 2(s^2 + 2)^2 = 2s^4 + 8s^2 + 10$

 (Note: $s^2 + 2 > 1$ for all s)

 Domain: $(-\infty, \infty)$

 Range: $[2, \infty)$

14. $h(t) = \cot t$

 Domain: all $t \neq k\pi$, k an integer

 Range: $(-\infty, \infty)$

20. $f(x) = \begin{cases} \sqrt{x+4}, & x \leq 5 \\ (x-5)^2, & x > 5 \end{cases}$

(a) $f(-3) = \sqrt{-3+4} = \sqrt{1} = 1$

(b) $f(0) = \sqrt{0+4} = 2$

(c) $f(5) = \sqrt{5+4} = 3$

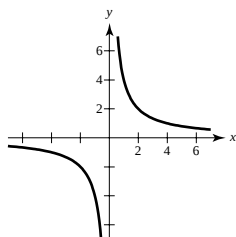
(d) $f(10) = (10-5)^2 = 25$

 Domain: $[-4, \infty)$

 Range: $[0, \infty)$

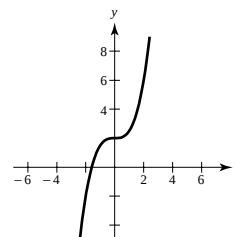
22. $g(x) = \frac{4}{x}$

 Domain: $(-\infty, 0), (0, \infty)$

 Range: $(-\infty, 0), (0, \infty)$


24. $f(x) = \frac{1}{2}x^3 + 2$

 Domain: $(-\infty, \infty)$

 Range: $(-\infty, \infty)$


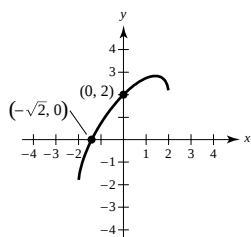
26. $f(x) = x + \sqrt{4-x^2}$

 Domain: $[-2, 2]$

Range:

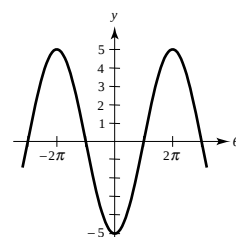
$$[-2, 2\sqrt{2}] \approx [-2, 2.83]$$

 y-intercept: $(0, 2)$

 x-intercept: $(-\sqrt{2}, 0)$


28. $h(\theta) = -5 \cos \frac{\theta}{2}$

 Domain: $(-\infty, \infty)$

 Range: $[-5, 5]$


30. $\sqrt{x^2 - 4} - y = 0 \Rightarrow y = \sqrt{x^2 - 4}$

y is a function of x . Vertical lines intersect the graph at most once.

32. $x^2 + y^2 = 4$

$$y = \pm \sqrt{4 - x^2}$$

y is not a function of x . Some vertical lines intersect the graph twice.

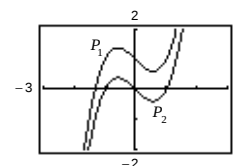
34. $x^2 + y = 4 \Rightarrow y = 4 - x^2$

y is a function of x since there is one value of y for each x .

36. $x^2y - x^2 + 4y = 0 \Rightarrow y = \frac{x^2}{x^2 + 4}$

y is a function of x since there is one value of y for each x .

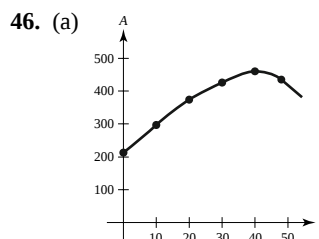
38. $p_1(x) = x^3 - x + 1$ has one zero. $p_2(x) = x^3 - x$ has three zeros. Every cubic polynomial has at least one zero. Given $p(x) = Ax^3 + Bx^2 + Cx + D$, we have $p \rightarrow -\infty$ as $x \rightarrow -\infty$ and $p \rightarrow \infty$ as $x \rightarrow \infty$ if $A > 0$. Furthermore, $p \rightarrow \infty$ as $x \rightarrow -\infty$ and $p \rightarrow -\infty$ as $x \rightarrow \infty$ if $A < 0$. Since the graph has no breaks, the graph must cross the x -axis at least one time.



40. The function is $f(x) = cx$. Since $(1, 1/4)$ satisfies the equation, $c = 1/4$. Thus, $f(x) = (1/4)x$.

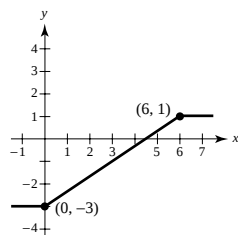
42. The function is $h(x) = c\sqrt{|x|}$. Since $(1, 3)$ satisfies the equation, $c = 3$. Thus, $h(x) = 3\sqrt{|x|}$.

44. The student travels $\frac{2-0}{4-0} = \frac{1}{2}$ mi/min during the first 4 minutes. The student is stationary for the following 2 minutes. Finally, the student travels $\frac{6-2}{10-6} = 1$ mi/min during the final 4 minutes.

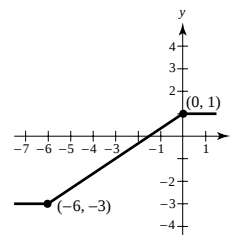


(b) $A(15) \approx 345$ acres/farm

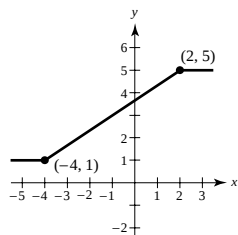
48. (a) $g(x) = f(x - 4)$
 $g(6) = f(2) = 1$
 $g(0) = f(-4) = -3$
 Shift f right 4 units



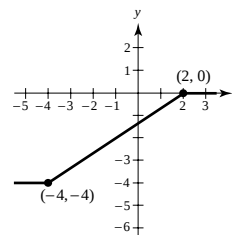
- (b) $g(x) = f(x + 2)$
 Shift f left 2 units



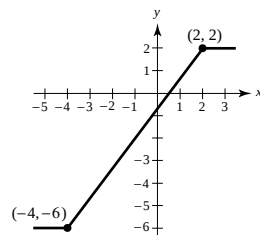
- (c) $g(x) = f(x) + 4$
 Vertical shift upwards
 4 units



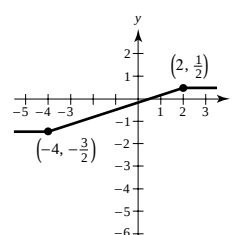
- (d) $g(x) = f(x) - 1$
 Vertical shift down 1 unit



- (e) $g(x) = 2f(x)$
 $g(2) = 2f(2) = 2$
 $g(-4) = 2f(-4) = -6$



- (f) $g(x) = \frac{1}{2}f(x)$
 $g(2) = \frac{1}{2}f(2) = \frac{1}{2}$
 $g(-4) = \frac{1}{2}f(-4) = -\frac{3}{2}$



50. (a) $h(x) = \sin(x + (\pi/2)) + 1$ is a horizontal shift $\pi/2$ units to the left, followed by a vertical shift 1 unit upwards.
 (b) $h(x) = -\sin(x - 1)$ is a horizontal shift 1 unit to the right followed by a reflection about the x -axis.

52. (a) $f(g(1)) = f(0) = 0$

(b) $g(f(1)) = g(1) = 0$

(c) $g(f(0)) = g(0) = -1$

(d) $f(g(-4)) = f(15) = \sqrt{15}$

(e) $f(g(x)) = f(x^2 - 1) = \sqrt{x^2 - 1}$

(f) $g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1 \quad (x \geq 0)$

54. $f(x) = x^2 - 1, g(x) = \cos x$

$(f \circ g)(x) = f(g(x)) = f(\cos x) = \cos^2 x - 1$

Domain: $(-\infty, \infty)$

$(g \circ f)(x) = g(x^2 - 1) = \cos(x^2 - 1)$

Domain: $(-\infty, \infty)$

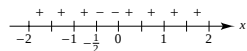
No, $f \circ g \neq g \circ f$.

56. $(f \circ g)(x) = f(\sqrt{x+2}) = \frac{1}{\sqrt{x+2}}$

Domain: $(-2, \infty)$

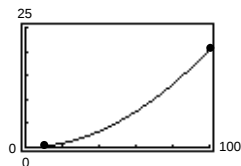
$(g \circ f)(x) = g\left(\frac{1}{x}\right) = \sqrt{\frac{1}{x} + 2} = \sqrt{\frac{1+2x}{x}}$

You can find the domain of $g \circ f$ by determining the intervals where $(1 + 2x)$ and x are both positive, or both negative.



Domain: $(-\infty, -\frac{1}{2}], (0, \infty)$

58. (a)



$$\begin{aligned} \text{(b) } H(1.6x) &= 0.002(1.6x)^2 + 0.005(1.6x) - 0.029 \\ &= 0.00512x^2 + 0.008x - 0.029 \end{aligned}$$

62. $f(-x) = \sin^2(-x) = \sin(-x) \sin(-x) = (-\sin x)(-\sin x) = \sin^2 x$

Even

60. $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$

Odd

64. (a) If f is even, then $(-4, 9)$ is on the graph.(b) If f is odd, then $(-4, -9)$ is on the graph.

$$\begin{aligned} \text{66. } f(-x) &= a_{2n}(-x)^{2n} + a_{2n-2}(-x)^{2n-2} + \cdots + a_2(-x)^2 + a_0 \\ &= a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0 \\ &= f(x) \end{aligned}$$

Even

68. Let $F(x) = f(x)g(x)$ where f is even and g is odd. Then

$$F(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -F(x).$$

Thus, $F(x)$ is odd.

70. (a) Let $F(x) = f(x) \pm g(x)$ where f and g are even. Then, $F(-x) = f(-x) \pm g(-x) = f(x) \pm g(x) = F(x)$. Thus, $F(x)$ is even.
- (b) Let $F(x) = f(x) \pm g(x)$ where f and g are odd. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \mp g(x) = -F(x)$. Thus, $F(x)$ is odd.
- (c) Let $F(x) = f(x) \pm g(x)$ where f is odd and g is even. Then, $F(-x) = f(-x) \pm g(-x) = -f(x) \pm g(x)$. Thus, $F(x)$ is neither odd nor even.

72. By equating slopes, $\frac{y-2}{0-3} = \frac{0-2}{x-3}$

$$y-2 = \frac{6}{x-3}$$

$$y = \frac{6}{x-3} + 2 = \frac{2x}{x-3},$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(\frac{2x}{x-3}\right)^2}.$$

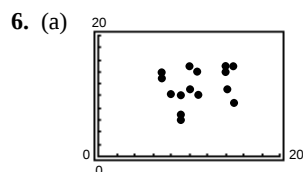
74. True

76. False; let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. Thus, $3f(x) \neq f(3x)$.

Section P.4 Fitting Models to Data

2. Trigonometric function

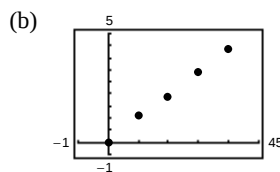
4. No relationship



No, the relationship does not appear to be linear.

- (b) Quiz scores are dependent on several variables such as study time, class attendance, etc. These variables may change from one quiz to the next.

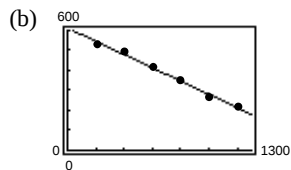
8. (a) $s = 9.7t + 0.4$



The model fits well.

- (c) If $t = 2.5$, $s = 24.65$ meters/second.

10. (a) Linear model: $H = -0.3323t + 612.9333$

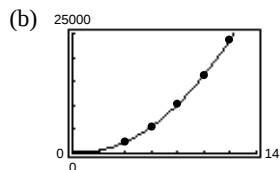


The fit is very good.

- (c) When $t = 500$,

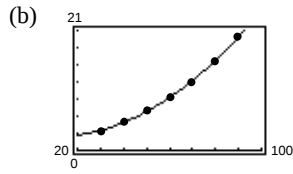
$$H = -0.3323(500) + 612.9333 \approx 446.78.$$

12. (a) $S = 180.89x^2 - 205.79x + 272$



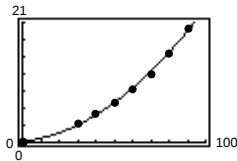
- (c) When $x = 2$, $S \approx 583.98$ pounds.

14. (a) $t = 0.00271s^2 - 0.0529s + 2.671$



(c) The curve levels off for $s < 20$.

(d) $t = 0.002s^2 + 0.0346s + 0.183$

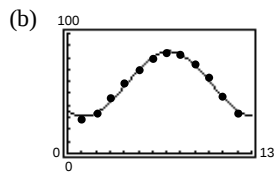


The model is better for low speeds.

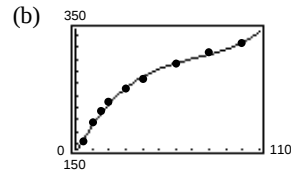
18. (a) $H(t) = 84.4 + 4.28 \sin\left(\frac{\pi t}{6} + 3.86\right)$

One model is

$$C(t) = 58 + 27 \sin\left(\frac{\pi t}{6} + 4.1\right).$$

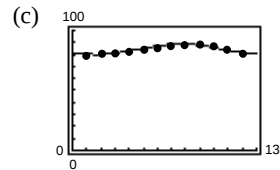


16. (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641 p^2 + 5.2826 p + 143.1$



(c) For $T = 300^\circ\text{F}$, $p \approx 68.29$ pounds per square inch.

(d) The model is based on data up to 100 pounds per square inch.



(d) The average in Honolulu is 84.4.

The average in Chicago is 58.

(e) The period is 12 months (1 year).

(f) Chicago has greater variability ($27 > 4.28$).

20. Answers will vary.

Review Exercises for Chapter P

2. $y = (x - 1)(x - 3)$

$$x = 0 \Rightarrow y = (0 - 1)(0 - 3) = 3 \Rightarrow (0, 3) \quad y\text{-intercept}$$

$$y = 0 \Rightarrow 0 = (x - 1)(x - 3) \Rightarrow x = 1, 3 \Rightarrow (1, 0), (3, 0) \quad x\text{-intercepts}$$

4. $xy = 4$

$x = 0$ and $y = 0$ are both impossible. No intercepts.

6. Symmetric with respect to y -axis since

$$y = (-x)^4 - (-x)^2 + 3$$

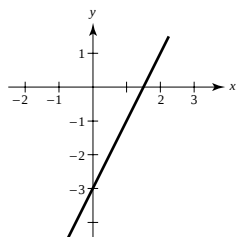
$$y = x^4 - x^2 + 3.$$

8. $4x - 2y = 6$

$$y = 2x - 3$$

Slope: 2

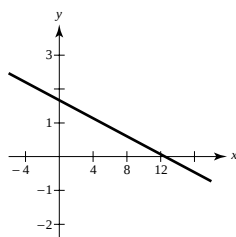
y-intercept: -3



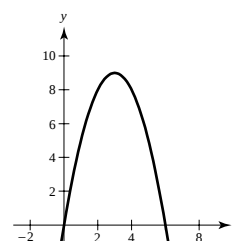
10. $0.02x + 0.15y = 0.25$

$$2x + 15y = 25$$

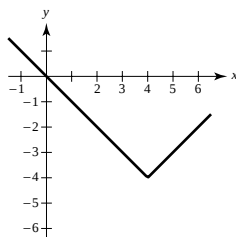
$$y = -\frac{2}{15}x + \frac{5}{3}$$

Slope: $-\frac{2}{15}$ y-intercept: $\frac{5}{3}$ 

12. $y = x(6 - x)$



14. $y = |x - 4| - 4$



16. $y = 8\sqrt[3]{x - 6}$

Xmin = -40
 Xmax = 40
 Xscl = 10
 Ymin = -40
 Ymax = 40
 Yscl = 10

18. $y = x + 1$

$$(x + 1) - x^2 = 7$$

$$0 = x^2 - x + 6$$

No real solution

No points of intersection

The graphs of $y = x + 1$ and $y = x^2 + 7$ do not intersect.

20. $y = kx^3$

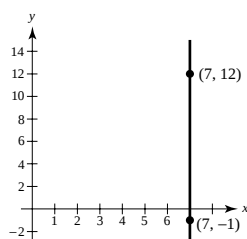
(a) $4 = k(1)^3 \Rightarrow k = 4$ and $y = 4x^3$

(c) $0 = k(0)^3 \Rightarrow$ any k will do!

(b) $1 = k(-2)^3 \Rightarrow k = -\frac{1}{8}$ and $y = -\frac{1}{8}x^3$

(d) $-1 = k(-1)^3 \Rightarrow k = 1 \Rightarrow y = x^3$

22.



The line is vertical and has no slope.

24. $\frac{3 - (-1)}{-3 - t} = \frac{3 - 6}{-3 - 8}$

$$\frac{4}{-3 - t} = \frac{-3}{-11}$$

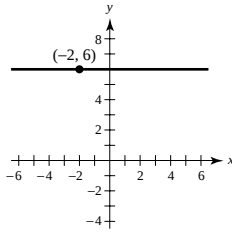
$$-44 = 9 + 3t$$

$$-53 = 3t$$

$$t = -\frac{53}{3}$$

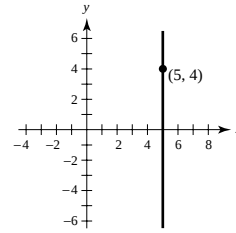
26. $y - 6 = 0(x - (-2))$

$y = 6$ Horizontal line



28. m is undefined. Line is vertical.

$x = 5$



30. (a) $y - 3 = -\frac{2}{3}(x - 1)$

$3y - 9 = -2x + 2$

$2x + 3y - 11 = 0$

(b) Slope of perpendicular line is 1.

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(c) $m = \frac{4 - 3}{2 - 1} = 1$

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(d) $y = 3$

$y - 3 = 0$

32. (a) $C = 9.25t + 13.50t + 36,500$

$= 22.75t + 36,500$

(b) $R = 30t$

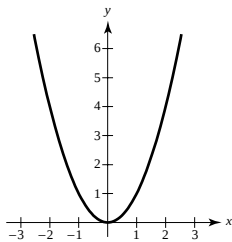
(c) $30t = 22.75t + 36,500$

$7.25t = 36,000$

$t \approx 5034.48$ hours to break even.

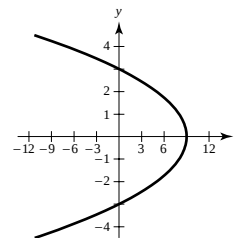
34. $x^2 - y = 0$

Function of x since there is one value for y for each x .



36. $x = 9 - y^2$

Not a function of x since there are two values of y for some x .



38. (a) $f(-4) = (-4)^2 + 2 = 18$ (because $-4 < 0$)

(b) $f(0) = |0 - 2| = 2$

(c) $f(1) = |1 - 2| = 1$

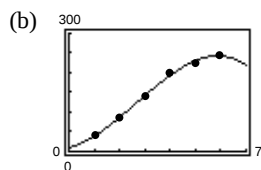
40. $f(x) = 1 - x^2$ and $g(x) = 2x + 1$

(a) $f(x) - g(x) = (1 - x^2) - (2x + 1) = -x^2 - 2x$

(b) $f(x)g(x) = (1 - x^2)(2x + 1) = -2x^3 - x^2 + 2x + 1$

(c) $g(f(x)) = g(1 - x^2) = 2(1 - x^2) + 1 = 3 - 2x^2$

15. (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$



(c) If $x = 4.5$, $y \approx 214$ horsepower.

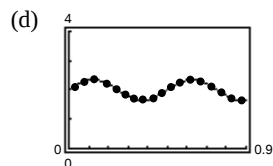
17. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.

19. Answers will vary.

Review Exercises for Chapter P

1. $y = 2x - 3$

$$x = 0 \Rightarrow y = 2(0) - 3 = -3 \Rightarrow (0, -3) \quad y\text{-intercept}$$

$$y = 0 \Rightarrow 0 = 2x - 3 \Rightarrow x = \frac{3}{2} \Rightarrow (\frac{3}{2}, 0) \quad x\text{-intercept}$$

3. $y = \frac{x-1}{x-2}$

$$x = 0 \Rightarrow y = \frac{0-1}{0-2} = \frac{1}{2} \Rightarrow (0, \frac{1}{2}) \quad y\text{-intercept}$$

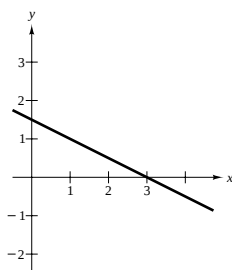
$$y = 0 \Rightarrow 0 = \frac{x-1}{x-2} \Rightarrow x = 1 \Rightarrow (1, 0) \quad x\text{-intercept}$$

5. Symmetric with respect to y -axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$

$$x^2y - x^2 + 4y = 0.$$

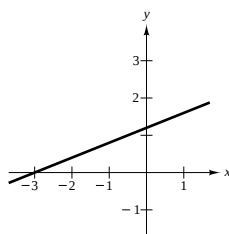
7. $y = -\frac{1}{2}x + \frac{3}{2}$



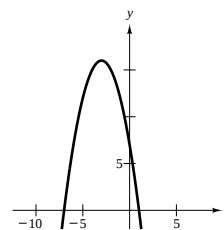
9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

$$-\frac{2}{5}x + y = \frac{6}{5}$$

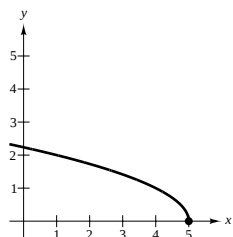
$$y = \frac{2}{5}x + \frac{6}{5}$$

Slope: $\frac{2}{5}$ y -intercept: $\frac{6}{5}$ 

11. $y = 7 - 6x - x^2$



13. $y = \sqrt{5 - x}$

Domain: $(-\infty, 5]$ 

15. $y = 4x^2 - 25$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $3x - 4y = 8$

$$\frac{4x + 4y = 20}{7x} = 28$$

$$7x = 28$$

$$x = 4$$

$$y = 1$$

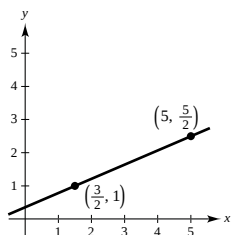
Point: (4, 1)

19. You need factors $(x + 2)$ and $(x - 2)$. Multiply by x to obtain origin symmetry

$$y = x(x + 2)(x - 2).$$

$$= x^3 - 4x.$$

21.



$$\text{Slope} = \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$$

23. $\frac{1 - t}{1 - 0} = \frac{1 - 5}{1 - (-2)}$

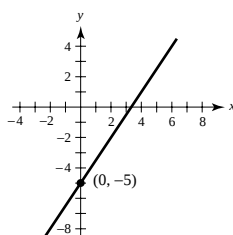
$$1 - t = -\frac{4}{3}$$

$$t = \frac{7}{3}$$

25. $y - (-5) = \frac{3}{2}(x - 0)$

$$y = \frac{3}{2}x - 5$$

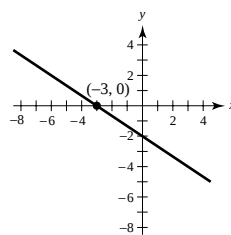
$$2y - 3x + 10 = 0$$



27. $y - 0 = -\frac{2}{3}(x - (-3))$

$$y = -\frac{2}{3}x - 2$$

$$3y + 2x + 6 = 0$$



29. (a) $y - 4 = \frac{7}{16}(x + 2)$

$$16y - 64 = 7x + 14$$

$$0 = 7x - 16y + 78$$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$$y = -2x$$

$$2x + y = 0$$

(b) Slope of line is $\frac{5}{3}$.

$$y - 4 = \frac{5}{3}(x + 2)$$

$$3y - 12 = 5x + 10$$

$$0 = 5x - 3y + 22$$

(d) $x = -2$

$$x + 2 = 0$$

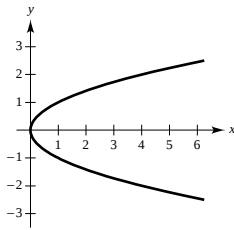
31. The slope is -850 . $V = -850t + 12,500$.

$$V(3) = -850(3) + 12,500 = \$9950$$

33. $x - y^2 = 0$

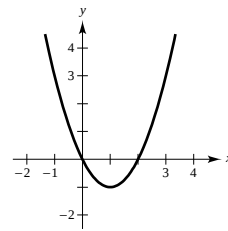
$$y = \pm \sqrt{x}$$

Not a function of x since there are two values of y for some x .



35. $y = x^2 - 2x$

Function of x since there is one value of y for each x .



37. $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b)
$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$$

$$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$$

39. (a) Domain: $36 - x^2 \geq 0 \Rightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

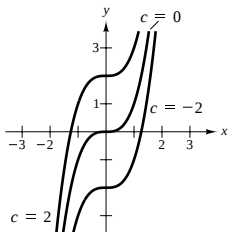
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

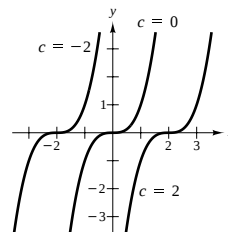
(c) Domain: all x or $(-\infty, \infty)$

Range: all y or $(-\infty, \infty)$

41. (a) $f(x) = x^3 + c, c = -2, 0, 2$



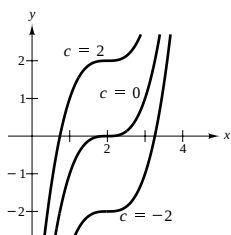
(b) $f(x) = (x - c)^3, c = -2, 0, 2$



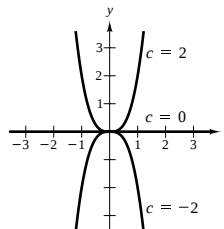
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41. —CONTINUED—

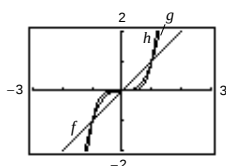
(c) $f(x) = (x - 2)^3 + c$, $c = -2, 0, 2$



(d) $f(x) = cx^3$, $c = -2, 0, 2$



43. (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

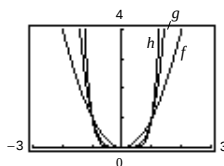


The graphs of f , g , and h all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

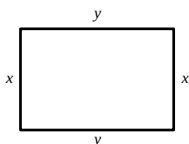
$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$



The graphs of f , g , and h all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

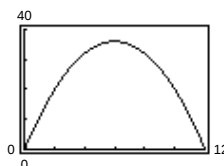
45. (a)



$2x + 2y = 24$

$y = 12 - x$

$A = xy = x(12 - x) = 12x - x^2$

(b) Domain: $0 < x < 12$ 

(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

47. (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

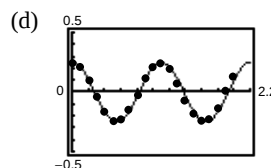
(d) 5, positive leading coefficient

49. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$(0.25 - (-0.25))/2 = 0.25.$

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4} \cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4} \cos(5.7t)$ 

Problem Solving for Chapter P

1. (a) $x^2 - 6x + y^2 - 8y = 0$
 $(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$
 $(x - 3)^2 + (y - 4)^2 = 25$

Center: (3, 4) Radius: 5

(c) Slope of line from (6, 0) to (3, 4) is $\frac{4 - 0}{3 - 6} = -\frac{4}{3}$.

Slope of tangent line is $\frac{3}{4}$. Hence,

$y - 0 = \frac{3}{4}(x - 6) \Rightarrow y = \frac{3}{4}x - \frac{9}{2}$ Tangent line

(b) Slope of line from (0, 0) to (3, 4) is $\frac{4}{3}$ Slope of tangent line is $-\frac{3}{4}$ Hence,

$y - 0 = -\frac{3}{4}(x - 0) \Rightarrow y = -\frac{3}{4}x$ Tangent line

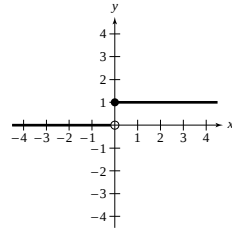
(d) $-\frac{3}{4}x = \frac{3}{4}x - \frac{9}{2}$

$\frac{3}{2}x = \frac{9}{2}$

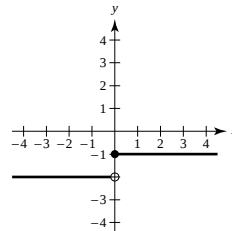
$x = 3$

Intersection: $\left(3, -\frac{9}{4}\right)$

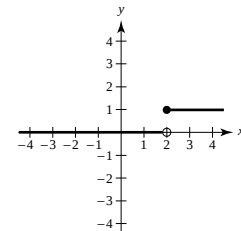
3. $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



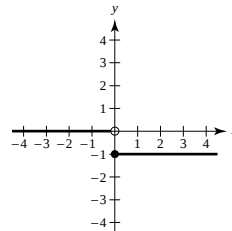
(a) $H(x) - 2$



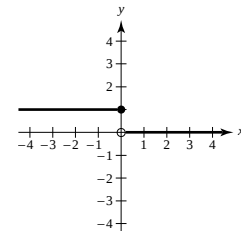
(b) $H(x - 2)$



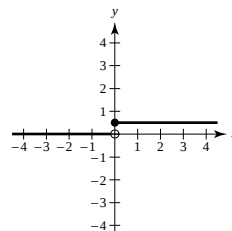
(c) $-H(x)$



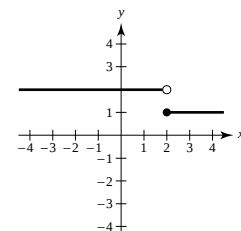
(d) $H(-x)$



(e) $\frac{1}{2}H(x)$



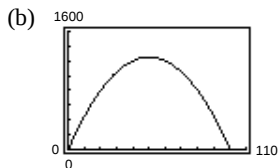
(f) $-H(x - 2) + 2$



5. (a) $x + 2y = 100 \Rightarrow y = \frac{100 - x}{2}$

$$A(x) = xy = x\left(\frac{100 - x}{2}\right) = -\frac{x^2}{2} + 50x$$

Domain: $0 < x < 100$



Maximum of 1250 m^2 at $x = 50 \text{ m}$, $y = 25 \text{ m}$.

$$\begin{aligned} \text{(c) } A(x) &= -\frac{1}{2}(x^2 - 100x) \\ &= -\frac{1}{2}(x^2 - 100x + 2500) + 1250 \\ &= -\frac{1}{2}(x - 50)^2 + 1250 \end{aligned}$$

$A(50) = 1250 \text{ m}^2$ is the maximum. $x = 50 \text{ m}$, $y = 25 \text{ m}$.

9. (a) Slope $= \frac{9 - 4}{3 - 2} = 5$. Slope of tangent line is less than 5.

(b) Slope $= \frac{4 - 1}{2 - 1} = 3$. Slope of tangent line is greater than 3.

(c) Slope $= \frac{4.41 - 4}{2.1 - 2} = 4.1$. Slope of tangent line is less than 4.1.

$$\begin{aligned} \text{(d) Slope} &= \frac{f(2 + h) - f(2)}{(2 + h) - 2} \\ &= \frac{(2 + h)^2 - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h, h \neq 0 \end{aligned}$$

(e) Letting h get closer and closer to 0, the slope approaches 4. Hence, the slope at $(2, 4)$ is 4.

11. (a) At $x = 1$ and $x = -3$ the sounds are equal.

(b) $\frac{I}{\sqrt{x^2 + y^2}} = \frac{2I}{\sqrt{(x - 3)^2 + y^2}}$

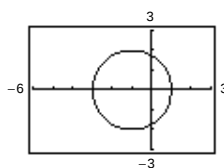
$$(x - 3)^2 + y^2 = 4(x^2 + y^2)$$

$$3x^2 + 3y^2 + 6x = 9$$

$$x^2 + 2x + y^2 = 3$$

$$(x + 1)^2 + y^2 = 4$$

Circle of radius 2 centered at $(-1, 0)$



7. The length of the trip in the water is $\sqrt{2^2 + x^2}$, and the length of the trip over land is $\sqrt{1 + (3 - x)^2}$. Hence, the total time is

$$T = \frac{\sqrt{4 + x^2}}{2} + \frac{\sqrt{1 + (3 - x)^2}}{4} \text{ hours.}$$

13.

$$d_1 d_2 = 1$$

$$[(x+1)^2 + y^2][(x-1)^2 + y^2] = 1$$

$$(x+1)^2(x-1)^2 + y^2[(x+1)^2 + (x-1)^2] + y^4 = 1$$

$$(x^2 - 1)^2 + y^2[2x^2 + 2] + y^4 = 1$$

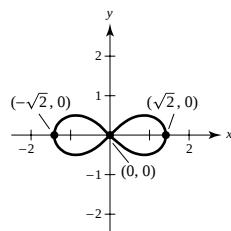
$$x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 = 1$$

$$(x^4 + 2x^2y^2 + y^4) - 2x^2 + 2y^2 = 0$$

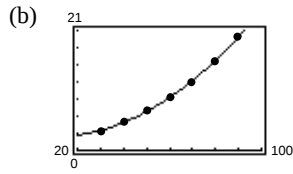
$$(x^2 + y^2)^2 = 2(x^2 - y^2)$$

Let $y = 0$. Then $x^4 = 2x^2 \Rightarrow x = 0$ or $x^2 = 2$.

Thus, $(0, 0)$, $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are on the curve.

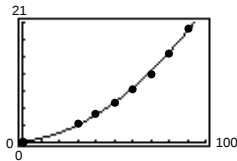


14. (a) $t = 0.00271s^2 - 0.0529s + 2.671$



(c) The curve levels off for $s < 20$.

(d) $t = 0.002s^2 + 0.0346s + 0.183$

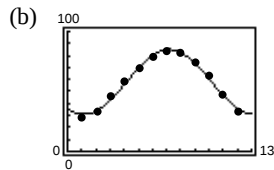


The model is better for low speeds.

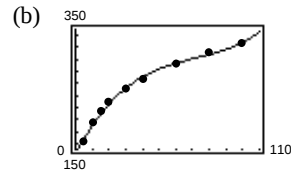
18. (a) $H(t) = 84.4 + 4.28 \sin\left(\frac{\pi t}{6} + 3.86\right)$

One model is

$$C(t) = 58 + 27 \sin\left(\frac{\pi t}{6} + 4.1\right).$$

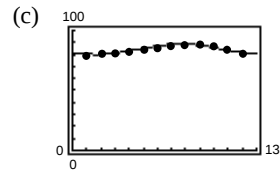


16. (a) $T = 2.9856 \times 10^{-4} p^3 - 0.0641 p^2 + 5.2826p + 143.1$



(c) For $T = 300^\circ\text{F}$, $p \approx 68.29$ pounds per square inch.

(d) The model is based on data up to 100 pounds per square inch.



(d) The average in Honolulu is 84.4.

The average in Chicago is 58.

(e) The period is 12 months (1 year).

(f) Chicago has greater variability ($27 > 4.28$).

20. Answers will vary.

Review Exercises for Chapter P

2. $y = (x - 1)(x - 3)$

$x = 0 \Rightarrow y = (0 - 1)(0 - 3) = 3 \Rightarrow (0, 3)$ y-intercept

$y = 0 \Rightarrow 0 = (x - 1)(x - 3) \Rightarrow x = 1, 3 \Rightarrow (1, 0), (3, 0)$ x-intercepts

4. $xy = 4$

$x = 0$ and $y = 0$ are both impossible. No intercepts.

6. Symmetric with respect to y-axis since

$$y = (-x)^4 - (-x)^2 + 3$$

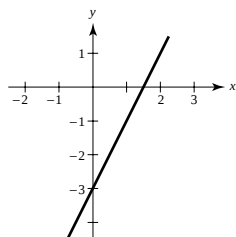
$$y = x^4 - x^2 + 3.$$

8. $4x - 2y = 6$

$$y = 2x - 3$$

Slope: 2

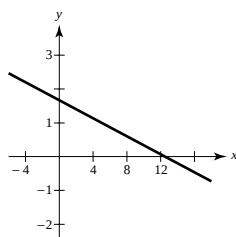
y-intercept: -3



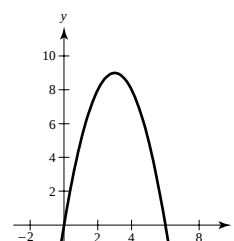
10. $0.02x + 0.15y = 0.25$

$$2x + 15y = 25$$

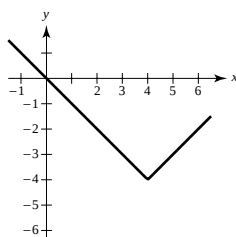
$$y = -\frac{2}{15}x + \frac{5}{3}$$

Slope: $-\frac{2}{15}$ y-intercept: $\frac{5}{3}$ 

12. $y = x(6 - x)$



14. $y = |x - 4| - 4$



16. $y = 8\sqrt[3]{x - 6}$

Xmin = -40
 Xmax = 40
 Xscl = 10
 Ymin = -40
 Ymax = 40
 Yscl = 10

18. $y = x + 1$

$$(x + 1) - x^2 = 7$$

$$0 = x^2 - x + 6$$

No real solution

No points of intersection

The graphs of $y = x + 1$ and $y = x^2 + 7$ do not intersect.

20. $y = kx^3$

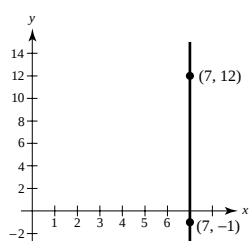
(a) $4 = k(1)^3 \Rightarrow k = 4$ and $y = 4x^3$

(c) $0 = k(0)^3 \Rightarrow$ any k will do!

(b) $1 = k(-2)^3 \Rightarrow k = -\frac{1}{8}$ and $y = -\frac{1}{8}x^3$

(d) $-1 = k(-1)^3 \Rightarrow k = 1 \Rightarrow y = x^3$

22.



The line is vertical and has no slope.

24. $\frac{3 - (-1)}{-3 - t} = \frac{3 - 6}{-3 - 8}$

$$\frac{4}{-3 - t} = \frac{-3}{-11}$$

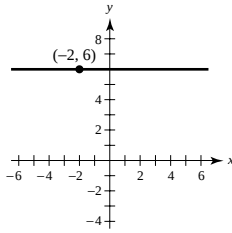
$$-44 = 9 + 3t$$

$$-53 = 3t$$

$$t = -\frac{53}{3}$$

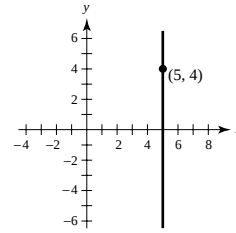
26. $y - 6 = 0(x - (-2))$

$y = 6$ Horizontal line



28. m is undefined. Line is vertical.

$x = 5$



30. (a) $y - 3 = -\frac{2}{3}(x - 1)$

$3y - 9 = -2x + 2$

$2x + 3y - 11 = 0$

(b) Slope of perpendicular line is 1.

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(c) $m = \frac{4 - 3}{2 - 1} = 1$

$y - 3 = 1(x - 1)$

$y = x + 2$

$0 = x - y + 2$

(d) $y = 3$

$y - 3 = 0$

32. (a) $C = 9.25t + 13.50t + 36,500$

$= 22.75t + 36,500$

(b) $R = 30t$

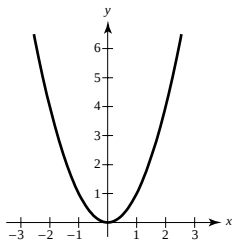
(c) $30t = 22.75t + 36,500$

$7.25t = 36,000$

$t \approx 5034.48$ hours to break even.

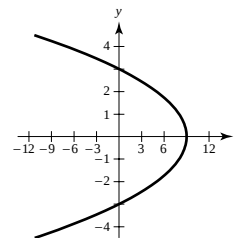
34. $x^2 - y = 0$

Function of x since there is one value for y for each x .



36. $x = 9 - y^2$

Not a function of x since there are two values of y for some x .



38. (a) $f(-4) = (-4)^2 + 2 = 18$ (because $-4 < 0$)

(b) $f(0) = |0 - 2| = 2$

(c) $f(1) = |1 - 2| = 1$

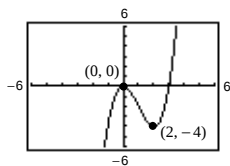
40. $f(x) = 1 - x^2$ and $g(x) = 2x + 1$

(a) $f(x) - g(x) = (1 - x^2) - (2x + 1) = -x^2 - 2x$

(b) $f(x)g(x) = (1 - x^2)(2x + 1) = -2x^3 - x^2 + 2x + 1$

(c) $g(f(x)) = g(1 - x^2) = 2(1 - x^2) + 1 = 3 - 2x^2$

42. $f(x) = x^3 - 3x^2$



- (a) The graph of g is obtained from f by a vertical shift down 1 unit, followed by a reflection in the x -axis:

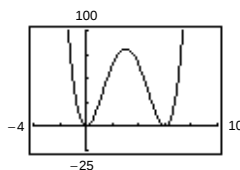
$$\begin{aligned} g(x) &= -[f(x) - 1] \\ &= -x^3 + 3x^2 + 1 \end{aligned}$$

- (b) The graph of g is obtained from f by a vertical shift upwards of 1 and a horizontal shift of 2 to the right.

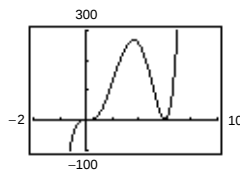
$$\begin{aligned} g(x) &= f(x - 2) + 1 \\ &= (x - 2)^3 - 3(x - 2)^2 + 1 \end{aligned}$$

46. For company (a) the profit rose rapidly for the first year, and then leveled off. For the second company (b), the profit dropped, and then rose again later.

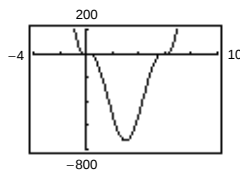
44. (a) $f(x) = x^2(x - 6)^2$



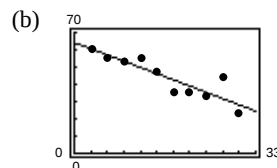
(b) $g(x) = x^3(x - 6)^2$



(c) $h(x) = x^3(x - 6)^3$



48. (a) $y = -1.204x + 64.2667$



- (c) The data point (27, 44) is probably an error. Without this point, the new model is

$$y = -1.4344x + 66.4387.$$

Problem Solving for Chapter P

2. Let $y = mx + 1$ be a tangent line to the circle from the point (0, 1). Then

$$x^2 + (y + 1)^2 = 1$$

$$x^2 + (mx + 1 + 1)^2 = 1$$

$$(m^2 + 1)x^2 + 4mx + 3 = 0$$

Setting the discriminant $b^2 - 4ac$ equal to zero,

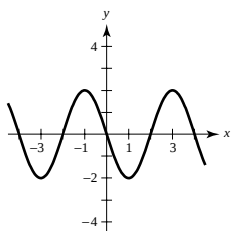
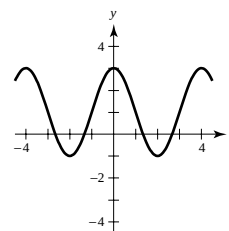
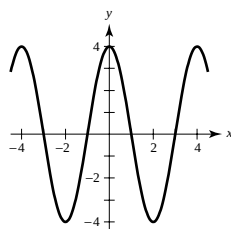
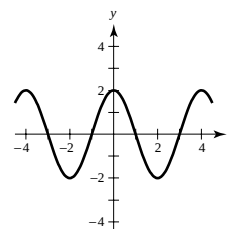
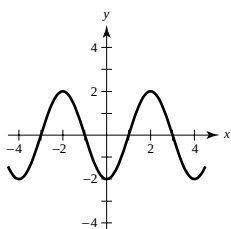
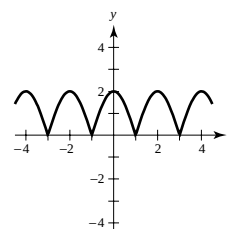
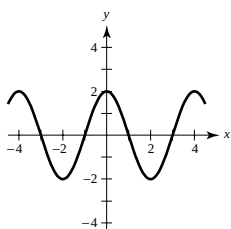
$$16m^2 - 4(m^2 + 1)(3) = 0$$

$$16m^2 - 12m^2 = 12$$

$$4m^2 = 12$$

$$m = \pm\sqrt{3}$$

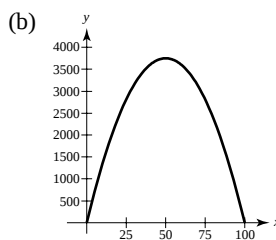
Tangent lines: $y = \sqrt{3}x + 1$ and $y = -\sqrt{3}x + 1$.

4. (a) $f(x + 1)$

 (b) $f(x) + 1$

 (c) $2f(x)$

 (d) $f(-x)$

 (e) $-f(x)$

 (f) $|f(x)|$

 (g) $f(|x|)$


6. (a) $4y + 3x = 300 \Rightarrow y = \frac{300 - 3x}{4}$

$$A(x) = x(2y) = x\left(\frac{300 - 3x}{2}\right) = \frac{-3x^2 + 300x}{2}$$

$$\text{Domain: } 0 < x < 100$$


 Maximum of 3750 ft² at $x = 50$ ft, $y = 37.5$ ft.

(c) $A(x) = -\frac{3}{2}(x^2 - 100x)$

$$= -\frac{3}{2}(x^2 - 100x + 2500) + 3750$$

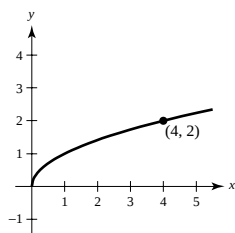
$$= -\frac{3}{2}(x - 50)^2 + 3750$$

$A(50) = 3750$ square feet is the maximum area,
where $x = 50$ ft and $y = 37.5$ ft.

8. Let d be the distance from the starting point to the beach.

$$\begin{aligned}
 \text{Average velocity} &= \frac{\text{distance}}{\text{time}} \\
 &= \frac{2d}{\frac{d}{120} + \frac{d}{60}} \\
 &= \frac{2}{\frac{1}{120} + \frac{1}{60}} \\
 &= 80 \text{ km/hr}
 \end{aligned}$$

10.



(a) Slope $= \frac{3-2}{9-4} = \frac{1}{5}$. Slope of tangent line is greater than $\frac{1}{5}$.

(b) Slope $= \frac{2-1}{4-1} = \frac{1}{3}$. Slope of tangent line is less than $\frac{1}{3}$.

(c) Slope $= \frac{2.1-2}{4.1-4} = \frac{10}{41}$. Slope of tangent line is greater than $\frac{10}{41}$.

(d) Slope $= \frac{f(4+h) - f(4)}{(4+h) - 4}$
 $= \frac{\sqrt{4+h} - 2}{h}$

(e) $\frac{\sqrt{4+h} - 2}{h} = \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$
 $= \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)}$
 $= \frac{1}{\sqrt{4+h} + 2}, h \neq 0$

As h gets closer to 0, the slope gets closer to $\frac{1}{4}$. The slope is $\frac{1}{4}$ at the point $(4, 2)$.

$$12. (a) \quad \frac{I}{\sqrt{x^2 + y^2}} = \frac{kI}{\sqrt{(x-4)^2 + y^2}}$$

$$(x-4)^2 + y^2 = k^2(x^2 + y^2)$$

$$(k^2 - 1)x^2 + (k^2 - 1)y^2 + 8x = 16$$

If $k = 1$, then $x = 2$ is a vertical line. So, assume $k^2 - 1 \neq 0$. Then

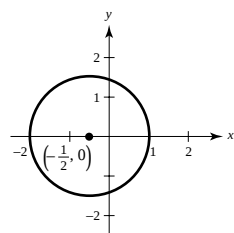
$$x^2 + y^2 + \frac{8x}{k^2 - 1} = \frac{16}{k^2 - 1}$$

$$\left(x + \frac{4}{k^2 - 1}\right)^2 + y^2 = \frac{16}{k^2 - 1} + \frac{16}{(k^2 - 1)^2}$$

$$\left(x + \frac{4}{k^2 - 1}\right)^2 + y^2 = \left(\frac{4k}{k^2 - 1}\right)^2, \text{ Circle}$$

$$(b) \text{ If } k = 3, \left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$$

(c) For large k , the center of the circle is near $(0, 0)$, and the radius becomes smaller.



$$14. f(x) = y = \frac{1}{1-x}$$

(a) Domain: all $x \neq 1$

Range: all $y \neq 0$

$$(b) f(f(x)) = f\left(\frac{1}{1-x}\right) = \frac{1}{1 - \left(\frac{1}{1-x}\right)} = \frac{1}{\frac{1-x-1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$$

Domain: all $x \neq 0, 1$

$$(c) f(f(f(x))) = f\left(\frac{x-1}{x}\right) = \frac{1}{1 - \left(\frac{x-1}{x}\right)} = \frac{1}{\frac{1}{x}} = x$$

Domain: all $x \neq 0, 1$

(d) The graph is not a line. It has holes at $(0, 0)$ and $(1, 1)$.

